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FIXED POINT THEOREMS FOR A-CONTRACTION MAPPINGS OF INTEGRAL TYPE IN COMPLETE G-METRIC SPACES

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Abstract

The aim of this paper is to prove an integral type fixed point theorems for A-contraction mappings in complete G-metric spaces. Our results improve, extend and generalize several previously known fixed point theorem in the existing literature.

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1 Introduction

Fixed point theory plays a crucial part in nonlinear functional analysis and is useful for proving the existence theorems for nonlinear differential and integral equations. Moreover, it is well known that the contraction mapping principle, formulated and proved in the PhD dissertation of Banach in 1920, which was published in 1922 is one of the most important theorems in classical functional analysis [2]. During the last five decades, this theorem has undergone various generalizations either by relaxing the condition of contractivity or changing the underlying space or sometimes both. Due to the importance, generalizations of Banach fixed point theorem have been investigated heavily by many authors.

After the classical result by Banach, Kannan [6] gave a substantially new contractive mapping to prove the fixed point theorem. Since then there have been many theorems emerged as generalizations under various contractive conditions. Such conditions involve linear and nonlinear expressions (rational, irrational, and general type).

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M. Akram et.al [1] introduced a new class of contraction maps, called A-contraction, which is a proper superclass of Kannan's [6], Bianchini's [3] and Reich's [12] type contraction. This definition can be stated as follows.

Let a non-empty set A consisting of all functions $\alpha : R_+^3 \to R_+$ satisfying:

(A1) α is continuous on the set R_+^3 of all triplets of non-negative reals (with respect to the Euclidean metric on R^3).

(A2) $a \le kb$ for some $k \in [0, 1)$ whenever $a \le \alpha(a, b, b)$ or $a \le \alpha(b, a, b)$ or $a \le \alpha(b, b, a)$ for all a, b.

Definition 1.1. *A self-map T on a metric space (X, d) is said to be A-contraction, if it satisfies the condition*

$$d(Tx, Ty) \le \alpha\Big(d(x, y), d(x, Tx), d(y, Ty)\Big),$$

for all $x, y \in X$ and some $\alpha \in A$.

Example 1.2. Let a self-map T on a metric space (X, d) satisfying

$$d(Tx, Ty) \le \beta \max\{d(Tx, x) + d(Ty, y), d(Ty, y) + d(x, y), d(Tx, x) + d(x, y)\}$$

for all $x, y \in X$ and some $\beta \in [0, \frac{1}{2})$, is an A-contraction. (see [1] for detail and comparison with other contraction maps.)

In 2002, Branciari [4] obtained a fixed point theorem for a single mapping satisfying an analogue of a Banach contraction principle for integral type inequality. Following Branciari's paper, a lot of research works have been carried out on generalizing contractive conditions of integral type for different contractive mapping satisfying various known properties. In 2012 Manta Soha and Debashis Dey [13] proved analogues of some fixed point result for A-contraction type mappings in integral setting.

On the other hand, in 2006, Mustafa and Sims [8, 9] introduced the notion of generalized metric spaces or simply G-metric spaces. Several authors studied a lot of fixed and common fixed point theorems in such spaces (see, e.g., [7, 10, 11]). The aim of this paper is to prove an integral type fixed point theorems for A-contraction mappings in complete G-metric spaces. Our results improve, extend and generalize several previously known fixed point theorem in the existing literature.

2 Preliminaries

We recall some basic definitions and results which we need in the sequel. First, we give the definition of a G-metric.

Definition 2.1. [9]. *Let X be a nonempty set and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:*
(G1) $G(x, y, z) = 0$ *if* $x = y = z$,
(G2) $0 < G(x, x, y)$ *for all* $x, y \in X$ *with* $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $z \neq y$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$, *symmetry in all three variables,*
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$.

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

Example 2.2 [9]. Let (X, d) be a metric space. The function $G : X \times X \times X \longrightarrow [0, +\infty)$, defined by $G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$, for all $x, y, z \in X$, is a G-metric on X.

Example 2.3˙ [9]. Let $X = [0, +\infty)$. The function $G : X \times X \times X \longrightarrow [0, +\infty)$, defined by $G(x, y, z) = |x - y| + |y - z| + |z - x|$, for all $x, y, z \in X$, is a G-metric on X.

