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EXISTENCE OF SOLUTIONS FOR THREE-POINT BOUNDARY VALUE PROBLEM FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

This paper deals with the existence and uniqueness of solutions for nonlinear fractional differential equations supplemented with three-point boundary conditions. Our results are based on some well-known tools of fixed point theory such as Banach contraction principle and the Leray-Schauder nonlinear alternative, and are illustrated with an example.

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1 Introduction

In this paper, we are interested in the existence of solutions for the nonlinear fractional differential equation

$$^{c}D^{\alpha}u\left(t\right)=f\left(t,u\left(t\right),u'\left(t\right)\right),\quad t\in\left[0,1\right], \tag{1}$$

subject to three-point boundary conditions

$$\begin{cases}\beta u\left(0\right)+\gamma u\left(1\right)=u\left(\eta\right),\\ u\left(0\right)=\int_{0}^{\eta}u\left(s\right)ds,\\ \beta^{c}D^{p}u\left(0\right)+\gamma^{c}D^{p}u\left(1\right)={}^{c}D^{p}u\left(\eta\right),\end{cases} \tag{2}$$

where $2<\alpha\leq 3,\ 1<p\leq 2,\ \ 0<\eta<1,\ \beta,\gamma\in\mathbb{R}^{+},\ \ f\in C\left(\left[0,1\right]\times\mathbb{R}^{2},\mathbb{R}\right)$ and $^{c}D^{\alpha}$ denotes the Caputo fractional derivative of order α.
The q-difference equations and operators have extensively been studied in the

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framework of quantum calculus (q-calculus). In fact, q-calculus has a rich history and the details of its basic notions, results and methods can be found in the texts [2, 11] and papers [1, 2]. The concept of fractional calculus has played an important role in improving the work based on integer-order (classical) calculus in several diverse disciplines of science and engineering. The nonlocal nature of a fractional order differential operator, which takes into account hereditary properties of various materials and processes, has help to improve the mathematical modeling of many natural phenomena and physical processes, see for example [12, 18]. The increasing interest in fractional differential equations and inclusions is motivated by their applications in various fields of science such as physics chemistry, biology, economics, fluid mechanics, control theory, etc, we refer the reader to [4, 5, 6, 8, 10, 13, 14, 15, 16, 19, 21] and the references therein.

In [3], Ahmad et al. studied the following nonlocal boundary value problems of nonlinear fractional q-difference equations

$$\begin{cases} \left({}^cD_q^\alpha u\right)(t) = f\left(t, u\left(t\right)\right), \ t \in [0,1], \ \alpha \in (1,2] \\ a_1 u(0) - b_1 \left(D_q u\right)(0) = c_1 u\left(\eta_1\right), \\ a_2 u(1) + b_2 \left(D_q u\right)(1) = c_2 u\left(\eta_2\right), \end{cases}$$

where ${}^cD_q^\alpha$ denotes the Caputo fractional q-derivative of order α and $a_i,\ b_i,\ c_i,\ \eta_i \in \mathbb{R}\ (i = 1, 2)$.

In [7], the authors studied the existence of solutions for the nonlinear fractional differential equation

$${}^cD^\alpha u(t) = f\left(t, u(t), u'(t)\right), \quad t \in [0,1],$$

subject to three-point boundary conditions

$$\begin{cases} \beta u(0) + \gamma u(1) = u(\eta), \\ \beta u'(0) + \gamma u'(1) = u'(\eta), \\ \beta {}^cD^p u(0) + \gamma {}^cD^p u(1) = {}^cD^p u(\eta), \end{cases}$$

where $2 < \alpha \leq 3,\ 1 < p \leq 2,\ \ 0 < \eta < 1,\ \beta, \gamma \in \mathbb{R}^+,\ f \in C\left([0,1] \times \mathbb{R}^2, \mathbb{R}\right)$ and ${}^cD^\alpha$ denotes the Caputo fractional derivative of order α.

Motivated greatly by the above mentioned works, we etablish the existence and uniqueness of solutions for nonlocal boundary value problem $(1) - (2)$ by using some well-known tools of fixed point theory such as Banach contraction principle and the Leray-Schauder nonlinear alternative (see [9, 21]). The paper is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel. Section 3, deals with main results and we give an example to illustrate our results.

2 Preliminaries

In this section, we introduce some necessary definitions and lemmas of fractional calculus to facilitate the analysis of the problem $(1) - (2)$. These details can be found in the recent literature; see [12, 19, 14] and the references therein.

Definition 2.1. *Let $\alpha > 0,\ n-1 < \alpha < n,\ n = [\alpha]+1$ and $u \in C([0,\infty),\mathbb{R})$. The Caputo derivative of fractional order α for the function u is defined by*

$$^{c}D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)\,ds.$$

where $\Gamma(\cdot)$ is the Eleur Gamma function.

