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ALMOST CONFORMAL RICCI SOLITON AND η-RICCI SOLITON ON 3-DIMENSIONAL (ϵ, δ) TRANS-SASAKIAN MANIFOLD

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Abstract

In this paper we have shown that if a 3-dimensional (ϵ, δ) trans-Sasakian manifold admits conformal Ricci soliton (g, V, λ) and the vector field V is point wise collinear with the unit vector field ξ then V is a constant multiple of ξ. Similarly, we have proved that under the same condition an almost conformal Ricci soliton becomes conformal Ricci soliton. We have also studied η-Ricci soliton on (ϵ, δ) trans-Sasakian manifold. Finally, we have shown that if a 3-dimensional (ϵ, δ) trans-Sasakian manifold admits conformal gradient shrinking Ricci soliton then the manifold is an Einstein manifold.

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1 Introduction

Hamilton [12] introduced the concept of Ricci flow and proved its existence in 1982. This concept was developed to answer Thurston's geometric conjecture which says that each closed three manifold admits a geometric decomposition. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S \tag{1}$$

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on a compact Riemannian manifold M with Riemannian metric g and S is the Ricci tensor of type $(0,2)$.

A self-similar solution to the Ricci flow [12], [21] is called a Ricci soliton [13] if it moves by a one parameter family of diffeomorphism and scaling. The Ricci soliton equation is given by

$$\pounds_X g + 2S = 2\lambda g, \tag{2}$$

where X is a vector field, $\pounds_X$ is the Lie derivative along X and λ is any scalar. The Ricci soliton is said to be shrinking, steady or expanding accordingly as λ is positive, zero or negative respectively.

A. E. Fischer developed the concept of conformal Ricci flow [10] in 2003-04 which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow on a smooth closed connected manifold M is defined by the equation [10] given below

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg \tag{3}$$

and $r(g) = -1$,
where p is a positive non-dynamical scalar field(only time dependent), $r(g)$ is the scalar curvature and n is the dimension of the manifold M.

The concept of conformal Ricci soliton was first studied by N. Basu and A. Bhattacharyya [2] in 2015 and the equation is given by

$$\pounds_X g + 2S = [2\lambda - (p + \frac{2}{n})]g, \tag{4}$$

where λ is constant.
The equation (1.4) is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation.

S. Pigola, M. Rigoli, M. Rimoldi, A. G. Setti introduced the concept of almost Ricci soliton in 2010 [18]. A. Barros, R. Batista, E. Ribeiro [1] have also worked on it. According to them, a Riemannian manifold (M^n, g) admits an almost Ricci soliton if there exists a complete vector field X and a smooth soliton function $\lambda : M^n \to \mathbb{R}$ satisfying

$$R_{ij} + \frac{1}{2}(X_{ij} + X_{ji}) = \lambda g_{ij},$$

where R_{ij} and $X_{ij} + X_{ji}$ stand for the Ricci tensor and the Lie derivative $\pounds_X g$ in local coordinates respectively. It is said to be expanding, steady or shrinking if

$\lambda < 0, \lambda = 0$ or $\lambda > 0$ respectively.

We have introduced the notion of almost conformal Ricci soliton and the equation is given by

$$\pounds_X g + 2S = [2\lambda - (p + \frac{2}{n})]g. \tag{5}$$

where $\lambda : M^n \to \mathbb{R}$ is a smooth function.

A gradient Ricci soliton on a Riemannian manifold (M^n, g_{ij}) is defined by [8]

$$S + \nabla\nabla f = \rho g, \tag{6}$$

where ρ is any constant, f is a smooth function called potential function and ∇ is the Levi-Civita connection on M. In particular a gradient shrinking Ricci soliton satisfies the equation,

$$S + \nabla\nabla f - \frac{1}{2\tau} g = 0,$$

where $\tau = T - t$ and T is the maximal time of the solution.

Again for conformal Ricci soliton if the vector field is the gradient of a function f then it is called conformal gradient shrinking Ricci soliton [5] and the equation is given by

$$S + \nabla\nabla f = (\frac{1}{2\tau} - \frac{2}{n} - p)g, \tag{7}$$

where $\tau = T - t$ and T is the maximal time of the solution and f is the Ricci potential function.

J.C. Cho and M. Kimura introduced the notion of η-Ricci soliton [9] which was used by C. Călin and M. Crasmareanu on Hopf hypersurfaces in complex space forms [7]. A Riemannian manifold (M, g) is called an η-Ricci soliton if there exists a smooth vector field ξ such that the Ricci tensor satisfies the following equation

$$2S + \pounds_\xi g + 2\lambda g + 2\mu\eta \otimes \eta = 0 \tag{8}$$

for some constant λ, μ. $\pounds_\xi$ is the Lie derivative operator along the vector field ξ and S is the Ricci tensor. If $\mu = 0$, then η-Ricci soliton becomes Ricci soliton.

