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A CONVERGENCE ANALYSIS OF THREE-STEP NEWTON-LIKE METHOD UNDER WEAK CONDITIONS IN BANACH SPACES

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Abstract

In this paper, we present a local and semi-local convergence analysis for three-step Newton-like method in order to approximate a solution of nonlinear equations in a Banach space. The convergence ball and error estimates are given for these methods. Also, we give some numerical examples for this study. Our results are generalizations of the analogous ones recently proved by Sahu et al. [3].

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1 Introduction

Throughout this paper, $\mathbb{N}$ will denote the set of all positive integers and $B(X,Y)$ will designate the space of all bounded linear operators from X to Y. Also, for given any $x \in X$ and $r > 0$, $B_r[x] = \{y \in X : \|y - x\| \leq r\}$ and $B_r(x) = \{y \in X : \|y - x\| < r\}$.

One of the most important problems in analysis is the solution of nonlinear equation

$$F(x) = 0. \tag{1}$$

in a Banach space X, where F is a Fréchet differentiable operator on some open convex subset C of X with values in another Banach space Y. To solve these equations we can use different iterative methods. One of them is the Newton method defined by

$$x_{n+1} = x_n - \left[F'(x_n)\right]^{-1} F(x_n),\ n \in \mathbb{N}, \tag{2}$$

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where x_0 is an initial point, $F'(x)$ denotes the Fréchet derivative of F at the point $x \in C$.

Argyros [1], Dennis [2] and Rheinboldt [6] considered the following modified Newton method to avoid the inverse of derivative which is required at each step in Newton method (2),

$$x_{n+1} = x_n - \left[F'(x_0)\right]^{-1} F(x_n),\ n \in \mathbb{N}. \tag{3}$$

There are two types of the convergence analysis of iterative methods, semi-local and local convergence analysis. The semi-local convergence analysis is, based on the information around an initial point while the local one is, based on the information around a solution.

In 2012, Sahu et al. [3] introduced the S-iteration process of Newton-like type for finding the solution of operator equation (1) as follows:

Let $x_0 \in X$ nad $\alpha \in (0,1)$. The sequence $\{x_{n+1}\}$ is defined by;

$$\begin{cases} x_{n+1} = z_n - [F'(x_0)]^{-1} F(z_n) \\ z_n = (1-\alpha)x_n + \alpha y_n, \\ y_n = x_n - [F'(x_0)]^{-1} F(x_n),\ n \in \mathbb{N}. \end{cases} \tag{4}$$

Moreover, he showed that the rate of convergence of this iterative process is faster than (3).

Let $x^* \in C$ be a solution of (1) such that $[F'(x^*)]^{-1} \in B(Y,X)$. For some $L_0, L_1, L_2 > 0$, $x_0 \in C$ and $\forall x \in C$, suppose that $[F'(x^*)]^{-1}$ and F satisfy the following conditions:

$$\left\|F'(x) - F'(x_0)\right\| \leq L_0 \left\|x - x_0\right\|, \tag{5}$$

$$\left\|\left[F'(x^*)\right]^{-1} \left(F'(x) - F'(x_0)\right)\right\| \leq L_1 \left\|x - x_0\right\|, \tag{6}$$

and

$$\left\|\left[F'(x^*)\right]^{-1} \left(F'(x) - F'(x^*)\right)\right\| \leq L_2 \left\|x - x^*\right\|. \tag{7}$$

In 2012, Sahu et al. [3] proved some theorems on the semi-local and local convergence of the iterative method (4) to solve the operator equation (1).

Recently, Karaca and Yildirim [10] introduced iteration process (8) for the class of nonexpansive mapping in Banach spaces and they proved that this iteration process is faster than Picard iteration, S and normal S-iterations [7].

Let X be a normed space, C a nonempty convex subset of X and $T : C \longrightarrow C$ an operator. Then, for arbitrary $x_0 \in C$

$$\begin{cases} x_{n+1} = Ty_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n T z_n, \\ z_n = (1-\beta_n)x_n + \beta_n T x_n,\ n \in \mathbb{N} \end{cases} \tag{8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.

Now, we introduce iteration process (8) of Newton-like type in order to find the solution of operator equation (1).