Definition 2.2. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0,\infty) \to \mathbb{R}$ is given by*

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} u(s)\,ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Eleur Gamma function, provided that the right side is pointwise defined on $(0,\infty)$.

Lemma 2.1. ([19]). *Let $\alpha,\ \beta \geq 0$ and $u \in L^p(0,1),\ 0 \leq p \leq +\infty$. Then the next formulas hold.*
(i) $\left(I^{\beta}I^{\alpha}u\right)(t) = I^{\alpha+\beta}u(t)$,
(ii) $\left(D^{\beta}I^{\alpha}u\right)(t) = I^{\alpha-\beta}u(t)$,
(iii) $\left(D^{\alpha}I^{\alpha}u\right)(t) = u(t)$.

Lemma 2.2. ([14]). *Let $\alpha > 0,\ n-1 < \alpha < n$ and the function $g : [0,T] \to \mathbb{R}$ be continuous for each $T > 0$. Then, the general solution of the fractional differential equation $^{c}D^{\alpha}g(t) = 0$ is given by*

$$g(t) = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},$$

where $c_0, c_1, ..., c_{n-1}$ are real constants and $n = [\alpha]+1$.

Lemma 2.3. ([12]). *Assume that $u \in C[0,1] \cap L^1(0,1)$ with a Caputo fractional derivative of order $\alpha > 0$ that belongs to $u \in C^n[0,1]$, then*

$$I^{\alpha c}D^{\alpha}u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},$$

where $c_0, c_1, ..., c_{n-1}$ are real constants and $n = [\alpha]+1$.

3 Main results

Let $X = \{u :\ u,\ u' \in C([0,1],\mathbb{R})\}$ endowed with the norm defined by $\|u\| = \underset{t\in[0,1]}{sup}|u(t)| + \underset{t\in[0,1]}{sup}|u'(t)|$ such that $\|u\| < \infty$. Then $(X, \|.\|)$ is a Banach space.

Lemma 3.1. *Let $y \in C([0,1], \mathbb{R})$. Then the integral solution of the linear fractional differential equation*

$$^{c}D^{\alpha}u(t) = y(t) \quad t \in [0,1], \ \alpha \in (2,3], \tag{3}$$

subject to three-point boundary conditions

$$\beta u(0) + \gamma u(1) = u(\eta), \quad \beta \geq 0, \ \gamma \geq 0, \tag{4}$$

$$u(0) = \int_0^{\eta} u(s)\, ds, \quad \eta \in (0,1), \tag{5}$$

$$\beta^{c}D^{p}u(0) + \gamma^{c}D^{p}u(1) = ^{c}D^{p}u(\eta) \quad p \in (1,2], \tag{6}$$

is given by

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + \frac{1}{1-\eta} \int_0^{\eta} \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\, d\tau \right) ds$$

$$-\frac{\Lambda_1(t)}{Q_1(1-\eta)} \int_0^{\eta} \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\, d\tau \right) ds$$

$$-\frac{\Lambda_2(t) M_1}{6(1-\eta) Q_1} \left[\int_0^{\eta} \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\, ds - \gamma \int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\, ds \right]$$

$$+\frac{\Lambda_1(t)}{Q_1(\beta+\gamma-1)} \left[\int_0^{\eta} \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds - \gamma \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds \right], \tag{7}$$

where

$$\Lambda_1(t) = (\beta+\gamma-1)\left(\eta^2 + 2(1-\eta)t\right), \quad M_1 = \frac{\Gamma(3-p)}{\gamma - \eta^{2-p}}$$

$$\Lambda_2(t) = \left(\eta^3(\beta+\gamma-1) + 3\left(\gamma-\eta^2\right)(1-\eta)\right)\left(\eta^2 + 2(1-\eta)t\right)$$

$$-Q_1\left(\eta^3 + 3(1-\eta)t^2\right)$$

and

$$Q_1 = 2(1-\eta)(\gamma-\eta) + \eta^2(\beta+\gamma-1) \neq 0.$$

Proof. In view of Lemma 2.1 and Lemma 2.3, the solution of equation (3) can be written as

$$u(t) = I^{\alpha}y(t) + c_0 + c_1 t + c_2 t^2 = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + c_0 + c_1 t + c_2 t^2, \tag{8}$$

where $c_0, c_1, c_2 \in \mathbb{R}$ are arbitrary constants.
Differentiating both sides of (8) and applying Definition 2.1, Lemma 2.1 and Lemma 2.3, we obtain

$$ {}^cD^p u(t) = I^{\alpha-p} y(t) + c_2 \frac{2t^{2-p}}{\Gamma(3-p)} = \int_0^t \frac{(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\, ds + \frac{2t^{2-p}}{\Gamma(3-p)} c_2, \quad (9) $$

where $\alpha \in (2,3]$ and $p \in (1,2]$.
Integrating both sides of (8), we obtain

$$ \int_0^\eta u(t)\, dt = \int_0^\eta \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds \right) dt + c_0 \eta + \frac{1}{2} c_1 \eta^2 + \frac{1}{3} c_2 \eta^3. \quad (10) $$