The study of manifolds with indefinite metrics is of interest from the standpoint of physics and relativity. Several authors have worked on it. In 1993, the concept of (ϵ)-Sasakian manifold was first introduced by Bejancu and Duggal [4]. R. Kumar, R. Rani, R. K. Nagaich [14] studied the sectional curvature of this manifold. The existence of a new structure on indefinite metrics influences the curvature. For further investigation, H.G. Nagaraja, R. C. Premalatha and G. Somashekara [15] have studied (ϵ, δ) trans-Sasakian manifold which generalizes (ϵ)-Sasakian manifold.

2 Preliminaries

Let (M, g) be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is the compatible Riemannian metric satisfying

$$\phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \eta \circ \phi = 0, \phi\xi = 0. \tag{9}$$

An almost contact metric manifold M is called an (ϵ)-almost contact metric manifold if

$$g(\xi, \xi) = \epsilon, \tag{10}$$

$$\epsilon g(X, \xi) = \eta(X), \tag{11}$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \qquad g(X, \phi Y) = -g(\phi X, Y) \tag{12}$$

for all vector fields $X, Y \in \chi(M)$, $\epsilon = g(\xi, \xi) = \pm 1$. An (ϵ)-almost contact metric manifold is called an (ϵ, δ) trans-Sasakian manifold if

$$(\nabla_X \phi)Y = \alpha[g(X, Y)\xi - \epsilon\eta(Y)X] + \beta[g(\phi X, Y)\xi - \delta\eta(Y)\phi X] \tag{13}$$

for some smooth functions α, β on M, $\epsilon = \pm 1 and \delta = \pm 1$. When $\alpha = 1$ and $\beta = 0$ the (ϵ, δ) trans-Sasakian manifold reduces to an (ϵ)-Sasakian manifold and for $\alpha = 0, \beta = 1$ the manifold reduces to a (δ)-Kenmotsu manifold.

An almost contact metric structure (ϕ, ξ, η, g) on M is called a trans-Sasakian structure [17], if $(M \times R, J, G)$ belongs to the class W_4 [11], where J is the almost complex structure on $M \times R$ defined by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$ for all vector field X on M and smooth function f on $M \times R$. From the expression of (2.5) we can write

$$\nabla_X \xi = -\epsilon\alpha\phi X - \beta\delta\phi^2(X), \tag{14}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \epsilon\delta\beta g(\phi X, \phi Y). \tag{15}$$

For a 3-dimensional (ϵ, δ) trans-Sasakian manifold the following relation holds:

$$2\alpha\beta + \xi\alpha = 0. \tag{16}$$

The Riemann curvature tensor R on a 3-dimensional (ϵ, δ) trans-Sasakian manifold is given by [16]

$$\begin{aligned} R(X,Y)Z &= (2(\frac{r}{2} - (\alpha^2 - \beta^2)) - \frac{r}{2})(g(Y,Z)X - g(X,Z)Y) \\ &+ (3(\alpha^2 - \beta^2) - \epsilon\frac{r}{2})(g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\xi \\ &+ (3(\alpha^2 - \beta^2) - \epsilon\frac{r}{2})\eta(Z)(\eta(Y)X - \eta(X)Y). \end{aligned} \tag{17}$$

Putting $Z = \xi$ in (2.9) we get

$$R(X,Y)\xi = \epsilon[(\alpha^2 - \beta^2) + (\frac{2-\epsilon}{2})r][\eta(Y)X - \eta(X)Y]. \tag{18}$$

Also the Ricci tensor S of type $(0,2)$ is given by

$$S(X,Y) = (\frac{r}{2} - (\alpha^2 - \beta^2))g(X,Y) + (3(\alpha^2 - \beta^2) - \epsilon\frac{r}{2})\eta(X)\eta(Y), \tag{19}$$

where $\epsilon\delta = 1$ and r is the scalar curvature of the manifold M.
Also

$$S(X,\xi) = (\frac{r}{2} + 2(\alpha^2 - \beta^2) - \epsilon\frac{r}{2})\eta(X). \tag{20}$$

Again,

$$\begin{aligned} (\pounds_\xi g)(X,Y) &= (\nabla_\xi g)(X,Y) - \epsilon\alpha g(\phi X,Y) - \epsilon\alpha g(X,\phi Y) + 2\beta\delta g(X,Y) \\ &- 2\beta\delta\eta(X)\eta(Y) \\ &= 2\beta\delta[g(X,Y) - \eta(X)\eta(Y)]. \end{aligned} \tag{21}$$

$[\because g(X,\phi Y) + g(\phi X,Y) = 0]$.

