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THERMOELASTICITY WITH FRACTIONAL ORDER STRAIN FOR DIPOLAR MATERIALS WITH VOIDS

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Abstract

The goal of this article is to combine the theory based on fractional order of strain with the theory of thermoelasticity of dipolar bodies with voids, in order to obtain the basic thermoelasticity equations with fractional order strain for dipolar materials with voids.

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1 Introduction

A theory of linear elastic materials with voids or vacuous pores was presented by S.C.Cowin and J.W Nunziato in [4] , where they applied the theory to some problems of technological interest.

In fact, this theory is a linearization of a theory described by them earlier in [14], where they presented a nonlinear theory for the behaviour of porous solids. They developed the first study on porous materials made by M.A.Goodman and S.C.Cowin in [5].

The presence of small pores (or voids) in the conventional continuum model is introduced by assigning an additional degree of freedom to each particle, namely the fraction of elementary volume that is possibly found void of mater.

This additional degree of freedom is useful in order to develop the mechanical behaviour of porous solids in which the matrix material is elastic and the interstices are voids of material [11].

D. Ieşan has studied a linear theory of thermoelastic materials with voids in [9], treating, on the one hand, some general theorems (uniqueness, reciprocal and variational theorems) and, on the other hand, some problems of equilibrium.

The first results on dipolar bodies theory, which is a part of a multipolar structures theory, were published by R.D.Mindlin in [13], A.E.Green and R.S.Rivlin in [6].

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In the theory of dipolar continua, each material point is constrained to deform homogeneously. In that case, the degrees of freedom for each particle are three translations and nine micro-deformations [12]. In [10] , M.Marin applied the general results from the theory of elliptic equations, in order to obtain the existence and uniqueness of the generalized solutions for the boundary value problems in elasticity of dipolar materials with voids.

H.M.Youssef has derived in [17] a new theory based on fractional order of strain (fraction order Duhamel-Neumann stress-strain relation). We combine this theory with the theory of thermoelasticity of dipolar bodies with voids, in order to obtain the basic equations of the thermoelasticity with fractional order strain for dipolar materials with voids.

2 Notations and basic equations

Considering a thermoelastic, anisotropic dipolar material with voids, which occupies a properly regular region Ω of the three - dimensional Euclidean space $\mathbb{R}^3$, we suppose that Ω is bounded by a $C^{1,1}$ boundary $\partial\Omega$, and the closure of Ω is noted by $\overline{\Omega}$.

The body's movement is reported to a fixed system of rectangular Cartesian axes Ox_i $(i = 1, 2, 3)$, and throughout this article we adopt the Cartesian tensor notation.

The convention, according to which the material time derivative is represented by a superposed dot stand, and a comma followed by a subscript representing partial derivatives, is used in this paper.

At the same time, on repeated indices we use the Einstein summation, and the time argument or the spatial argument of a function will be omitted when the confusion is ruled out.

In Ω the points are denoted by x_i or x. The variable t is the time and $t \in [0, t_0)$.

The thermoelastic dipolar body with voids has a behaviour characterized by the following kinematic variables:

$$u_i = u_i(x,t),\ \Phi_{ij} = \Phi_{ij}(x,t),\ \nu = \nu(x,t),\ (x,t) \in \Omega \times [0, t_0)$$

where $u = (u_i)_{i=\overline{1,3}}$ is the displacement vector field, $\Phi = (\Phi_{ij})_{1\leq,j\leq 3}$ is the dipolar tensor field and ν is the volume fraction field corresponding to voids.

We have the following fundamental equations:

– the equations of motion:

$$\begin{aligned} (t_{ji} + \eta_{ji})_{,j} + \rho_0 F_i = \rho_0 \ddot{u}_i \quad & \text{in } \Omega \times (0, \infty) \\ \mu_{ijk,i} + \eta_{jk} + \rho_0 M_{jk} = I_{ks} \ddot{\Phi}_{js} \quad & \text{in } \Omega \times (0, \infty) \end{aligned} \tag{1}$$