Let C be an open convex subset of a Banach space X. Assume that $F : X \to Y$ is a Frechet differentiable operator where Y is a Banach space. Let $\alpha, \beta \in (0, 1)$. Starting with $x_0 \in X$ and $\{x_{n+1}\}$ is defined by:

$$\begin{cases} x_{n+1} = y_n - [F'(x_0)]^{-1} F(y_n) \\ y_n = z_n - \alpha [F'(x_0)]^{-1} F(z_n) \\ z_n = x_n - \beta [F'(x_0)]^{-1} F(x_n), \ n \in \mathbb{N}, \end{cases} \tag{9}$$

which can be written in the following compact form:

$$\begin{cases} x_{n+1} = y_n - [F'(x_0)]^{-1} F(y_n) \\ y_n = (1-\alpha) z_n - \alpha \left[z_n - [F'(x_0)]^{-1} F(z_n) \right] \\ z_n = (1-\beta) x_n + \beta \left[x_n - [F'(x_0)]^{-1} F(x_n) \right], \ n \in \mathbb{N}. \end{cases}$$

The aim of this paper is to prove semi-local and local convergence analysis of iteration process (9). It is shown that the rate of convergence of (9) is faster than (4). Applications to numerical examples are included.

Definition 1. *Let C be a nonempty subset of normed space X. A mapping $T : C \longrightarrow X$ is called:*

(i) *Contraction if there exists a constant $L \in (0, 1)$ such that $\|T(x) - T(y)\| \leq L \|x - y\|$ for all $x, y \in C$.*

(ii) *Quasi-contraction [8] if there exists a constant $L \in (0, 1)$ and $F(T) = \{x \in C : T(x) = x\} \neq \emptyset$ such that $\|T(x) - p\| \leq L \|x - p\|$ for all $x \in C$ and $p \in F(T)$.*

(iii) *Lipschitzian if there exists a constant $L > 0$ such that $\|T(x) - T(y)\| \leq L \|x - y\|$ for all $x, y \in C$.*

In the sequel, we also need the following lemmas.

Lemma 1. *[4] Let T be a bounded linear operator on a Banach space X. Then the following are equivalent:*

(a) *There is a bounded linear operator P on X such that P^{-1} exists, and $\|P - T\| < \frac{1}{\|P^{-1}\|}$.*

(b) *T^{-1} exists. Moreover, if T^{-1} exists, then*

$$\|T^{-1}\| \leq \frac{\|P^{-1}\|}{1 - \|1 - P^{-1}T\|} \leq \frac{\|P^{-1}\|}{1 - \|P^{-1}\| \, \|P - T\|}.$$

Lemma 2. *[5] Let X and Y be Banach spaces, C be a nonempty open convex subset of X and $F : X \to Y$ be a Fréchet differentiable operator. Assume that $x^* \in C$ is a solution of (1) such that $[F'(x^*)]^{-1} \in B(Y,X)$ and the operator F satisfies the conditions (7). Suppose that $B_r(x^*) \subseteq C$, where $r = \frac{1}{L_2}$. Then, for any $x \in B_r(x^*)$, $F'(x^*)$ is invertible, and the following inequality holds:*

$$\left\| \left(\left[F'(x^*) \right]^{-1} F'(x^*) \right)^{-1} \right\| \leq \frac{1}{1 - L_2 \|x - x^*\|}.$$

Lemma 3. *[9] Let X and Y be Banach spaces, C be a nonempty open convex subset of X and $F : X \to Y$ be a Fréchet differentiable operator. Then, for all $x, y \in C$,*

$$F(x) - F(y) = \int_0^1 F'(y + t(x-y))(x-y)dt.$$

2 Main results

Before studying convergence analysis of (9), we will give the following theorem.

Let the operators $T, S_\alpha, S_\beta : B_r[x_0] \longrightarrow X$ be defined as follows: for all $x \in B_r[x_0]$ and for fixed $\alpha, \beta \in (0,1)$,

$$\begin{aligned} T(x) &= x - \left[F'(x_0) \right]^{-1} F(x) \\ S_\alpha(x) &= x - \alpha \left[F'(x_0) \right]^{-1} F(x) \\ S_\beta(x) &= x - \beta \left[F'(x_0) \right]^{-1} F(x). \end{aligned} \tag{10}$$

Theorem 1. *Let F be a Fréchet differentiable operator defined on an open convex subset C of a Banach space X with values in a Banach space Y. Suppose that $[F'(x_0)]^{-1} \in B(Y,X)$ for some $x_0 \in C$ and the operator F satisfies the conditions (5) and $\left\| [F'(x_0)]^{-1} F(x_0) \right\| \leq k$ and $\left\| [F'(x_0)]^{-1} \right\| \leq l$ for some $k, l > 0$. Moreover assume that $\gamma = klL_0 < \frac{1}{2}$ and $B_r[x_0] \subseteq C$, where $r = \frac{1-\sqrt{1-2\gamma}}{\gamma} k$. Then,*

(i) The operators (10) are contractions on $B_r[x_0]$ with constant δ, $(1 - \alpha(1-\delta))$ and $(1 - \beta(1-\delta))$ and the operator equation (1) has a uniqe solution in $B_r[x_0]$.