Now for conformal Ricci soliton we have

$$S = \frac{1}{2}[2\lambda - (p + \frac{2}{n})]g - \frac{1}{2}\pounds_x g.$$

Using (2.13) in the above expression and taking $n = 3$ we get

$$\begin{aligned} S(X,Y) &= \frac{1}{2}[2\lambda - (p + \frac{2}{3})]g(X,Y) - \beta\delta[g(X,Y) - \eta(X)\eta(Y)] \\ &= Ag(X,Y) - \beta\delta[g(X,Y) - \eta(X)\eta(Y)], \end{aligned} \tag{22}$$

where $A = \frac{1}{2}[2\lambda - (p + \frac{2}{3})]$.

Hence the manifold is η-Einstein manifold satisfying conformal Ricci soliton.
So we can state the following proposition.

Proposition 2.1 : If a 3-dimensional (ϵ, δ) trans-Sasakian manifold admits conformal Ricci soliton (g, ξ, λ) then the manifold becomes an η-Einstein manifold.

Also,

$$QX = AX - \beta\delta[X - \eta(X)\xi]. \tag{23}$$

$$S(X, \xi) = \frac{1}{2}[2\lambda - (p + \frac{2}{3})]\eta(X). \tag{24}$$

Again for almost conformal Ricci soliton

$$\begin{aligned} S(X,Y) &= \lambda g(X,Y) - \frac{1}{2}(p + \frac{2}{3})g(X,Y) - \beta\delta g(X,Y) + \beta\delta\eta(X)\eta(Y) \\ &= (\lambda - \beta\delta)g(X,Y) - \frac{1}{2}(p + \frac{2}{3})g(X,Y) + \beta\delta\eta(X)\eta(Y), \end{aligned}$$

where λ is a smooth function.

Thus we can state the following proposition.

Proposition 2.2 : A 3-dimensional (ϵ, δ) trans-Sasakian manifold admitting almost conformal Ricci soliton (g, ξ, λ) is an η-Einstein manifold.

Example of a 3-dimensional (ϵ, δ) trans-Sasakian manifold:

In this section we construct an example of a 3-dimensional (ϵ, δ) trans-Sasakian manifold. To construct this, we consider a three dimensional manifold $M = \{(x, y, z) \in R^3 : z \neq 0\}$ where (x, y, z) are the standard coordinates in R^3. The vector fields

$$e_1 = e^{-z}(\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}), e_2 = e^{-z}\frac{\partial}{\partial y}, e_3 = (\epsilon + \delta)\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1,$$

$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0.$$

Let η be the 1-form which satisfies the relation

$$\eta(e_3) = 1.$$

Let ϕ be the $(1, 1)$ tensor field defined by $\phi(e_1) = -e_1, \phi(e_2) = -e_2, \phi(e_3) = 0$. Then we have

$$\phi^2(Z) = -Z + \eta(Z)e_3$$

for any $Z, W \in \chi(M^3)$. Thus for $e_3 = \xi, (\phi, \xi, \eta, g)$ defines an almost contact metric structure on M. Now, after calculating we have

$$[e_1, e_3] = (\epsilon + \delta)e_1, [e_1, e_2] = ye^{-z}e_2 + \frac{e^{-2z}}{(\epsilon + \delta)}e_3, [e_2, e_3] = (\epsilon + \delta)e_2.$$

By Koszul's formula we get

$$\nabla_{e_1}e_1 = -(\epsilon+\delta)e_3, \nabla_{e_2}e_1 = -ye^{-z}e_2 - \frac{1}{2(\epsilon+\delta)}e^{-2z}e_3, \nabla_{e_3}e_1 = -\frac{1}{2(\epsilon+\delta)}e^{-2z}e_2,$$

$$\nabla_{e_1}e_2 = \frac{1}{2(\epsilon+\delta)}e^{-2z}e_3, \nabla_{e_2}e_2 = ye^{-z}e_1 - (\epsilon+\delta)e_3, \nabla_{e_3}e_2 = \frac{1}{2(\epsilon+\delta)}e^{-2z}e_1,$$

$$\nabla_{e_1}e_3 = (\epsilon+\delta)e_1 - \frac{1}{2(\epsilon+\delta)}e^{-2z}e_2, \nabla_{e_2}e_3 = \frac{1}{2(\epsilon+\delta)}e^{-2z}e_1 + (\epsilon+\delta)e_2, \nabla_{e_3}e_3 = 0.$$

From the above we have found that $\alpha = (\epsilon + \delta), \beta = 0$ and we can see that $M^3(\phi, \xi, \eta, g)$ is a (ϵ, δ) trans-Sasakian manifold.