– the equation of energy:

$$\rho_0 T \dot{\eta} = \rho_0 Q + q_{i,i} \tag{2}$$

– the geometric equations:

$$
\begin{aligned}
2\varepsilon_{ij} &= u_{i,j} + u_{j,i} \\
\gamma_{ij} &= u_{j,i} - \Phi_{ij} \\
\chi_{ijk} &= \Phi_{jk,i}
\end{aligned}
\tag{3}
$$

– the balance of equilibrated forces:

$$
\sigma_{i,i} + \xi + \rho_0 L = \rho_0 k \ddot{\nu} \qquad \text{in } \Omega \times (0, \infty)
\tag{4}
$$

– the initial conditions:

$$
\begin{aligned}
u_i(x,0) &= u_i^0(x) \\
\dot{u}_i(x,0) &= u_i^1(x) \\
\Phi_{ij}(x,0) &= \Phi_{ij}^0(x) \\
\dot{\Phi}_{ij}(x,0) &= \Phi_{ij}^1(x) \qquad x \in \overline{\Omega} \\
\nu(x,0) &= \nu^0(x) \\
\dot{\nu}(x,0) &= \nu^1(x) \\
\theta(x,0) &= \theta^0(x)
\end{aligned}
\tag{5}
$$

– the boundary conditions:

$$
\begin{aligned}
u_i(x,t) &= \widetilde{u}_i \\
\Phi_{ij}(x,t) &= \widetilde{\Phi}_{ij} \qquad (x,t) \in \partial\Omega \times [0, \infty) \\
\nu(x,t) &= \widetilde{\nu} \\
\theta(x,t) &= \widetilde{\theta}
\end{aligned}
\tag{6}
$$

where $u_i^0, u_i^1, \Phi_{ij}^0, \Phi_{ij}^1, \nu^0, \nu^1, \theta^0, \widetilde{u}_i, \widetilde{\Phi}_{ij}, \widetilde{\nu}$ and $\widetilde{\theta}$ are prescribed functions.

We postulate the conservation of energy in the form:

$$
\begin{aligned}
&\int_\Omega (\rho_0 \dot{u}_i \ddot{u}_i + I_{ks} \dot{\Phi}_{jk} \ddot{\Phi}_{js} + \rho_0 k \dot{\nu} \ddot{\nu}) dV + \int_\Omega \rho_0 \dot{e} dV = \\
&= \int_\Omega \rho_0 (F_i \dot{u}_i + M_{jk} \dot{\Phi}_{jk} + L \dot{\nu} + Q) dV + \int_{\partial\Omega} (t_i \dot{u}_i + \mu_{jk} \dot{\Phi}_{jk} + \sigma \dot{\nu} + q) dA \ .
\end{aligned}
\tag{7}
$$

The following notations were used in the above equations:
– ρ_0 is the constant reference density;
– $t_{ij}, \eta_{ij}, \mu_{ijk}$ are the components of stress tensors;
– F_i are the components of the body force per unit mass;
– M_{jk} are the components of the dipolar body force per unit mass;
– k is the coefficient of equilibrated inertia;
– L is the extrinsic equilibrated body force per unit mass associated to voids;
– Q is the heat supply per unit mass;
– t_i are the components of the stress vector;
– μ_{jk} is the dipolar stress tensor;
– q is the heat flux vector;
– I_{ks} are the coefficients of microinertia;
– σ is the equilibrated stress corresponding to ν.

– θ is the temperature measured from the constant absolute temperature T_0 of the body considered in the reference state.

Using the Green and Rivlin method, we consider a second motion, different from the given motion only by a superposed constant rigid translation.

All characteristics of the body remain unchanged for the new motion.

The principle of conservation of energy and the other characteristics of the body are still valid if we replace $\dot{u}$ by $\dot{u}_i + \alpha_i$ respectively $\dot{\Phi}_{ij}$ by $\dot{\Phi}_{ij} + \beta_{ij}$, where α_i and β_{ij} are arbitrary constants.

Using the divergence theorem and considering the fact that constants α_i and β_{ij} are arbitrary, we obtain the relations:

$$\left.\begin{aligned} t_i &:= (t_{ji} + \eta_{ji})n_j \\ \mu_{jk} &:= \mu_{ijk} n_i \\ q &:= q_i n_i \\ \sigma &:= \sigma_i n_i \end{aligned}\right\} \quad \text{on } \partial\Omega \tag{8}$$

where n_i are the components of the outward unit normal to the boundary surface.