(ii) The operator $G : B_r[x_0] \longrightarrow X$, $G(x) = TS_\alpha S_\beta(x)$ generated by (10) is a contraction on $B_r[x_0]$ with constant $\delta(1 - \alpha(1-\delta))(1 - \beta(1-\delta))$.

Proof. (i) Denote by $\delta := lrL_0$. This implies that $\delta < 1$. For all $x, y \in B_r[x_0]$, we obtain that

$$
\begin{aligned}
& \left\| S_\beta(x) - S_\beta(y) \right\| \\
= \; & \left\| x - y - \beta \left[F'(x_0) \right]^{-1} \left(F(x) - F(y) \right) \right\| \\
= \; & \beta \left\| x - y - \left[F'(x_0) \right]^{-1} \left(F(x) - F(y) \right) \right\| + (1-\beta) \left\| x - y \right\| \\
= \; & \beta \left\| x - y - \left[F'(x_0) \right]^{-1} \int_0^1 F'(y + t(x-y))(x-y)dt \right\| + (1-\beta) \left\| x - y \right\| \\
= \; & \beta \left\| \left[F'(x_0) \right]^{-1} \left[\int_0^1 F'(y + t(x-y))(x-y)dt - \int_0^1 F'(x_0)(x-y)dt \right] \right\| \\
& + (1-\beta) \left\| x - y \right\| \\
\leq \; & \beta \left\| \left[F'(x_0) \right]^{-1} \right\| \left\| \int_0^1 F'(y + t(x-y))(x-y)dt - \int_0^1 F'(x_0)(x-y)dt \right\| \\
& + (1-\beta) \left\| x - y \right\| \\
\leq \; & \beta \left\| \left[F'(x_0) \right]^{-1} \right\| \int_0^1 \left\| F'(y + t(x-y))(x-y) - F'(x_0)(x-y) \right\| dt \\
& + (1-\beta) \left\| x - y \right\| \\
\leq \; & \beta l \int_0^1 \left\| F'(y + t(x-y)) - F'(x_0) \right\| \left\| x - y \right\| dt + (1-\beta) \left\| x - y \right\| \\
\leq \; & \beta l \int_0^1 L_0 \left\| y + t(x-y) - x_0 \right\| \left\| x - y \right\| dt + (1-\beta) \left\| x - y \right\| \\
\leq \; & \beta l r L_0 \left\| x - y \right\| + (1-\beta) \left\| x - y \right\| \\
= \; & (1 - \beta(1 - lrL_0)) \left\| x - y \right\| \\
= \; & (1 - \beta(1 - \delta)) \left\| x - y \right\|.
\end{aligned}
$$

Since $\delta < 1$, we get that $(1 - \beta(1-\delta)) < 1$ Thus, the operator S_β is a contraction. Now we need to show that $S_\beta(B_r[x_0]) \subseteq B_r[x_0]$. For $x \in B_r[x_0]$, we get that

$$
\begin{aligned}
& \left\| S_\beta(x) - x_0 \right\| \leq \left\| S_\beta(x) - S_\beta(x_0) \right\| + \left\| S_\beta(x_0) - x_0 \right\| \\
= \; & \left\| x - x_0 - \beta \left[F'(x_0) \right]^{-1} \left(F(x) - F(x_0) \right) \right\| \\
& + \left\| x_0 - \beta \left[F'(x_0) \right]^{-1} F(x_0) - x_0 \right\| \\
\leq \; & \beta \left\| x - x_0 - \left[F'(x_0) \right]^{-1} \left(F(x) - F(x_0) \right) \right\| + (1-\beta) \left\| x - x_0 \right\| + \beta k \\
\leq \; & \beta \left\| \left[F'(x_0) \right]^{-1} \right\| \int_0^1 \left\| F'(x_0 + t(x-x_0)) - F'(x_0) \right\| \left\| x - x_0 \right\| dt \\
& + (1-\beta) \left\| x - x_0 \right\| + \beta k \\
\leq \; & \beta l \int_0^1 L_0 \left\| t(x-x_0) \right\| \left\| x - x_0 \right\| dt + (1-\beta) \left\| x - x_0 \right\| + \beta k \\
\leq \; & \beta l L_0 \left\| x - x_0 \right\|^2 \int_0^1 t dt + (1-\beta) \left\| x - x_0 \right\| + \beta k
\end{aligned}
$$