3 Some results of conformal Ricci soliton and almost conformal Ricci soliton on 3-dimensional (ϵ, δ) trans-Sasakian manifold

In this section we shall prove two important theorems:

Theorem 1. *If a 3-dimensional (ϵ, δ) trans-Sasakian manifold admits almost conformal Ricci soliton and if V is point-wise collinear with ξ, then V is a constant multiple of ξ and λ becomes a constant function i.e. almost conformal Ricci soliton becomes conformal Ricci soliton. Also the manifold is an η-Einstein manifold and the value of $\lambda = \frac{1}{2}p + \frac{2}{3}\beta\gamma\delta$, provided β is constant.*

Proof. A conformal Ricci soliton equation on a Riemannian manifold M of dimension 3 is defined by

$$\pounds_V g + 2S = [2\lambda - (p + \frac{2}{3})]g,$$

where V is a vector field.

Let V be pointwise collinear with ξ (i.e. the direction of V is along ξ) i.e. $V = \gamma\xi$

where γ is a function on 3-dimensional (ϵ,δ) trans-Sasakian manifold. Then

$$(\pounds_V g+2S-[2\lambda-(p+\frac{2}{3})]g)(X,Y)=0,$$

which implies

$$(\pounds_{\gamma\xi}g)(X,Y)+2S(X,Y)-[2\lambda-(p+\frac{2}{3})]g(X,Y)=0.$$

Applying the property of Lie derivative and Levi-Civita connection we have

$$\gamma g(\nabla_X\xi,Y)+(X\gamma)g(\xi,Y)+(Y\gamma)g(\xi,X)+\gamma g(\nabla_Y\xi,X)+2S(X,Y)$$
$$-[2\lambda-(p+\frac{2}{3})]g(X,Y)=0.$$

Using (2.6) in the above equation we obtain

$$2\beta\delta\gamma g(X,Y)-2\gamma\beta\delta\eta(X)\eta(Y)+(X\gamma)\eta(Y)+(Y\gamma)\eta(X)$$
$$+2S(X,Y)-[2\lambda-(p+\frac{2}{3})]g(X,Y)=0. \tag{25}$$

Replacing Y by ξ in (3.1) we get

$$X\gamma+(\xi\gamma)\eta(X)+2S(X,\xi)-[2\lambda-(p+\frac{2}{3})]g(X,Y)=0. \tag{26}$$

Again putting $X=\xi$ in (3.2) we have

$$2\xi\gamma+2S(\xi,\xi)-[2\lambda-(p+\frac{2}{3})]=0. \tag{27}$$

Using (2.12) in (3.3) we get

$$\xi\gamma=\frac{1}{2}[2\lambda-(p+\frac{2}{3})]-[\frac{r}{2}+2(\alpha^2-\beta^2)-\frac{\epsilon r}{2}]. \tag{28}$$

Using the value of $\xi\gamma$ from (3.4) in (3.2) we have

$$X\gamma+(\frac{1}{2}[2\lambda-(p+\frac{2}{3})]-2(\frac{r}{2}+2(\alpha^2-\beta^2)-\frac{\epsilon r}{2}))\eta(X)+2(\frac{r}{2}+2(\alpha^2-\beta^2)-\frac{\epsilon r}{2})\eta(X)$$
$$-[2\lambda-(p+\frac{2}{3})]\eta(X)=0,$$

which implies

$$X\gamma=\frac{1}{2}[2\lambda-(p+\frac{2}{3})]\eta(X)-2(\frac{r}{2}+2(\alpha^2-\beta^2)-\frac{\epsilon r}{2})\eta(X), \tag{29}$$

or it can be written as

$$d\gamma=[\frac{1}{2}[2\lambda-(p+\frac{2}{3})]-2(\frac{r}{2}+2(\alpha^2-\beta^2)-\frac{\epsilon r}{2})]\eta, \tag{30}$$

where we put $\nabla_X = d$. Applying exterior differentiation in (3.6) and considering λ as constant we have

$$[\lambda - \frac{1}{2}(p + \frac{2}{3}) - \frac{r}{2} - 2(\alpha^2 - \beta^2) + \frac{\epsilon r}{2}]d\eta = 0. \tag{31}$$

Since $d\eta \neq 0$, so we have

$$[\lambda - \frac{1}{2}(p + \frac{2}{3}) - \frac{r}{2} - 2(\alpha^2 - \beta^2) + \frac{\epsilon r}{2}] = 0. \tag{32}$$

Using (3.8) in (3.6) we have

$$d\gamma = 0$$

implies γ is constant.