By using the boundary condition (4) in (8), we obtain

$$ c_0(\beta+\gamma-1) + c_1(\gamma-\eta) + c_2\left(\gamma-\eta^2\right) = \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds $$
$$ -\gamma \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds \quad (11) $$

By using the boundary condition (5) in (8) and (10), we obtain

$$ (1-\eta) c_0 - \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds - \frac{1}{2} c_1 \eta^2 - \frac{1}{3} \eta^3 = 0. \quad (12) $$

By using the boundary condition (6) in (9), we obtain

$$ c_2 = \frac{M_1}{2} \left(\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\, ds - \gamma \int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\, ds \right). \quad (13) $$

Solving the above system of the equations (11), (12) and (13) for c_0, c_1, c_2, we get

$$ c_2 = \frac{M_1}{2} \left(I^{\alpha-p} y(\eta) - \gamma I^{\alpha-p} y(1) \right), $$
$$ = \frac{M_1}{2} \left(\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\, ds - \gamma \int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\, ds \right), $$

$$ c_0 = -\frac{2\eta^2(\beta+\gamma-1)}{2(1-\eta) Q_1} \int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\, d\tau \right) ds $$

$$+\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds$$
$$-\frac{\left(\eta^2\left[\eta^3(\beta+\gamma-1)+3\left(\gamma-\eta^2\right)(1-\eta)\right]-\eta^3Q_1\right)M_1}{2(1-\eta)Q_1}\left[I^{\alpha-p}y(\eta)-\gamma I^{\alpha-p}y(1)\right]$$
$$+\frac{\eta^2}{Q_1}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds-\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\right],$$

and

$$c_1=\frac{-2(\beta+\gamma-1)}{Q_1}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds$$
$$-\frac{\left(\eta^3(\beta+\gamma-1)+3\left(\gamma-\eta^2\right)(1-\eta)\right)M_1}{3Q_1}\left[I^{\alpha-p}y(\eta)-\gamma I^{\alpha-p}y(1)\right]$$
$$+\frac{2(1-\eta)}{Q_1}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds-\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\right],$$

where

$$I^{\alpha-p}y(\eta)-\gamma I^{\alpha-p}y(1)=\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds-\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds.$$

Substituting the values of constants c_0, c_1 and c_2 in (8), we get (7). The proof is complete. □

Throughout the paper, we let

$$M=\frac{\Gamma(3-p)}{\left|\gamma-\eta^{2-p}\right|}\neq 0,\ |\beta+\gamma-1|\neq 0,\ \left|\gamma-\eta^2\right|\neq 0,$$

$$Q=\left|2(1-\eta)(\gamma-\eta)+\eta^2|\beta+\gamma-1|\right|\neq 0,$$
$$A(t)=|\beta+\gamma-1|\left(\eta^2+2(1-\eta)t\right),$$

and

$$B(t)=\left(\eta^3|\beta+\gamma-1|+3\left|\gamma-\eta^2\right|(1-\eta)\right)\left(\eta^2+2(1-\eta)t\right)-Q\left(\eta^3+3(1-\eta)t^2\right).$$

The following inequalities hold:

$$|A(t)|\leq|\beta+\gamma-1|\left(\eta^2+2(1-\eta)\right)=A_1,$$

$$|B(t)| \leq \Big| \left(\eta^3 |\beta+\gamma-1| + 3\left|\gamma-\eta^2\right|(1-\eta)\right)\left(\eta^2+2(1-\eta)\right)$$
$$-Q\left(\eta^3+3(1-\eta)\right)\Big| = B_1,$$

$$\left|A'(t)\right| \leq 2|\beta+\gamma-1|(1-\eta) = A_1',$$

and

$$\left|B'(t)\right| \leq 2(1-\eta)\left|\left(\eta^3|\beta+\gamma-1| + 3\left|\gamma-\eta^2\right|(1-\eta)\right) - 3Q\right| = B_1'.$$

To simplify the proofs in the forthcoming theorems, we etablish the bounds for the integrals arising in the sequel.

Lemma 3.2. *For $y \in C([0,1],\mathbb{R})$, we have*

$$\left|\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\,d\tau\right) ds\right| \leq \frac{\eta^{\alpha+1}}{\Gamma(\alpha+2)}\|y\|.$$

Proof. Obviously,

$$\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} d\tau = \left[-\frac{(s-\tau)^{\alpha}}{\Gamma(\alpha)}\right]_0^s = \frac{s^\alpha}{\alpha\Gamma(\alpha)} = \frac{s^\alpha}{\Gamma(\alpha+1)}.$$

Hence

$$\left|\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\,d\tau\right) ds\right| \leq \|y\| \int_0^\eta \frac{s^\alpha}{\Gamma(\alpha+1)} ds = \frac{\eta^{\alpha+1}}{\Gamma(\alpha+2)}\|y\|.$$