$$\begin{aligned} &\leq \frac{\beta l r^2 L_0}{2} + (1-\beta)\, r + \beta k \\ &\leq r, \end{aligned}$$

which means that $S_\beta\,(B_r[x_0]) \subseteq B_r[x_0]$.

Similarly, we obtain that

$$\|S_\alpha\,(x) - S_\alpha\,(y)\| \leq (1-\alpha\,(1-\delta))\,\|x-y\| \text{ and } \|T\,(x) - T\,(y)\| \leq \delta\,\|x-y\|\,. \tag{11}$$

Thus S_α and T are contractions with constant $(1-\alpha\,(1-\delta))$ and δ, respectively.

Also, for $x \in B_r[x_0]$, we get that

$$\|S_\alpha\,(x) - x_0\| \leq r \text{ and } \|T\,(x) - x_0\| \leq r.$$

This implies that $S_\alpha\,(B_r[x_0]) \subseteq B_r[x_0]$ and $T\,(B_r[x_0]) \subseteq B_r[x_0]$. Therefore, by Banach contraction principle, S_β, S_α and T have a uniqe fixed point in $B_r[x_0]$.

(ii) From (11) and (11),

$$\begin{aligned} \|G\,(x) - G\,(y)\| &= \|TS_\alpha S_\beta\,(x) - TS_\alpha S_\beta\,(y)\| \\ &\leq \delta\,\|S_\alpha S_\beta\,(x) - S_\alpha S_\beta\,(y)\| \\ &\leq \delta\,(1-\alpha\,(1-\delta))\,\|S_\beta\,(x) - S_\beta\,(y)\| \\ &\leq \delta\,(1-\alpha\,(1-\delta))\,(1-\beta\,(1-\delta))\,\|x-y\|\,. \end{aligned}$$

Thus, the operator G is a contraction with constant
$\delta\,(1-\alpha\,(1-\delta))\,(1-\beta\,(1-\delta))\,.$ □

Theorem 2. *Let F be a Fréchet differentiable operator defined on an open convex subset C of a Banach space X with values in a Banach space Y. Suppose that $[F'\,(x_0)]^{-1} \in B(Y,X)$ for some $x_0 \in C$ and the operator F satisfies the conditions (5) and $\left\|[F'\,(x_0)]^{-1}\,F(x_0)\right\| \leq k$ and $\left\|[F'\,(x_0)]^{-1}\right\| \leq l$ for some $k,l > 0$. Moreover assume that $\gamma = klL_0 < \frac{1}{2}$ and $B_r[x_0] \subseteq C$, where $r = \frac{1-\sqrt{1-2\gamma}}{\gamma}k$. Then,*

(i) The sequence $\{x_n\}$ defined by (9) is in $B_r[x_0]$ and it converges strongly to x^*.

(ii) The following inequality holds:

$$\|x_{n+1} - x^*\| \leq \lambda^{n+1}\,\|x_0 - x^*\|\,, \tag{12}$$

where $\lambda = \delta\,(1-\alpha\,(1-\delta))\,(1-\beta\,(1-\delta))$ and $\delta = lrL_0$.

Proof. (i) From Theorem 1, there is a unique solution of the equation (1). Let

$x^* \in B_r[x_0]$. Using (9), we get

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|G(x_n) - G(x^*)\| \\ &= \|TS_\alpha S_\beta(x_n) - TS_\alpha S_\beta(x^*)\| \\ &\leq \delta \|S_\alpha S_\beta(x_n) - S_\alpha S_\beta(x^*)\| \\ &\leq \delta (1 - \alpha (1 - \delta)) \|S_\beta(x_n) - S_\beta(x^*)\| \\ &\leq \delta (1 - \alpha (1 - \delta)) (1 - \beta (1 - \delta)) \|x_n - x^*\| \\ &\quad \vdots \\ &\leq [\delta (1 - \alpha (1 - \delta)) (1 - \beta (1 - \delta))]^{n+1} \|x_0 - x^*\| . \end{aligned} \tag{13}$$

Thus, $x_n \longrightarrow x^*$, as $n \longrightarrow \infty$.
(ii) It follows from (13). □

Remark 1. *From the inequality (3.9) in Theorem 3.5 in [3] and (12)*

$$\lambda = \delta (1 - \alpha (1 - \delta)) (1 - \beta (1 - \delta)) < \delta (1 - \alpha + \alpha\delta) . \tag{14}$$

The inequality (14) shows that the error estimate in Theorem 2 is sharper than that of Theorem 3.5 in [3].