Hence from (3.1) we have

$$2\beta\gamma\delta g(X,Y) - 2\gamma\beta\delta\eta(X)\eta(Y) + 2S(X,Y) - [2\lambda - (p + \frac{2}{3})]g(X,Y) = 0,$$

i.e.

$$S(X,Y) = \frac{1}{2}[2\lambda - (p + \frac{2}{3})]g(X,Y) - \beta\gamma\delta g(X,Y) + \gamma\beta\delta\eta(X)\eta(Y).$$

So the manifold is an η-Einstein manifold. Putting $X = Y = e_i$ where $\{e_i\}$ is the orthonormal basis of the tangent space TM where TM is a tangent bundle of M and summing over i we get,

$$r = \frac{3}{2}[2\lambda - (p + \frac{2}{3})] - 2. \tag{33}$$

Now for conformal Ricci soliton $r = -1$, so using this value in the above equation we get

$$\lambda = \frac{1}{2}p + \frac{2}{3}\beta\gamma\delta. \tag{34}$$

Again for almost conformal Ricci soliton we consider that λ is a smooth function. Then applying the exterior derivative in (3.6) we get

$$[\lambda - \frac{1}{2}(p + \frac{2}{3}) - \frac{r}{2} - 2(\alpha^2 - \beta^2) + \frac{\epsilon r}{2}]d\eta + (d\lambda)\eta = 0,$$

which gives

$$[\lambda - \frac{1}{2}(p + \frac{2}{3}) - \frac{r}{2} - 2(\alpha^2 - \beta^2) + \frac{\epsilon r}{2}] = 0$$

and

$$d\lambda = 0. \tag{35}$$

So λ is a constant function and from (3.6) and (3.11) we get γ is constant. □

Theorem 2. *A 3-dimensional (ϵ, δ) trans-Sasakian manifold admitting a conformal Ricci soliton (g, ξ, λ) satisfies the following relations:*
1. For $\alpha > \beta$, the conformal Ricci soliton is shrinking.
2. For $\alpha < -\beta$ and $\frac{p}{2} + \frac{1}{3} > 2(\alpha^2 - \beta^2)$ the conformal Ricci soliton becomes shrinking.
3. For $\alpha < -\beta$ and $\frac{p}{2} + \frac{1}{3} < 2(\alpha^2 - \beta^2)$ the conformal Ricci soliton becomes expanding.
4. For $2(\alpha^2 - \beta^2) = -(\frac{p}{2} + \frac{1}{3})$ the conformal Ricci soliton becomes steady.

Proof. Now, from conformal Ricci soliton equation we have

$$(\pounds_\xi g)(X, Y) = 2\beta\delta[g(X, Y) - \eta(X)\eta(Y)].$$

Using conformal Ricci soliton equation and (2.11) in the above equation we have

$$2\beta\delta[g(X,Y)-\eta(X)\eta(Y)]+2[(\frac{r}{2}-(\alpha^2-\beta^2))g(X,Y)+(3(\alpha^2-\beta^2)-\frac{r\epsilon}{2})\eta(X)\eta(Y)]$$

$$-[2\lambda - (p + \frac{2}{3})]g(X, Y) = 0. \tag{36}$$

For conformal Ricci soliton we have $r = -1$, so the above equation becomes

$$[2\beta\delta - 1 - 2(\alpha^2 - \beta^2) - 2\lambda + p + \frac{2}{3}]g(X, Y)$$

$$+[6(\alpha^2 - \beta^2) - 2\beta\delta + \epsilon]\eta(X)\eta(Y) = 0. \tag{37}$$

Now taking $X = Y = \xi$ in (3.13) and using the notion $g(\xi, \xi) = \epsilon$ we get

$$2\beta\delta\epsilon - 2\beta\delta - 2\epsilon(\alpha^2 - \beta^2) + 6(\alpha^2 - \beta^2) - 2\lambda\epsilon + p\epsilon + \frac{2}{3}\epsilon = 0.$$

Now two cases may arise for two different values of ϵ. Here we consider that $\epsilon = 1$. Then $\lambda = 2(\alpha^2 - \beta^2) + \frac{p}{2} + \frac{1}{3}$. Since $\alpha^2 \neq \beta^2$ so
(1) Suppose $\alpha^2 \geq \beta^2$, then $(\alpha + \beta)(\alpha - \beta) > 0$ which implies α always greater than β. Then $\lambda > 0$ and the conformal Ricci soliton is shrinking.
(2) Suppose $\alpha^2 < \beta^2$ and $\frac{p}{2} + \frac{1}{3} > 2(\alpha^2 - \beta^2)$, then $(\alpha + \beta)(\alpha - \beta) < 0$ which implies α always less than $-\beta$. Then $\lambda > 0$ and the conformal Ricci soliton becomes shrinking.
(3) Suppose $\alpha^2 < \beta^2$ and $\frac{p}{2} + \frac{1}{3} < 2(\alpha^2 - \beta^2)$, then $(\alpha + \beta)(\alpha - \beta) < 0$ which implies α always less than $-\beta$. Then $\lambda < 0$ and the conformal Ricci soliton becomes expanding.
(4) Suppose $2(\alpha^2 - \beta^2) = -(\frac{p}{2} + \frac{1}{3})$, then $\lambda = 0$ and the conformal Ricci soliton becomes steady.
For $\epsilon = -1$, similar conditions of λ will arise depending on δ which the reader may work out. □

4 η Ricci soliton on 3-dimensional (ϵ, δ) trans-Sasakian manifold

An η-Ricci soliton equation on a Riemannian manifold M of dimension 3 is defined by

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0,$$

where V is a vector field.