In this article we use the heat conduction equations introduced by Cattaneo, see [2],[15],[16] equations that take the next form:

$$q_i + \tau_0 \dot{q}_i = -K_{ij} T_{,j}, \quad i = 1, 2, 3; \tag{9}$$

where K_{ij} is the thermal conductvity tensor, and τ_0 is the relaxation time [7],[8].

The fractional derivative with respect to time, introduced by Caputo [1] and used in [15],[17], is defined as:

$$D_t^{\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\beta}} d\tau, \quad 0 \leq \beta \leq 1 \tag{10}$$

where Γ is the Gamma function.

Multiplying, following [3], the $(1)_1$ relation by $\dot{u}_i$, we obtain:

$$(t_{ji} + \eta_{ji})_{,j} \dot{u}_i + \rho_0 F_i \dot{u}_i = \rho_0 \ddot{u}_i \dot{u}_i, \tag{11}$$

then, we have:

$$\int_{\Omega} \rho_0 \dot{u}_i \ddot{u}_i dV = \int_{\Omega} (t_{ji} + \eta_{ji})_{,j} \dot{u}_i dV + \int_{\Omega} \rho_0 F_i \dot{u}_i dV \ . \tag{12}$$

Multiplying , the $(1)_2$ relation by $\dot{\Phi}_{jk}$, we get the relation:

$$\mu_{ijk,i} \dot{\Phi}_{jk} + \eta_{jk} \dot{\Phi}_{jk} + \rho_0 M_{jk} \dot{\Phi}_{jk} = I_{ks} \ddot{\Phi}_{js} \dot{\Phi}_{jk} \ . \tag{13}$$

Naturally, we deduce:

$$\int_{\Omega} I_{ks} \ddot{\Phi}_{js} \dot{\Phi}_{jk} dV = \int_{\Omega} (\mu_{ijk,i} \dot{\Phi}_{jk} + \eta_{jk} \dot{\Phi}_{jk} + \rho_0 M_{jk} \dot{\Phi}_{jk}) dV \ . \tag{14}$$

Multiplying the (4) relation by $\dot{\nu}$ we obtain:

$$\sigma_{i,i}\dot{\nu}+\xi\dot{\nu}+\rho_0 L\dot{\nu}=\rho_0 k\ddot{\nu}\dot{\nu} \tag{15}$$

so, we can write:

$$\int_{\Omega}\rho_0 k\dot{\nu}\ddot{\nu}dV=\int_{\Omega}(\sigma_{i,i}\dot{\nu}+\xi\dot{\nu}+\rho_0 L\dot{\nu})dV\ . \tag{16}$$

Introducing the relations: (8),(12),(14),(16) into (7) we have the following relations:

$$\begin{aligned}&\int_{\Omega}(t_{ji}+\eta_{ji})_{,j}\dot{u}_i dV+\int_{\Omega}\rho_0\dot{e}dV+\int_{\Omega}\mu_{ijk,i}\dot{\Phi}_{jk}dV+\int_{\Omega}\eta_{jk}\dot{\Phi}_{jk}dV+\\&+\int_{\Omega}\sigma_{i,i}\dot{\nu}dV+\int_{\Omega}\xi\dot{\nu}dV=\int_{\Omega}\rho_0 QdV+\int_{\partial\Omega}(t_{ji}+\eta_{ji})n_j\dot{u}_i dA+\\&+\int_{\partial\Omega}\mu_{ijk}\dot{\Phi}_{jk}n_i dA+\int_{\partial\Omega}\sigma_i\dot{\nu}n_i dA+\int_{\partial\Omega}q_i n_i dA\ .\end{aligned} \tag{17}$$