□

For the sake of brevity, we set

$$\triangle_1 = \frac{\eta^{\alpha+1}}{(1-\eta)\Gamma(\alpha+2)} + \frac{A_1\eta^{\alpha+1}}{Q(1-\eta)\Gamma(\alpha+2)} + \frac{MB_1(\eta^{\alpha-p}+\gamma)}{(1-\eta)Q\Gamma(\alpha-p+1)}$$
$$+\frac{A_1(\eta^\alpha+\gamma)}{Q|\beta+\gamma-1|\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}, \tag{14}$$

and

$$\triangle_2 = \frac{A_1'\eta^{\alpha+1}}{Q(1-\eta)\Gamma(\alpha+2)} + \frac{MB_1'(\eta^{\alpha-p}+\gamma)}{(1-\eta)Q\Gamma(\alpha-p+1)}$$
$$+\frac{A_1'(\eta^\alpha+\gamma)}{Q|\beta+\gamma-1|\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}. \tag{15}$$

In view of Lemma 3, we define the operator $F : X \to X$ by

$$(Fu)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\,d\tau\right) ds$$

$$-\frac{B(t)M}{6(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds-\gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds\right]$$

$$+\frac{A(t)}{Q|\beta+\gamma-1|}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds-\gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds.\right]$$

$$-\frac{A(t)}{Q(1-\eta)}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds. \tag{16}$$

Observe that the boundary value problem $(1)-(2)$ has solutions if the operator equation $u=Fu$ has fixed points, where F is given by (16).
Now we are in the position to present the first main results of this paper. The existence result is based on Leray-Schauder nonlinear alternative.

Theorem 3.1. ([13]) *(Nonlinear alternative for single valued maps). Let E be a Banach space, C a closed, convex subset of E and U an open subset of C with $0\in U$. Suppose that $F:\overline{U}\to C$ is a continuous and compact (that is $F\left(\overline{U}\right)$ a relatively compact subset of C) map. Then either*
(i) F has a fixed point in $\overline{U}$, or
(ii) there is a $u\in\partial U$ (the boundary of U in C) and $\lambda\in(0,1)$ with $u=\lambda F(u)$.

Theorem 3.2. *Assume that $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function. Further, it is assumed that the following conditions hold,*
(H_1) there exists a function $\phi\in C([0,1],\mathbb{R}^+)$ and a nondecreasing function $\psi:\mathbb{R}^+\to\mathbb{R}^+$such that $|f(t,u,u')|\le\phi(t)\psi(\|u\|)$, for all $(t,u)\in[0,1]\times\mathbb{R}$,
(H_2) there exists a constant $N>0$ such that

$$\frac{N}{\|\phi\|\,\psi(N)(\triangle_1+\triangle_2)}>1, \tag{17}$$

where $\triangle_1$ and $\triangle_2$ are given by (14) and (15) respectively. Then the boundary value problem $(1)-(2)$ has at least one solution on $[0,1]$.

Proof. It is clear that F is a continuous operator where F is defined by (16). Now, we show that F maps bounded sets into bounded subsets of X. For a positive number r, let $B_r=\{u\in X:\|u\|\le r\}$ be a bounded set in X. Then, by (14) and by (15) we have

$$|(Fu)(t)|\le\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[\phi(s)\psi(\|u\|)]\,ds$$

$$+\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}[\phi(\tau)\psi(\|u\|)]\,d\tau\right)ds$$

$$+\frac{MB_1}{6(1-\eta)Q}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\left[\phi(s)\psi(\|u\|)\right]ds\right.$$

$$\left.+\gamma\int_0^{1}\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\left[\phi(s)\psi(\|u\|)\right]ds\right]$$

$$+\frac{A_1}{Q|\beta+\gamma-1|}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\phi(s)\psi(\|u\|)\right]ds\right.$$

$$\left.+\gamma\int_0^{1}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\phi(s)\psi(\|u\|)\right]ds\right]$$

$$+\frac{A_1}{Q(1-\eta)}\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left[\phi(\tau)\psi(\|u\|)\right]d\tau\right)ds,$$

$$\le\|\phi\|\psi(r)\left\{\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\frac{1}{1-\eta}\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}d\tau\right)ds\right.$$

$$+\frac{A_1}{Q(1-\eta)}\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}d\tau\right)ds$$

$$+\frac{MB_1}{6(1-\eta)Q}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds+\gamma\int_0^{1}\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds\right]$$

$$\left.+\frac{A_1}{Q|\beta+\gamma-1|}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\gamma\int_0^{1}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right]\right\}$$

$$=\|\phi\|\psi(r)\triangle_1.$$

Also,

$$\left|\left(F'u\right)(t)\right|\le\int_0^{t}\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)}\left[\phi(s)\psi(\|u\|)\right]ds$$