In this section we shall prove a theorem of η Ricci soliton on 3-dimensional (ϵ, δ) trans-Sasakian manifold.

Theorem 3. *In a 3-dimensional (ϵ, δ) trans-Sasakian manifold admitting η-Ricci soliton if V is point-wise collinear with ξ, then V is a constant multiple of ξ and the manifold is η-Einstein manifold.*

Proof. Let V be pointwise collinear with ξ (i.e. the direction of V is along ξ) i.e. $V = b\xi$ where b is a smooth function on 3-dimensional (ϵ, δ) trans-Sasakian manifold. Then

$$(\pounds_{b\xi} g + 2S + 2\lambda g)(X, Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{38}$$

Applying the property of Lie derivative and Levi-Civita connection we have

$$\begin{aligned} bg(\nabla_X \xi, Y) + (Xb)g(\xi, Y) + (Yb)g(\xi, X) + bg(\nabla_Y \xi, X) + 2S(X, Y) \\ +2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0. \end{aligned} \tag{39}$$

Using (2.6) in (4.2) we get

$$\begin{aligned} 2\beta\delta bg(X, Y) - 2b\beta\delta\eta(X)\eta(Y) + (Xb)\eta(Y) + (Yb)\eta(X) \\ +2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0. \end{aligned} \tag{40}$$

Putting $Y = \xi$ in (4.3) we get

$$Xb + (\xi b)\eta(X) + 2S(X, \xi) + 2\lambda\eta(X) + 2\mu\eta(X) = 0. \tag{41}$$

Again putting $X = \xi$ in (4.4) we have

$$b\xi + S(\xi, \xi) + \lambda + \mu = 0. \tag{42}$$

Using (2.12) in (4.5) we get

$$b\xi = -\lambda - \mu - [\frac{r}{2} + 2(\alpha^2 - \beta^2) - \frac{\epsilon r}{2}]. \tag{43}$$

Using the value of $b\xi$ in (4.3) we have

$$Xb = (\frac{\epsilon r}{2} - \frac{r}{2} - \lambda - \mu - 2(\alpha^2 - \beta^2))\eta(X). \tag{44}$$

Or

$$db = (\frac{\epsilon r}{2} - \frac{r}{2} - \lambda - \mu - 2(\alpha^2 - \beta^2))\eta. \tag{45}$$

where we put $\nabla_X = d$.
Applying exterior differentiation in (4.8) we have

$$(\frac{\epsilon r}{2} - \frac{r}{2} - \lambda - \mu - 2(\alpha^2 - \beta^2))d\eta = 0. \tag{46}$$

As $d\eta \neq 0$, we have

$$\frac{\epsilon r}{2} - \frac{r}{2} - \lambda - \mu - 2(\alpha^2 - \beta^2) = 0. \tag{47}$$

Hence from (4.8) it can be easily seen that b is constant.

So from (4.3) we have

$$2b\beta\delta g(X,Y) - 2b\beta\delta\eta(X)\eta(Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Therefore

$$S(X,Y) = (\beta\delta b - \mu)\eta(X)\eta(Y) - (\lambda + b\beta\delta)g(X,Y). \tag{48}$$

So the manifold is an η-Einstein manifold. □

5 Conformal gradient shrinking Ricci soliton on 3-dimensional (ϵ, δ) trans-Sasakian manifold

Theorem 4. *If a 3-dimensional (ϵ, δ) trans-Sasakian manifold admits conformal gradient shrinking Ricci soliton, then the manifold is an Einstein manifold.*

Proof. The conformal gradient shrinking Ricci soliton equation is given by

$$S + \nabla\nabla f = (\frac{1}{2\tau} - \frac{2}{3} - p)g, \tag{49}$$

which can be reduced to the following equation

$$\nabla_Y Df + QY = (\frac{1}{2\tau} - \frac{2}{3} - p)Y, \tag{50}$$

where D is the gradient operator of g.
From (5.2) we have

$$\nabla_X\nabla_Y Df + \nabla_X QY = (\frac{1}{2\tau} - \frac{2}{3} - p)\nabla_X Y.$$

Now

$$R(X,Y)Df = \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df$$

$$= (\frac{1}{2\tau} - \frac{2}{3} - p)[\nabla_X Y - \nabla_Y X - [X,Y]] - \nabla_X(QY) + \nabla_Y(QX) + Q[X,Y],$$

where R is the curvature tensor.