Using the theorem of divergence, the previous relation becomes:

$$\begin{gathered}\int\eta_{jk}\dot{\Phi}_{jk}dV+\int_{\Omega}\xi\dot{\nu}dV+\int_{\Omega}\rho_0\dot{e}dV=\\\int_{\Omega}\rho_0 QdV+\int_{\Omega}(t_{ji}+\eta_{ji})\dot{u}_{i,j}dV+\int_{\Omega}\mu_{ijk}\dot{\Phi}_{jk,i}dV+\int_{\Omega}\sigma_i\dot{\nu}_{,i}dV+\int_{\Omega}q_{i,i}dV\ .\end{gathered} \tag{18}$$

Rewriting the above relation, we get:

$$\int_{\Omega}[\rho_0\dot{e}+\eta_{jk}\dot{\Phi}_{jk}+\xi\dot{\nu}-\rho_0 Q-(t_{ji}+\eta_{ji})\dot{u}_{i,j}-\mu_{ijk}\dot{\Phi}_{jk,i}-\sigma_i\dot{\nu}_{,i}-q_{i,i}]dV=0. \tag{19}$$

Considering that Ω is an arbitrary domain, we deduce from (19) the following equality:

$$\rho_0\dot{e}=(t_{ji}+\eta_{ji})\dot{u}_{i,j}+\mu_{ijk}\dot{\Phi}_{jk,i}-\eta_{jk}\dot{\phi}_{jk}+q_{i,i}+\rho_0 Q-\xi\dot{\nu}+\sigma_i\dot{\nu}_{,i} \tag{20}$$

The Helmholtz free energy $\Phi=e-T\eta$

$$\begin{aligned}\Phi&=\Phi(\widetilde{\varepsilon}_{ij},\gamma_{ij},\chi_{ijk},T,T_{,i},\nu,\nu_{,i})\\e&=e(\widetilde{\varepsilon}_{ij},\gamma_{ij},\chi_{ijk},T,T_{,i},\nu,\nu_{,i})\\\eta&=\eta(\widetilde{\varepsilon}_{ij},\gamma_{ij},\chi_{ijk},T,T_{,i},\nu,\nu_{,i})\\q&=q(\widetilde{\varepsilon}_{ij},\gamma_{ij},\chi_{ijk},T,T_{,i},\nu,\nu_{,i})\ .\end{aligned} \tag{21}$$

were $\widetilde{\varepsilon}_{ij}=\left(1+\tau^{\beta}D_t^{\beta}\right)\varepsilon_{ij}$ and τ is the mechanical relaxation parameter.

Replacing $\dot{e} = \dot{\Phi} + \dot{T}\eta + T\dot{\eta}$ and $\dot{u}_{i,j}$ by $\dot{\tilde{\varepsilon}}_{ij}$ into relation (20) we have:

$$\rho_0\dot{\Phi}+\rho_0\dot{T}\eta+\rho_0T\dot{\eta} = (t_{ji}+\eta_{ji})\dot{\tilde{\varepsilon}}_{ij}+\mu_{ijk}\dot{\Phi}_{jk,i}-\eta_{jk}\dot{\Phi}_{jk}+q_{i,i}+\rho_0Q-\xi\dot{\nu}+\sigma_i\dot{\nu}_{,i}. \quad (22)$$

Using $\dot{\chi}_{ijk} = \dot{\Phi}_{jk,i}$; $\quad \dot{\gamma}_{ij} = \dot{u}_{j,i} - \dot{\Phi}_{ij}$ the relation (22) becomes:

$$\rho_0\dot{\Phi} = t_{ij}\dot{\tilde{\varepsilon}}_{ij} + \eta_{ij}\dot{\gamma}_{ij} + \mu_{ijk}\dot{\chi}_{ijk} + q_{i,i} + \rho_0Q - \xi\dot{\nu} + \sigma_i\dot{\nu}_{,i} - \rho_0\dot{T}\eta - \rho_0T\dot{\eta}. \quad (23)$$

From relation (21) , we get:

$$\begin{aligned}\rho_0\dot{\Phi} =&\rho_0\frac{\partial\Phi}{\partial\tilde{\varepsilon}_{ij}}\dot{\tilde{\varepsilon}}_{ij} + \rho_0\frac{\partial\Phi}{\partial\gamma_{ij}}\dot{\gamma}_{ij} + \rho_0\frac{\partial\Phi}{\partial\chi_{ijk}}\dot{\chi}_{ijk} + \rho_0\frac{\partial\Phi}{\partial T}\dot{T}+\\ &+ \rho_0\frac{\partial\Phi}{\partial T_{,i}}\dot{T}_{,i} + \rho_0\frac{\partial\Phi}{\partial\nu}\dot{\nu} + \rho_0\frac{\partial\Phi}{\partial\nu_{,i}}\dot{\nu}_{,i}\ .\end{aligned} \quad (24)$$