$$+\frac{A_1'}{Q(1-\eta)}\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left[\phi(\tau)\psi(\|u\|)\right]d\tau\right)ds$$

$$+\frac{MB_1'}{6(1-\eta)Q}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\left[\phi(s)\psi(\|u\|)\right]ds\right.$$

$$+\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\left[\phi(s)\,\psi(\|u\|)\right]ds\Bigg]$$

$$+\frac{MB_1'}{6(1-\eta)Q}\Bigg[\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\left[\phi(s)\,\psi(\|u\|)\right]ds$$

$$+\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\left[\phi(s)\,\psi(\|u\|)\right]ds\Bigg]$$

$$+\frac{A_1'}{Q|\beta+\gamma-1|}\Bigg[\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\phi(s)\,\psi(\|u\|)\right]ds$$

$$+\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\phi(s)\,\psi(\|u\|)\right]ds\Bigg|$$

$$\le\|\phi\|\,\psi(r)\Bigg\{\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)}ds+\frac{A_1'}{Q(1-\eta)}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}d\tau\right)ds$$

$$+\frac{MB_1'}{6(1-\eta)Q}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds+\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds\right]$$

$$+\frac{A_1'}{Q|\beta+\gamma-1|}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right]\Bigg\}$$

$$=\|\phi\|\,\psi(r)\,\triangle_2.$$

Consequently,

$$\|Fu\|\le\|\phi\|\,\psi(r)\left(\triangle_1+\triangle_2\right). \tag{18}$$

Now, we show that F maps bounded sets into equicontinuous sets of X. Let $t_1,\ t_2\in[0,1]$ with $t_1<t_2$ and $u\in B_r$ is a bounded set of X. Then

$$|(Fu)(t_2)-(Fu)(t_1)|\le\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\phi(s)\,\psi(\|u\|)\right]ds$$

$$+\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\phi(s)\,\psi(\|u\|)\right]ds$$

$$+\frac{|A(t_2)-A(t_1)|}{Q(1-\eta)}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left[\phi(\tau)\psi(\|u\|)\right]d\tau\right)ds$$

$$+\frac{A(t_2)-A(t_1)}{Q|\beta+\gamma-1|}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\phi(s)\psi(\|u\|)\right]ds\right.$$

$$\left.-\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\phi(s)\psi(\|u\|)\right]ds\right]$$

$$+\frac{(B(t_2)-B(t_1))M}{6(1-\eta)Q}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\left[\phi(s)\psi(\|u\|)\right]ds\right.$$

$$\left.+\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\left[\phi(s)\psi(\|u\|)\right]ds\right].$$

Obviously, the right-hand side of the above inequality tends to zero as $t_2\to t_1$. Similarly, we have

$$\left|\left(F'u\right)(t_2)-\left(F'u\right)(t_1)\right|\le\int_0^{t_1}\frac{(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left|f\left(s,u(s),u'(s)\right)\right|ds$$

$$+\frac{|A'(t_2)-A'(t_1)|}{Q(1-\eta)}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left[\phi(\tau)\psi(\|u\|)\right]d\tau\right)ds$$

$$+\frac{(B'(t_2)-B'(t_1))M}{6(1-\eta)Q}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\left[\phi(s)\psi(\|u\|)\right]ds\right.$$

$$\left.+\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\left[\phi(s)\psi(\|u\|)\right]ds\right]$$

$$+\frac{A'(t_2)-A'(t_1)}{Q|\beta+\gamma-1|}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\phi(s)\psi(\|u\|)\right]ds\right.$$

$$\left.-\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\phi(s)\psi(\|u\|)\right]ds\right].$$

Again, it is seen that the right-hand side of the above inequality tends to zero as $t_2 \to t_1$. Thus, $\|(Fu)(t_2) - (Fu)(t_1)\| \to 0$, as $t_2 \to t_1$. This shows that the operator F is completely continuous, by the Ascoli-Arzela theorem. Thus, the operator F satisfies all the conditions of Theorem 3.1, and hence by its conclusion, either condition (i) or condition (ii) holds. We show that the condition (ii) is not possible. Let $U = \{u \in X : \|u\| < N\}$ with N given by (17). In view of condition (H_2) and by (18), we have

$$\|Fu\| \leq \|\phi\| \psi(r)(\triangle_1 + \triangle_2) < N,$$

where we have used (17).

Now, suppose there exists $u \in \partial U$ and $\lambda \in (0,1)$ such that $u = \lambda Fu$. Then for such a choice of u and the constant λ, we have

$$N = \|u\| = \lambda \|Fu\| < \|\phi\| \psi(\|u\|)[\triangle_1 + \triangle_2] = \|\phi\| \psi(N)[\triangle_1 + \triangle_2] < N,$$

which is a contradiction. Consequently, by the Leray-Schauder, alternative, we deduce that F has a fixed point $u \in \overline{U}$ which is a solution of the boundary value problem $(1) - (2)$. The proof is completed.