Our next result is as follows:

Theorem 3. *Let F be a Fréchet differentiable operator defined on an open convex subset C of a Banach space X with values in a Banach space Y. Let $x^* \in C$ be a solution of (1) such that $[F'(x^*)]^{-1} \in B(Y, X)$. Suppose that $[F'(x^*)]^{-1}$ and F satisfy the conditions (6) and (7) with $B_r(x^*) \subseteq C$ and $x_0 \in B_r(x^*)$, where $r_1 = \frac{1}{L_2}$ and $r = \frac{2}{2L_2 + 3L_1}$. Moreover, assume that the operators S_β, S_α and T are defined by (10). Then*

(i) For $x \in B_r(x^*)$,

$$\begin{aligned} \|T(x) - x^*\| &\leq \gamma_x \|x - x^*\| , \\ \|S_\alpha(x) - x^*\| &\leq (\alpha\gamma_x + 1 - \alpha) \|x - x^*\| , \\ \|S_\beta(x) - x^*\| &\leq (\beta\gamma_x + 1 - \beta) \|x - x^*\| , \end{aligned} \tag{15}$$

where $\gamma_x = \frac{L_1}{2(1 - rL_2)} (\|x - x^*\| + 2\|x_0 - x^*\|)$.

(ii) T, S_α and S_β are quasi-contraction operators on $B_r(x^*)$ with constant γ, $1 - (1 - \gamma)\alpha$ and $1 - (1 - \gamma)\beta$, respectively, where $\gamma = \sup_{x \in B_r(x^*)} \{\gamma_x\}$.

Proof. (i) Let $x \in B_r(x^*)$ with $x \neq x^*$. From (10),

$$\begin{aligned} \|T(x) - x^*\| &= \left\| x - [F'(x_0)]^{-1} F(x) - x^* \right\| \\ &= \left\| [F'(x_0)]^{-1} \big((F(x) - F(x^*)) - F'(x_0)(x - x^*) \big) \right\| \\ &= \Big\| ([F'(x^*)]^{-1} F'(x_0))^{-1} \int_0^1 [F'(x^*)]^{-1} \\ &\quad \big(F'(tx + (1 - t)x^*)(x - x^*) - F'(x_0)(x - x^*) \big) dt \Big\| \end{aligned} \tag{16}$$

It follows from (6), (16) and Lemma 2,

$$\begin{aligned}\|T(x)-x^*\| &\leq \frac{L_1}{2(1-rL_2)}\left(\|x-x^*\|+2\|x_0-x^*\|\right)\|x-x^*\| \\ &= \gamma_x\|x-x^*\|.\end{aligned}$$

Using the operator S_α defined by (10),

$$\begin{aligned}&\|S_\alpha(x)-x^*\| \\ =\;& \left\|x-\alpha\left[F'(x_0)\right]^{-1}F(x)-x^*\right\| \\ =\;& \left\|\alpha(x-x^*)-\alpha\left[F'(x_0)\right]^{-1}(F(x)-F(x^*))+(1-\alpha)(x-x^*)\right\| \\ \leq\;& \alpha\left\|\left[F'(x_0)\right]^{-1}(F(x)-F(x^*))-(x-x^*)\right\|+(1-\alpha)\|x-x^*\| \\ =\;& \alpha\Bigg\|(\left[F'(x^*)\right]^{-1}F'(x_0))^{-1}\int_0^1\left[F'(x^*)\right]^{-1}F'(tx+(1-t)x^*) \\ & \big((x-x^*)-F'(x_0)(x-x^*)\big)\,dt\Bigg\|+(1-\alpha)\|x-x^*\| \\ \leq\;& \alpha\left\|(\left[F'(x^*)\right]^{-1}F'(x_0))^{-1}\right\|\int_0^1\left\|\left[F'(x^*)\right]^{-1}\left(F'(tx+(1-t)x^*)-F'(x_0)\right)\right\| \\ & \|x-x^*\|\,dt+(1-\alpha)\|x-x^*\|\end{aligned}$$