Comparing relations (23) and (24), we obtain

$$\begin{aligned} t_{ij} &= \rho_0\frac{\partial\Phi}{\partial\tilde{\varepsilon}_{ij}}\\ \eta_{ij} &= \rho_0\frac{\partial\Phi}{\partial\gamma_{ij}}\\ \mu_{ijk} &= \rho_0\frac{\partial\Phi}{\partial\chi_{ijk}}\\ \eta &= -\frac{\partial\Phi}{\partial T}\\ \xi &= -\rho_0\frac{\partial\Phi}{\partial\nu}\\ \sigma_i &= \rho_0\frac{\partial\Phi}{\partial\nu_{,i}}\\ \frac{\partial\Phi}{\partial T_{,i}} &= 0\end{aligned} \quad (25)$$

and

$$q_{i,i} + \rho_0Q = \rho_0T\dot{\eta} \quad (26)$$

which is the energy equation in our case.

The free energy is given by the relation below:

$$
\begin{aligned}
&\rho_0\Phi(\widetilde{\varepsilon}_{ij},\gamma_{ij},\chi_{ijk},\theta,\nu,\nu_{,i}) = \frac{1}{2}C_{ijmn}\widetilde{\varepsilon}_{ij}\widetilde{\varepsilon}_{mn} + G_{mnij}\widetilde{\varepsilon}_{mn}\gamma_{ij}+\\
&\quad+\frac{1}{2}B_{ijmn}\gamma_{ij}\gamma_{mn} + F_{mnrij}\widetilde{\varepsilon}_{ij}\chi_{mnr} + D_{mnijk}\gamma_{mn}\chi_{ijk}+\\
&\quad+\frac{1}{2}A_{ijkmnr}\chi_{ijk}\chi_{mnr} + d_{ijm}\widetilde{\varepsilon}_{ij}\nu_{,m} + e_{ijm}\gamma_{ij}\nu_{,m} + f_{ijkm}\chi_{ijk}\nu_{,m}+\\
&\quad+\frac{1}{2}g_{im}\nu_{,i}\nu_{,m} + a_{ij}\widetilde{\varepsilon}_{ij}\nu + b_{ij}\gamma_{ij}\nu + c_{ijk}\chi_{ijk}\nu + d_i\nu_{,i}\nu + \frac{1}{2}\omega\nu^2-\\
&\quad-\alpha_{ij}\widetilde{\varepsilon}_{ij}\theta - \beta_{ij}\gamma_{ij}\theta - \gamma_{ijk}\chi_{ijk}\theta - \frac{1}{2}a\theta^2 - a_i\nu_{,i}\theta - b\nu\theta\ .
\end{aligned}
\tag{27}
$$

The free energy expression has the constitutive coefficients as functions in $C^1(\Omega)$ and they satisfy the following symmetry relations:

$$
\begin{aligned}
C_{ijmn} &= C_{mnij} = C_{jimn},\\
B_{ijmn} &= B_{mnij},\\
G_{ijmn} &= G_{ijnm},\\
F_{ijkmn} &= F_{jikmn},\\
A_{ijkmnr} &= A_{mnrijk},\\
a_{ij} &= a_{ji},\\
g_{im} &= g_{mi},\\
\alpha_{ij} &= \alpha_{ji}\ .
\end{aligned}
$$