□

Now, we are in a position to present the second main result of this paper.

Theorem 3.3. *Assume that $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a jointly continuous satisfying the condition*
(H_3) $|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq L(|u_1 - v_1| + |u_2 - v_2|)$, *for* $t \in [0,1]$, $u_i, v_i \in \mathbb{R}$, $i = 1,2$,
where $L > 0$ is a constant. Then the boundary value problem $(1) - (2)$ has a unique solution on $[0,1]$ provided

$$(\triangle_1 + \triangle_2) L < 1, \tag{19}$$

where $\triangle_1$ and $\triangle_1$ are given by (14) and (15) respectively.

Proof. Let us set $\sup\limits_{t\in[0,1]} |f(t,0,0)| = N < \infty$. For $u \in X$ we observe that

$$|f(t,u(t),v(t))| \leq |f(t,u(t),v(t)) - f(t,0,0)| + |f(t,0,0)| \leq L\|u\| + N.$$

Then for $u \in X$, we have

$$\|(Fu)(t)\| \leq \frac{1}{1-\eta} \int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} [L\|u\| + N] d\tau \right) ds$$

$$+ \frac{|A(t)|}{Q(1-\eta)} \int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} [L\|u\| + N] d\tau \right) ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [L\|u\| + N] ds$$

$$+\frac{M\left|B\left(t\right)\right|}{6\left(1-\eta\right)Q}\left[\int_0^{\eta}\frac{\left(\eta-s\right)^{\alpha-p-1}}{\Gamma\left(\alpha-p\right)}\left[L\left\|u\right\|+N\right]ds+\gamma\int_0^{1}\frac{\left(1-s\right)^{\alpha-p-1}}{\Gamma\left(\alpha-p\right)}\left[L\left\|u\right\|+N\right]ds\right]$$

$$+\frac{\left|A\left(t\right)\right|}{Q\left|\beta+\gamma-1\right|}\left[\int_0^{\eta}\frac{\left(\eta-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}\left[L\left\|u\right\|+N\right]ds+\gamma\int_0^{1}\frac{\left(1-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}\left[L\left\|u\right\|+N\right]ds\right],$$

$$\leq\left(L\left\|u\right\|+N\right)\left\{\frac{1}{1-\eta}\int_0^{\eta}\left(\int_0^{s}\frac{\left(s-\tau\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}d\tau\right)ds\right.$$

$$+\frac{A_1}{Q\left(1-\eta\right)}\int_0^{\eta}\left(\int_0^{s}\frac{\left(s-\tau\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}d\tau\right)ds+\int_0^{t}\frac{\left(t-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}ds$$

$$+\frac{MB_1}{6\left(1-\eta\right)Q}\left[\int_0^{\eta}\frac{\left(\eta-s\right)^{\alpha-p-1}}{\Gamma\left(\alpha-p\right)}ds+\gamma\int_0^{1}\frac{\left(1-s\right)^{\alpha-p-1}}{\Gamma\left(\alpha-p\right)}ds\right]$$

$$\left.+\frac{A_1}{Q\left|\beta+\gamma-1\right|}\left[\int_0^{\eta}\frac{\left(\eta-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}ds+\gamma\int_0^{1}\frac{\left(1-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}ds\right]\right\},$$

$$\leq\left(L\left\|u\right\|+N\right)\triangle_1<\infty.$$

Also,

$$\left\|\left(F'u\right)\left(t\right)\right\|\leq\frac{\left|A'\left(t\right)\right|}{Q\left(1-\eta\right)}\int_0^{\eta}\left(\int_0^{s}\frac{\left(s-\tau\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}\left[L\left\|u\right\|+N\right]d\tau\right)ds$$

$$+\int_0^{t}\frac{\left(t-s\right)^{\alpha-2}}{\Gamma\left(\alpha-1\right)}\left[L\left\|u\right\|+N\right]ds$$

$$+\frac{M\left|B'\left(t\right)\right|}{6\left(1-\eta\right)Q}\left[\int_0^{\eta}\frac{\left(\eta-s\right)^{\alpha-p-1}}{\Gamma\left(\alpha-p\right)}\left[L\left\|u\right\|+N\right]ds+\gamma\int_0^{1}\frac{\left(1-s\right)^{\alpha-p-1}}{\Gamma\left(\alpha-p\right)}\left[L\left\|u\right\|+N\right]ds\right]$$

$$+\frac{\left|A'\left(t\right)\right|}{Q\left|\beta+\gamma-1\right|}\left[\int_0^{\eta}\frac{\left(\eta-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}\left[L\left\|u\right\|+N\right]ds+\gamma\int_0^{1}\frac{\left(1-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}\left[L\left\|u\right\|+N\right]ds\right],$$

$$\leq\left(L\left\|u\right\|+N\right)\left\{\frac{A_1'}{Q\left(1-\eta\right)}\int_0^{\eta}\left(\int_0^{s}\frac{\left(s-\tau\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}d\tau\right)ds+\int_0^{t}\frac{\left(t-s\right)^{\alpha-2}}{\Gamma\left(\alpha-1\right)}ds\right.$$