As ∇ is Levi-Civita connection, from the above equation we get

$$R(X,Y)Df = -\nabla_X(QY) + \nabla_Y(QX) + Q[X,Y] = (\nabla_Y Q)X - (\nabla_X Q)Y. \quad (51)$$

Again from (2.14) we have

$$QX = (\frac{r}{2} - (\alpha^2 - \beta^2))X + (3(\alpha^2 - \beta^2) - \frac{\epsilon r}{2})\eta(X)\xi.$$

Differentiating the above equation with respect to W and then putting $W = \xi$ we get

$$(\nabla_\xi Q)X = \frac{dr(\xi)}{2}X - \epsilon\frac{dr(\xi)}{2}\eta(X)\xi.$$

Then

$$g((\nabla_\xi Q)X - (\nabla_X Q)\xi, \xi) = g(\frac{dr(\xi)}{2}(X - \eta(X)\xi), \xi) = 0. \quad (52)$$

Using (5.4) in (5.3) we get

$$g(R(\xi, X)Df, \xi) = 0. \quad (53)$$

Now from (2.10) we obtain

$$g(R(\xi, X)Df, \xi) = \epsilon[(\alpha^2 - \beta^2) + (\frac{2-\epsilon}{2})r](g(X, Df) - \eta(X)\eta(Df)). \quad (54)$$

From (5.5) and (5.6) we have

$$\epsilon[(\alpha^2 - \beta^2) + (\frac{2-\epsilon}{2})r](g(X, Df) - \eta(X)\eta(Df)) = 0.$$

Now since $\alpha^2 \neq \beta^2$ and $\epsilon = \pm 1$, we have

$$g(X, Df) = \eta(X)g(Df, \xi), \quad (55)$$

which implies

$$Df = (\xi f)\xi. \quad (56)$$

Now from (5.2) we have

$$g(\nabla_Y Df, X) + g(QY, X) = (\frac{1}{2\tau} - \frac{2}{3} - p)g(Y, X).$$

Using (5.8) and (2.6) we have

$$\begin{aligned} S(X,Y)-(\frac{1}{2\tau}-\frac{2}{3}-p)g(Y,X) &= -g(\nabla_Y(\xi f)\xi,X) \\ &= \epsilon\alpha(\xi f)g(\phi Y,X)+\beta\delta(\xi f)\eta(Y)\eta(X) \\ &- Y(\xi f)\eta(X)-\beta\delta(\xi f)g(X,Y). \end{aligned} \tag{57}$$

Putting $X=\xi$ in (5.9) we get

$$S(Y,\xi)-(\frac{1}{2\tau}-\frac{2}{3}-p)\eta(Y)=-Y(\xi f).$$

So

$$(\frac{r}{2}+2(\alpha^2-\beta^2)-\frac{\epsilon r}{2})\eta(Y)-(\frac{1}{2\tau}-\frac{2}{3}-p)\eta(Y)=-Y(\xi f). \tag{58}$$

Now from (5.9) and interchanging X and Y we obtain

$$\begin{aligned} S(X,Y)-(\frac{1}{2\tau}-\frac{2}{3}-p)g(Y,X) &= \epsilon\alpha(\xi f)g(\phi X,Y)-\beta\delta(\xi f)g(X,Y) \\ &+ \beta\delta(\xi f)\eta(Y)\eta(X)-X(\xi f)\eta(Y). \end{aligned} \tag{59}$$

Adding (5.9) and (5.11) we get

$$\begin{aligned} 2S(X,Y)-2(\frac{1}{2\tau}-\frac{2}{3}-p)g(Y,X) &= -2\beta(\xi f)\delta g(X,Y)+2\beta\delta(\xi f)\eta(Y)\eta(X) \\ &- (\xi f)(Y\eta(X)+X\eta(Y)). \end{aligned} \tag{60}$$

Putting the value of $Y(\xi f)$ from (5.10) in (5.12) we get

$$\begin{aligned} QY-(\frac{1}{2\tau}-\frac{2}{3}-p)Y &= -\beta\delta(\xi f)Y+\beta\delta(\xi f)\eta(Y)\xi \\ &-\frac{r}{2}\eta(Y)\xi-2(\alpha^2-\beta^2)\eta(Y)\xi+(\frac{1}{2\tau}-\frac{2}{3}-p)\eta(Y)\xi+\frac{\epsilon r}{2}\eta(Y)\xi \end{aligned} \tag{61}$$

From (5.2) we can write

$$\nabla_Y Df=\beta\delta(\xi f)[Y-\eta(Y)\xi]+[2(\alpha^2-\beta^2)-(\frac{1}{2\tau}-\frac{2}{3}-p)]\eta(Y)\xi+(\frac{r}{2}-\frac{\epsilon r}{2})\eta(Y)\xi \tag{62}$$