Taking into account (25), we deduce the following constitutive equations of the linear theory of thermoelasticity of dipolar bodies with voids using fractional order strain:

$$
\begin{aligned}
t_{ij} &= C_{ijmn}\widetilde{\varepsilon}_{mn} + G_{ijmn}\gamma_{mn} + F_{mnrij}\chi_{mnr} + d_{ijm}\nu_{,m} + a_{ij}\nu - \alpha_{ij}\theta =\\
&= C_{ijmn}(1+\tau^\beta D_t^\beta)\varepsilon_{mn} + G_{ijmn}\gamma_{mn} + F_{mnrij}\chi_{mnr} + d_{ijm}\nu_{,m} + a_{ij}\nu - \alpha_{ij}\theta\\
\eta_{ij} &= B_{ijmn}\gamma_{mn} + D_{ijmnr}\chi_{mnr} + G_{ijmn}\widetilde{\varepsilon}_{mn} + e_{ijm}\nu_{,m} + b_{ij}\nu - \beta_{ij}\theta =\\
&= B_{ijmn}\gamma_{mn} + D_{ijmnr}\chi_{mnr} + G_{ijmn}(1+\tau^\beta D_t^\beta)\varepsilon_{mn} + e_{ijm}\nu_{,m} + b_{ij}\nu - \beta_{ij}\theta\\
\mu_{ijk} &= A_{ijkmnr}\chi_{mnr} + D_{mnijk}\gamma_{mn} + F_{ijkmn}\widetilde{\varepsilon}_{mn} + f_{ijkm}\nu_{,m} + c_{ijk}\nu - \gamma_{ijk}\theta =\\
&= A_{ijkmnr}\chi_{mnr} + D_{mnijk}\gamma_{mn} + F_{ijkmn}(1+\tau^\beta D_t^\beta)\varepsilon_{mn} + f_{ijkm}\nu_{,m} + c_{ijk}\nu - \gamma_{ijk}\theta\\
\rho_0\eta &= \alpha_{ij}\widetilde{\varepsilon}_{ij} + \beta_{ij}\gamma_{ij} + \gamma_{ijk}\chi_{ijk} + a\theta + a_i\nu_{,i} + b\nu =\\
&= \alpha_{ij}(1+\tau^\beta D_t^\beta)\varepsilon_{ij} + \beta_{ij}\gamma_{ij} + \gamma_{ijk}\chi_{ijk} + a\theta + a_i\nu_{,i} + b\nu\\
\xi &= -a_{ij}\widetilde{\varepsilon}_{ij} - b_{ij}\gamma_{ij} - c_{ijk}\chi_{ijk} - d_i\nu_{,i} - \omega\nu + b\theta =\\
&= -a_{ij}(1+\tau^\beta D_t^\beta)\varepsilon_{ij} - b_{ij}\gamma_{ij} - c_{ijk}\chi_{ijk} - d_i\nu_{,i} - \omega\nu + b\theta\\
\sigma_i &= d_{mni}\widetilde{\varepsilon}_{mn} + e_{mni}\gamma_{mn} + f_{mnri}\chi_{mnr} + g_{im}\nu_{,m} + d_i\nu - a_i\theta =\\
&= d_{mni}(1+\tau^\beta D_t^\beta)\varepsilon_{mn} + e_{mni}\gamma_{mn} + f_{mnri}\chi_{mnr} + g_{im}\nu_{,m} + d_i\nu - a_i\theta.
\end{aligned}
\tag{28}
$$

Using relation $(25)_4$ into (26), we have:

$$
\begin{aligned}
q_{i,i} = & -\rho_0 Q + \rho_0 T \left(-\frac{\partial^2 \Phi}{\partial \widetilde{\varepsilon}_{ij} \partial T} \cdot \widetilde{\varepsilon}_{ij} - \frac{\partial^2 \Phi}{\partial \gamma_{ij} \partial T} \cdot \dot{\gamma}_{ij} - \frac{\partial^2 \Phi}{\partial \chi_{ijk} \partial T} \cdot \dot{\chi}_{ijk} - \right. \\
& \left. - \frac{\partial^2 \Phi}{\partial T^2} \cdot \dot{T} - \frac{\partial^2 \Phi}{\partial \nu \partial T} \cdot \dot{\nu} - \frac{\partial^2 \Phi}{\partial \nu_{,i} \partial T} \cdot \dot{\nu}_{,i} \right) .
\end{aligned} \tag{29}
$$