$$+\frac{MB_1'}{6\left(1-\eta\right)Q}\left[\int_0^{\eta}\frac{\left(\eta-s\right)^{\alpha-p-1}}{\Gamma\left(\alpha-p\right)}ds+\gamma\int_0^{1}\frac{\left(1-s\right)^{\alpha-p-1}}{\Gamma\left(\alpha-p\right)}ds\right]$$

$$+\frac{A_1'}{Q\left|\beta+\gamma-1\right|}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\gamma\int_0^{1}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right]\Bigg\},$$
$$\leq (L\left\|u\right\|+N)\bigtriangleup_2<\infty,$$

which implies that $\left\|Fu\right\|\leq (L\left\|u\right\|+N)(\bigtriangleup_1+\bigtriangleup_2)<\infty$.
For u, $v\in X$ and for each $t\in[0,1]$, it follows from assumption (H_3) that

$$\left\|(Fu)-(Fv)\right\|=\sup_{t\in[0,1]}\left|(Fu)(t)-(Fv)(t)\right|,$$

$$\leq L\left\|u-v\right\|\Bigg\{\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\left|A(t)\right|}{Q(1-\eta)}\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}d\tau\right)ds$$
$$+\frac{\left|B(t)\right|M}{6(1-\eta)Q}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds+\gamma\int_0^{1}\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds\right]$$
$$+\frac{\left|A(t)\right|}{Q\left|\beta+\gamma-1\right|}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}ds-\gamma\int_0^{1}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right]\Bigg\},$$
$$\leq L\bigtriangleup_1\left\|u-v\right\|.$$

Also,

$$\left\|(F'u)-(F'v)\right\|=\sup_{t\in[0,1]}\left|(F'u)(t)-(F'v)(t)\right|$$

$$\leq L\left\|u-v\right\|\Bigg\{\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{A_1'(t)}{Q(1-\eta)}\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}d\tau\right)ds$$
$$+\frac{B_1'(t)M}{6(1-\eta)Q}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds+\gamma\int_0^{1}\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds\right]$$
$$+\frac{A_1'(t)}{Q\left|\beta+\gamma-1\right|}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}ds-\gamma\int_0^{1}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right]\Bigg\},$$
$$\leq L\bigtriangleup_2\left\|u-v\right\|.$$

Thus $\left\|(Fu)-(Fv)\right\|\leq L(\bigtriangleup_1+\bigtriangleup_2)\left\|u-v\right\|$. Since $L(\bigtriangleup_1+\bigtriangleup_2)<1$, thus F is a contraction. Hence it follows by Banach's contraction principle that the boundary value problem $(1)-(2)$ has a unique solution on $[0,1]$.

$\square$

We construct an example to illustrate the applicability of the results presented.

Example 3.1. Consider the following fractional differential equation

$$^{c}D^{3}u(t)=\frac{t}{8}\left((cost)\,sin\left(\frac{|u(t)|+|u'(t)|}{2}\right)\right)+\frac{e^{-(u(t)+u'(t))^{2}}}{1+t^{2}},\quad t\in[0,1],\quad (20)$$

subject to the three-point boundary conditions

$$\begin{cases} \frac{1}{100}u(0)+\frac{1}{10}u(1)=u\left(\frac{1}{2}\right), \\ u(0)=\int_{0}^{0,5}u(s)\,ds, \\ \frac{1}{100}{}^{c}D^{\frac{3}{2}}u(0)+\frac{1}{10}{}^{c}D^{\frac{3}{2}}u(1)={}^{c}D^{\frac{3}{2}}u\left(\frac{1}{2}\right), \end{cases}\quad (21)$$

where $f(t,u,u')=\frac{t}{8}\left((cost)\,sin\left(\frac{|u|+|u'|}{2}\right)\right)+\frac{e^{-(u+u')^{2}}}{1+t^{2}}$, $t\in[0,1]$, $\eta=0,5$, $\beta=0,01$, $\gamma=0,1$ *and* $p=1,5$.
It can be easily found that $M=1,4597546147$ and $Q=\frac{249}{400}$.
For every $u_i,\ v_i\in\mathbb{R},\ i=1,2$, we have

$$|f(t,u_1,u_2)-f(t,v_1,v_2)|\le\frac{1}{16}\left(|u_1-v_1|+|u_2-v_2|\right),$$

where $L=\frac{1}{16}$. On the other hand, we have

$$|f(t,u(t),v(t))|\le\frac{t}{16}\psi\left(|u(t)|+|v(t)|\right),\quad t\in[0,1].$$