Now,

$$\begin{aligned} R(X,Y)Df &= \nabla_X\nabla_Y Df-\nabla_Y\nabla_X Df-\nabla_{[X,Y]}Df \\ &= \nabla_X[\beta\delta(\xi f)(Y-\eta(Y)\xi)+[2(\alpha^2-\beta^2)-(\frac{1}{2\tau}-\frac{2}{3}-p) \\ &\quad +(\frac{r}{2}-\frac{\epsilon r}{2})]\eta(Y)\xi)]-\nabla_Y[\beta\delta(\xi f)(X-\eta(X)\xi)+[2(\alpha^2-\beta^2) \\ &\quad -(\frac{1}{2\tau}-\frac{2}{3}-p)+(\frac{r}{2}-\frac{\epsilon r}{2})]\eta(X)\xi]-\nabla_{[X,Y]}Df \end{aligned}$$

$$\begin{aligned} &= \beta\delta(\xi f)[X,Y] - \beta\delta(\xi f)[\eta(Y)\nabla_X\xi - \eta(X)\nabla_Y\xi] \\ &+2(\alpha^2-\beta^2)[\eta(Y)\nabla_X\xi - \eta(X)\nabla_Y\xi] \\ &+(\frac{r}{2} - \frac{\epsilon r}{2})[\eta(Y)\nabla_X\xi - \eta(X)\nabla_Y\xi] \\ &-(\frac{1}{2\tau} - \frac{2}{3} - p)[\eta(Y)\nabla_X\xi - \eta(X)\nabla_Y\xi] - \nabla_{[X,Y]}Df. \end{aligned} \tag{63}$$

Also

$$\begin{aligned} \nabla_{[X,Y]}Df &= \beta\delta(\xi f)([X,Y] - \eta([X,Y])\xi) + (2(\alpha^2-\beta^2) \\ &-(\frac{1}{2\tau} - \frac{2}{3} - p))\eta([X,Y])\xi + (\frac{r}{2} - \frac{\epsilon r}{2})\eta([X,Y])\xi \\ &= \beta\delta(\xi f)[X,Y] - \beta\delta(\xi f)\nabla_X\eta(Y)\xi \\ &+\beta\delta(\xi f)\xi(\nabla_X\eta)Y + \beta\delta(\xi f)\nabla_Y\eta(X)\xi \\ &-\beta\delta(\xi f)\xi(\nabla_Y\eta)X + [2(\alpha^2-\beta^2) - (\frac{1}{2\tau} - \frac{2}{3} - p)]\nabla_X\eta(Y)\xi \\ &-[2(\alpha^2-\beta^2) - (\frac{1}{2\tau} - \frac{2}{3} - p)]\xi(\nabla_X\eta)Y \\ &-[2(\alpha^2-\beta^2) - (\frac{1}{2\tau} - \frac{2}{3} - p)]\nabla_Y\eta(X)\xi + (\frac{r}{2} - \frac{\epsilon r}{2})\nabla_X\eta(Y)\xi \\ &+[2(\alpha^2-\beta^2) - (\frac{1}{2\tau} - \frac{2}{3} - p)]\xi(\nabla_Y\eta)X - (\frac{r}{2} - \frac{\epsilon r}{2})\xi(\nabla_X\eta)Y \\ &+(\frac{r}{2} - \frac{\epsilon r}{2})\xi(\nabla_Y\eta)X - (\frac{r}{2} - \frac{\epsilon r}{2})\nabla_Y\eta(X)\xi. \end{aligned} \tag{64}$$

Putting (5.15) in (5.12) and taking inner product with ξ we have

$$2(\alpha^2-\beta^2) - (\frac{1}{2\tau} - \frac{2}{3} - p) - \beta(\xi f) + (\frac{r}{2} - \frac{\epsilon r}{2}) = 0, \tag{65}$$

since $\epsilon \neq 0$.
From (5.10) we obtain

$$\beta\delta(\xi f)\eta(Y) = -Y(\xi f). \tag{66}$$

Putting the value of (5.17) in (5.11) we have

$$S(X,Y) - (\frac{1}{2\tau} - \frac{2}{3} - p)g(X,Y) = \beta(\xi f)g(X,Y). \tag{67}$$

After contraction (5.18) reduces to

$$(\xi f) = \frac{-1}{3\beta} - \frac{1}{\beta}(\frac{1}{2\tau} - \frac{2}{3} - p) = C, \tag{68}$$

where C is a constant.

So from (5.8) we get

$$Df = (\xi f)\xi = C\xi. \tag{69}$$

Therefore

$$g(Df, X) = g(C\xi, X)$$

which implies

$$df(X) = C\eta(X).$$

Taking exterior differentiation we get
$Cd\eta = 0$ as $d^2 f(X) = 0$.

So from (5.20) we have found that f is constant.

Also from (5.1) we get

$$S(X, Y) = (\frac{1}{2\tau} - \frac{2}{3} - p)g(X, Y). \tag{70}$$

Hence M is an Einstein manifold. □

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