Again using (6) and Lemma 2, we obtain that

$$\begin{aligned}&\|S_\alpha(x)-x^*\| \\ \leq\;& \frac{\alpha L_1}{1-L_2\|x_0-x^*\|}\int_0^1\|t(x-x^*)+(x^*-x_0)\|\,\|x-x^*\|\,dt+(1-\alpha)\|x-x^*\| \\ \leq\;& \frac{\alpha L_1}{1-L_2\|x_0-x^*\|}\int_0^1(t\|x-x^*\|+\|x_0-x^*\|)\|x-x^*\|\,dt+(1-\alpha)\|x-x^*\| \\ \leq\;& \frac{\alpha L_1}{2(1-rL_2)}(\|x-x^*\|+2\|x_0-x^*\|)\|x-x^*\|+(1-\alpha)\|x-x^*\| \\ =\;& (\alpha\gamma_x+1-\alpha)\|x-x^*\|.\end{aligned}$$

Similarly, taking β instead of α, we deduce that

$$\|S_\beta(x)-x^*\|\leq(\beta\gamma_x+1-\beta)\|x-x^*\|.$$

(ii) The operators T, S_α and S_β are quasi-contractions. Indeed,

$$\begin{aligned}\gamma &= \sup_{x\in B_r(x^*)}\{\gamma_x\}=\frac{L_1}{2(1-rL_2)}\left(\sup_{x\in B_r(x^*)}\|x-x^*\|+2\|x_0-x^*\|\right) \\ &\leq \frac{L_1}{2(1-rL_2)}(r+2\|x_0-x^*\|) \\ &< \frac{3rL_1}{2(1-rL_2)} \\ &= 1\end{aligned}$$

Since $\gamma < 1$, we get that $1-(1-\gamma)\alpha < 1$ and $1-(1-\gamma)\beta < 1$. $\square$

Finally, we will give the following theorem on the local convergence analysis for iteration process (9).

Theorem 4. *Let F be a Fréchet differentiable operator defined on an open convex subset C of a Banach space X with values in a Banach space Y. Let $x^* \in C$ be a solution of (1) such that $[F'(x^*)]^{-1} \in B(Y,X)$. Suppose that $[F'(x^*)]^{-1}$ and F satisfy the conditions (6) and (7). Also assume that $B_{r_1}(x^*) \subseteq C$, where $r_1 = \frac{1}{L_2}$ and for any $x_0 \in B_r(x^*)$, where $r = \frac{2}{2L_2+3L_1}$. Then*

(i) The sequence $\{x_n\}$ defined by (9) is in $B_r(x^*)$ and it converges strongly to the unique solution x^* in $B_{r_1}(x^*)$.

(ii) The following inequality holds:

$$\|x_{n+1} - x^*\| \leq (\lambda')^{n+1} \|x_0 - x^*\| \tag{17}$$

where $\lambda' = (\alpha\gamma_0 + 1 - \alpha)(\beta\gamma_0 + 1 - \beta)\gamma_0$ and $\gamma_0 = \frac{3L_1}{2(1-rL_2)} \|x_0 - x^*\|$.

Proof. (i) Firstly, we will show that x^* is unique solution of (1) in $B_{r_1}(x^*)$. On the contrary, suppose that y^* is another solution of (1) in $B_{r_1}(x^*)$. From Lemma 3, we obtain

$$0 = F(x^*) - F(y^*) = \int_0^1 F'(y^* + t(x^*-y^*))(x^*-y^*)dt.$$

Now, we can define an operator L by

$$L(h) = \int_0^1 F'(y^* + t(x^*-y^*))\, h dt,$$

for all $h \in X$. Thus, we get

$$\begin{aligned}
\left\| I - [F'(x^*)]^{-1} L \right\| &= \left\| \int_0^1 [F'(x^*)]^{-1} (F'(x^*) - F'(y^* + t(x^*-y^*)))dt \right\| \\
&\leq \frac{L_2}{2} \|x^* - y^*\| \\
&< \frac{L_2 r_1}{2} \\
&= \frac{1}{2}.
\end{aligned}$$

From Lemma 1, we know that the operator L is invertible. Thus, $x^* = y^*$, which is a contradiction. That is, x^* is the unique solution of (1) in $B_{r_1}(x^*)$.