Put $\psi(t)=t$ and $\phi(t)=\frac{t}{16}$. Clearly, $\|\phi\|=\frac{1}{16}$ and the function ψ is nondecreasing and continuous on $[0,1]$. It can be easily found that $\Delta_1\cong 0,4141664514$ and $\Delta_2\cong 0,9758011659$.
Finally. Firstly, since $\|L\|(\triangle_1+\triangle_2)\cong 0,0868729761<1$, thus all assumptions and conditions of Theorem 3.3 are satisfied. Hence, Theorem 3.3 implies that the three-point boundary value problem $(20)-(21)$ has a unique solution on $[0,1]$.
Secondly, we check the conditions of Theorem 3.2. Clearly, assumption (H_1) holds with $\|\phi\|=\frac{1}{16}$, $\psi(\|u\|)=\|u\|$ .i.e $\psi(N)=N$ and by assumption (H_2) we found that there exists $N>0$. Thus the conclusion of Theorem 3.2 applies, and hence the problem $(20)-(21)$ has at least one solution on $[0,1]$.

References

[1] Adamas, C. R., *On the linear ordinary q-difference equation*, Ann. Math, **30** (1928), 195-205.

[2] Agrawal. O., *Some generalized fractional calculus operators and their applications in integral equations*, Fract. Cal. Appl. Anal. **15** (2012), 700-711.

[3] Ahmad, B., Ntouyas, S. K., and Purnaras, I., *Existence results for nonlocal boundary value problems of nonlinear fractional q-difference equations*, Adv. Differ. Equ-NY. **140** (2012), 15 pages.

[4] Agarwal, R. P., Ntouyas, S. K., Ahmad, B., and Alhouthuali, M. S., *Existence of solutions for integro-differential equations of fractional order with nonlocal tree-point fractional boundary condition*, Adv. Difference Equ. **53** (2013), 1.

[5] Agarwal, R. P., Baleanu, D., Hedayati, V., and Rezapour, S., *Two fractional derivative inclusion problems via integral boundary condition*, Appl. Math. Comput. **257** (2015), 205-212.

[6] Baleanu, D., Mohammadi, H., and Rezapour, S., *The existence of solutions for a nonlinear mixed problem of singular fractional equations*, Adv. Difference Equa. **1** (2013), 12 pages.

[7] Etemad, S., Ntouyas, S. K., and Tariboon, J., *Existence results for three-point boundary value problems for nonlinear fractional diffrential equations*, J. Nonlinear Sci. Appl. **9** (2016), 20105-2116.

[8] Ford, N. J., and Morgado, M. L., *Fractional boundary value problems: analysis and numerical methods*, Frac. Calc. Appl. Anal. **14** (2011), 554-567.

[9] Granas, A., and Dugundji, J., *Fixed point theory*, Springer, New-York(2003).

[10] Ghorbanian, R., Hedayati, V., Postolache, M., and Rezapour, S., *On a fractional differential inclusion via a new integral boundary condition*, J. Inequal. Appl. **53** (2014), 20 pages.

[11] Kac, V., and P. Cheung., *Quantum Calculus*, Springer, New York, (2002). 1.

[12] Kilbas, A. A., Srivastava, H. M., and Trijullo, J. J., *Theory and applications of fractional differential equations*, Elsevier Science b. V, Amsterdam, 2006.

[13] Li, X., Han, Z., and Sun, S., *Existence of positive solutions for nonlinear fractional q-difference equation with parameter*, Adv. Difference Equ. **260** (2013), 13 pages.

[14] Miller, S and Ross, B., *An introduction to the fractional calculus and fractional differential equations, John Wiley and Sons*, Inc. New York, 1993.

[15] Ntouyas, S. K., Etemad, K., Tariboon, S. J., and Sutsutad, W., *Boundary value problems for Riemann-Liouville nonlinear fractional differential inclusions with nonlocal Hadamard fractional integral conditions*, Medittter. J. Math. **13** (2016), no. 3, 939-954.

[16] Ntouyas, S. K., Etemad, S., and Tariboon, J., *Existence results for multi-term fractional differential inclusions*, Adv. Difference Equ. (2015), article 140, 15 pages.

[17] Ntouyas, S. K., Etemad, S., and Tariboon, J., *Existence of solutions for fractional differential inclusions with integral boundary conditions*, Bound. Value Prob. (2015), article 92, 14 pages.

[18] Pongarm, N., Asawasamrit, S., and Tariboon, J., *Sequential derivatives of nonlinear q-difference equations with three-point q-integral boundary conditions*, J. Appl. Math. (2013), Article ID 605169, 9 pages.

[19] Rajkovic, P. M., Marancovic, S. D., and Stankovic, M. S., *On q-analogues of Caputo derivative and Mettag-Leffter function*, Frac. Calc. Appl. Anal. **10** (2007), 359-373.

[20] Ral, A., Gupta, V., and Agarwal, R. P., *Applications of q-calculus in operator theory*, Springer, New York, 2013.

[21] Smart, D. R., *Fixed Point Theorems*, Combridge University Press, London-New York, 1974.