The relation (29) can be rewritten as:

$$
\begin{aligned}
q_{i,i} = & -\rho_0 Q - T \left[\frac{\partial}{\partial T} \left(\rho_0 \frac{\partial \Phi}{\partial \widetilde{\varepsilon}_{ij}} \right) \cdot \dot{\widetilde{\varepsilon}}_{ij} + \frac{\partial}{\partial T} \left(\rho_0 \frac{\partial \Phi}{\partial \gamma_{ij}} \right) \cdot \dot{\gamma}_{ij} + \right. \\
& + \frac{\partial}{\partial T} \left(\rho_0 \frac{\partial \Phi}{\partial \chi_{ijk}} \right) \cdot \dot{\chi}_{ijk} - \rho_0 \frac{\partial}{\partial T} \left(-\frac{\partial \Phi}{\partial T} \right) \cdot \dot{T} - \\
& \left. - \frac{\partial}{\partial T} \left(-\rho_0 \frac{\partial \Phi}{\partial \nu} \right) \cdot \dot{\nu} + \frac{\partial}{\partial T} \left(\rho_0 \frac{\partial \Phi}{\partial \nu_{,i}} \right) \cdot \dot{\nu}_{,i} \right] .
\end{aligned} \tag{30}
$$

Replacing relations (25) into (30), the last one takes the following form:

$$
\begin{aligned}
q_{i,i} = & -\rho_0 Q - T \left(\frac{\partial t_{ij}}{\partial T} \cdot \widetilde{\varepsilon}_{ij} + \frac{\partial \eta_{ij}}{\partial T} \cdot \dot{\gamma}_{ij} + \frac{\partial \mu_{ijk}}{\partial T} \cdot \dot{\chi}_{ijk} - \right. \\
& \left. - \rho_0 \frac{\partial \eta}{\partial T} \cdot \dot{T} - \frac{\partial \xi}{\partial T} \cdot \dot{\nu} + \frac{\partial \sigma_i}{\partial T} \cdot \dot{\nu}_{,i} \right) .
\end{aligned} \tag{31}
$$

Using the constitutive equations (28), the relation (31) becomes:

$$
q_{i,i} = -\rho_0 Q + \alpha_{ij} T \dot{\widetilde{\varepsilon}}_{ij} + \beta_{ij} T \dot{\gamma}_{ij} + \gamma_{ijk} T \dot{\chi}_{ijk} + a T \dot{T} + b T \dot{\nu} + a_i T \dot{\nu}_{,i} \ . \tag{32}
$$

Let $T \approx T_0$ for linearity, so:

$$
q_{i,i} = -\rho_0 Q + \alpha_{ij} T_0 (1 + \tau^\beta D_t^\beta) \dot{\varepsilon}_{ij} + \beta_{ij} T_0 \dot{\gamma}_{ij} + \gamma_{ijk} T_0 \dot{\chi}_{ijk} + a T_0 \dot{T} + b T_0 \dot{\nu} + a_i T_0 \dot{\nu}_{,i}. \tag{33}
$$

Using (6), the relations of Cattaneo, we can deduce:

$$
q_{i,i} + \tau_0 \dot{q}_{i,i} = (-K_{ij} T_{,j})_{,i} \qquad i,j = 1,2,3 \tag{34}
$$

so:

$$
\begin{aligned}
(-K_{ij} T_{,j})_{,i} = & -\rho_0 Q + \alpha_{ij} T_0 (1 + \tau^\beta D_t^\beta) \dot{\varepsilon}_{ij} + \beta_{ij} T_0 \dot{\gamma}_{ij} + \gamma_{ijk} T_0 \dot{\chi}_{ijk} + \\
& + a T_0 \dot{T} + b T_0 \dot{\nu} + a_i T_0 \dot{\nu}_{,i} - \tau_0 \left[\rho_0 \dot{Q} - \alpha_{ij} T_0 (1 + \tau^\beta D_t^\beta) \ddot{\varepsilon}_{ij} - \right. \\
& \left. - \beta_{ij} T_0 \ddot{\gamma}_{ij} - \gamma_{ijk} T_0 \ddot{\chi}_{ijk} - a T_0 \ddot{T} - b T_0 \ddot{\nu} - a_i T_0 \ddot{\nu}_{,i} \right] .
\end{aligned} \tag{35}
$$

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