Since the S_α and S_β defined by (10) are operators on $B_r[x^*]$, from (9), we get

$$z_0 = S_\beta(x_0),\ y_0 = S_\alpha S_\beta(x_0) \in B_r(x^*).$$

Also the operator V defined by (10) is an operator on $B_r[x^*]$ from Theorem 3. Therefore, (9) can be written as

$$x_{n+1} = TS_\alpha S_\beta(x_n). \tag{18}$$

From (18), we have

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|TS_\alpha S_\beta(x_n)-x^*\|\\ &\leq \gamma_{y_n}\|S_\alpha S_\beta(x_n)-x^*\|\\ &\leq \gamma_{y_n}(\alpha\gamma_{z_n}+1-\alpha)\|S_\beta(x_n)-x^*\|\\ &\leq \gamma_{y_n}(\alpha\gamma_{z_n}+1-\alpha)(\beta\gamma_{z_n}+1-\beta)\|x_n-x^*\|.\end{aligned}$$

By using the definition of γ_x, we obtain

$$\begin{aligned}\gamma_{y_n} &= \frac{L_1}{2(1-rL_2)}\left(\|S_\alpha S_\beta(x_n)-x^*\|+2\|x_0-x^*\|\right)\\ &\leq \frac{L_1}{2(1-rL_2)}\left[(\alpha\gamma_{z_n}+1-\alpha)\|S_\beta(x_n)-x^*\|+2\|x_0-x^*\|\right]\\ &\leq \frac{L_1}{2(1-rL_2)}\left[\|S_\beta(x_n)-x^*\|+2\|x_0-x^*\|\right]\\ &\leq \frac{L_1}{2(1-rL_2)}\left[(\beta\gamma_{z_n}+1-\beta)\|x_n-x^*\|+2\|x_0-x^*\|\right]\\ &\leq \frac{L_1}{2(1-rL_2)}\left[\|x_0-x^*\|+2\|x_0-x^*\|\right]\\ &\leq \frac{3L_1}{2(1-rL_2)}\|x_0-x^*\|\\ &= \gamma_0.\end{aligned}$$

Hence, we have

$$\|x_{n+1}-x^*\| \leq [(\alpha\gamma_0+1-\alpha)(\beta\gamma_0+1-\beta)\gamma_0]^{n+1}\|x_0-x^*\|. \tag{19}$$

which implies that $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$.
(ii) From (19), we obtain that desired error estimate. □

3 Numerical Examples

In this section, we will give two numerical examples in order to support our main results. The first example shows numerically that our method (9) is faster than the modified Newton method (3) and SIP of Newton-like (4).

Example 1. *Let $X=\mathbb{R}$, $C=(1,4)$ and $F: C \to \mathbb{R}$ an operator defined by*

$$F(x)=x^2-4 \quad , \quad \forall x \in C.$$

Then F is Frechet differentiable and its Frechet derivative $F^{'}(x)$ at any point $x \in C$ is given by $F^{'}(x)=2x$. For $x_0=3$, we have $[F'(x_0)]^{-1}=\frac{1}{6}$. Set $k=0.83333333333$, $l=0.16666666667$ and $L_0=2$, we have $\gamma=klL_0<\frac{1}{2}$ and

$$\begin{aligned}\left\|[F'(x_0)]^{-1}\right\| &\leq l,\\ \left\|[F'(x_0)]^{-1}F(x_0)\right\| &\leq k,\\ \left\|F^{'}(x)-F'(x_0)\right\| &\leq L_0\|x-x_0\|.\end{aligned}$$

Because all the conditions of Theorem 2 are satisfied, the sequence $\{x_n\}$ generated by (9) is in $B_r[x_0]$ and it converges to a unique $x^* \in B_r[x_0]$. From the following table and figure, we see that the new method (9) is faster than the modified Newton method (3) and SIP of Newton-like (4).

n	modified Newton method	SIP of Newton-like	our method
1	3.000000000000000	3.000000000000000	3.000000000000000
2	2.166666666666667	2.137731481481481	2.098513533003972
3	2.050925925925926	2.028722838184148	2.013991346662300
4	2.016543066986740	2.006299263040182	2.002060351400436
5	2.005468743484692	2.001395799303914	2.000304966616767
6	2.001817929969014	2.000309979239996	2.000045174307603
7	2.000605425844776	2.000068874489454	2.000006692359848
8	2.000201747524850	2.000015304958953	2.000000991457861
9	2.000067242391272	2.000003401078132	2.000000146882583
10	2.000022413376834	2.000000755793962	2.000000021760382
11	2.000007471041885	2.000000167954156	2.000000003223760
12	2.000002490337992	2.000000037323143	2.000000000477594
13	2.000000830111631	2.000000008294032	2.000000000070755
14	2.000000276703762	2.000000001843118	2.000000000010482
15	2.000000092234575	2.000000000409582	2.000000000001553
16	2.000000030744857	2.000000000091018	2.000000000000230
17	2.000000010248285	2.000000000020226	2.000000000000034
18	2.000000003416095	2.000000000004495	2.000000000000005
19	2.000000001138699	2.000000000000999	2.000000000000001
20	2.000000000379566	2.000000000000222	2.000000000000000

The following example shows the convergence of (9) in infinite dimensional space.

Example 2. *Let $X = C[0, \frac{1}{4}]$ be the space of real-valued continuous functions defined on the interval $[0, \frac{1}{4}]$ with norm $\|x\| = \max_{0\leq t\leq \frac{1}{4}} |x(t)|$. Consider the integral equation of Fredholm type $F(x) = 0$, where*

$$F(x)(s) = \cos 2\pi s - \frac{\pi}{2}\int_0^{1/4} s \sin 2\pi t x^2(t)dt,$$

with $s \in [0, \frac{1}{4}]$ and $x \in C[0, \frac{1}{4}]$. It is easy to find the Frèchet derivative of F as

$$F^{'}(x)\, h(s) = h(s) + \pi \int_0^{1/4} s \sin 2\pi t x(t) h(t) dt,\ \forall h \in X.$$

Now, we have

$$\begin{aligned}
\|F(x_0)\| &= \left\| \cos 2\pi s - \frac{\pi}{2}\int_0^{1/4} s \sin 2\pi t x_0^2(t)dt \right\| \\
&\leq 1 + \left|\frac{\pi}{2}\right| \max_{s\in[0,\frac{1}{4}]} \left| \int_0^{1/4} s \sin 2\pi t dt \right| \|x_0\|^2 \\
&\leq 1 + \frac{\|x_0\|^2}{8}
\end{aligned}$$

and

$$\begin{aligned}\left\|I - F'(x_0)\right\| &= \left\|\pi \int_0^{1/4} s \sin 2\pi t x_0(t) dt\right\| \\ &\le |\pi| \max_{s\in[0,\frac{1}{4}]} \left|\int_0^{1/4} s \sin 2\pi t dt\right| \|x_0\| \\ &\le \frac{\|x_0\|}{4}.\end{aligned}$$

Moreover, we get

$$\begin{aligned}\left\|F'(x) - F'(x_0)\right\| &= \left\|\pi \int_0^{1/4} s \sin 2\pi t x(t) dt - \pi \int_0^{1/4} s \sin 2\pi t x_0(t) dt\right\| \\ &\le |\pi| \left\|\int_0^{1/4} s \sin 2\pi t \left(x(t) - x_0(t)\right) dt\right\| \\ &\le \pi \max_{s\in[0,\frac{1}{4}]} \left|\int_0^{1/4} s \sin 2\pi t dt\right| \|x - x_0\| \\ &\le \frac{\|x - x_0\|}{4}\end{aligned}$$

and if $\frac{\|x_0\|}{4} < 1$, *then by Lemma 1, we obtain*

$$\left\|\left[F'(x_0)\right]^{-1}\right\| \le \frac{4}{4 - \|x_0\|}.$$

Thus, we get

$$\left\|\left[F'(x_0)\right]^{-1} F(x_0)\right\| \le \frac{8 + \|x_0\|^2}{8 - 4\|x_0\|}.$$

For the initial point $x_0(s) = \frac{1}{2}$, *we obtain*

$$\begin{aligned}\left\|\left[F'(x_0)\right]^{-1}\right\| &\le l = 1.142857 \\ \left\|\left[F'(x_0)\right]^{-1} F(x_0)\right\| &\le k = 1.178571 \\ L_0 &= 0.25.\end{aligned}$$

Hence, $\gamma = klL_0 = 0.336734 < \frac{1}{2}$. *So the conditions of Theorem 2 are satisfied. Therefore, the sequence* $\{x_n\}$ *generated by (9) is in* $B_r[x_0]$ *and it converges to a uniqe solution* $x^* \in B_r[x_0]$ *of the integral equation.*

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