Definition 2.4 [9]. *A G-metric space (X, G) is said to be symmetric if* $G(x, y, y) = G(y, x, x)$ *for all* $x, y \in X$.

Proposition 2.5 [9]. *Let (X, G) be a G-metric space. If $G(x, y, z) = 0$, then $x = y = z$ for all $x, y, z \in X$.*

Definition 2.6 *[9]. Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X.We say that the sequence $\{x_n\}$ is G- convergent to x if* $\lim_{n,m\to+\infty} G(x, x_n, x_m) = 0$, *That is, for any $\epsilon > 0$, there exist $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$ for all $m, n \geq N$. We call x the limit of the sequence and write $x_n \longrightarrow x$ or* $\lim_{n\to+\infty} x_n = x$.

Proposition 2.7 [9]. *Let (X, G) be a G-metric space. Then the following are equivalent:*
(1) $\{x_n\}$ *is G-convergent to* x *;*
(2) $G(x_n, x_n, x) \to 0$ *as* $n \to +\infty$ *;*
(3) $G(x_n, x, x) \to 0$ *as* $n \to +\infty$ *;*
(4) $G(x_n, x_m, x) \to 0$ *as* $n, m \to +\infty$.

Definition 2.8 [9]. *Let (X, G) be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq N$; that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.*

Proposition 2.9 [9]. *Let (X, G) be a G-metric space. Then the following are equivalent:*
(1) the sequence $\{x_n\}$ *is G-Cauchy ;*

(2) for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq N$.

Definition 2.10 [9]. *A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).*

Theorem 2.11 [5] *Let (X, G) be a G-metric space. Let $d : X \times X \longrightarrow [0, +\infty)$ be the function defined by $d(x, y) = G(x, y, y)$. Then*
(1) $\{x_n\} \subset X$ is a G-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent to x in (X, d);
(2) $\{x_n\} \subset X$ is a G-Cauchy to $x \in X$ if and only if $\{x_n\}$ is Cauchy to x in (X, d);
(3) (X, G) is a G-complete if and only if (X, d) is complete.

3 Main result

Now, we will prove our main result.

Theorem 3.1. *Let T be a self-mapping of a complete G-metric space (X, G) satisfying the following condition:*

$$\begin{aligned} &\int_0^{G(Tx,Ty,Ty)} \varphi(t)dt \\ \leq\ & \alpha\Big(\int_0^{G(x,y,y)} \varphi(t)dt, \int_0^{G(x,Tx,Tx)} \varphi(t)dt, \int_0^{G(y,Ty,Ty)} \varphi(t)dt\Big), \end{aligned} \tag{1}$$

for each $x, y \in X$ with some $\alpha \in A$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lesbesgue-integrable mapping which is summable on each compact subset of $[0, +\infty)$ non-negative, and such that for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0 \tag{2}$$

Then T has a unique fixed point $z \in X$.

Proof. Let $x_0 \in X$ be arbitrary and for brevity, define $x_{n+1} = Tx_n$. for each integer $n \geq 1$, from (1) we get,

$$\begin{aligned} &\int_0^{G(x_n,x_{n+1},x_{n+1})} \varphi(t)dt = \int_0^{G(Tx_{n-1},Tx_n,Tx_n)} \varphi(t)dt \\ \leq\ & \alpha\Big(\int_0^{G(x_{n-1},x_n,x_n)} \varphi(t)dt, \int_0^{G(x_{n-1},Tx_{n-1},Tx_{n-1})} \varphi(t)dt, \int_0^{G(x_n,Tx_n,Tx_n)} \varphi(t)dt\Big) \\ \leq\ & \alpha\Big(\int_0^{G(x_{n-1},x_n,x_n)} \varphi(t)dt, \int_0^{G(x_{n-1},x_n,x_n)} \varphi(t)dt, \int_0^{G(x_n,x_{n+1},x_{n+1})} \varphi(t)dt\Big). \end{aligned}$$

That by the axiom $(A2)$ of function α,

$$\int_0^{G(x_n,x_{n+1},x_{n+1})} \varphi(t)dt \leq k \int_0^{G(x_{n-1},x_n,x_n)} \varphi(t)dt, \tag{3}$$

for some $k \in [0,1)$ as $\alpha \in A$. In this way, on can obtain

$$\begin{aligned}\int_0^{G(x_n,x_{n+1},x_{n+1})}\varphi(t)dt &\leq k\int_0^{G(x_{n-1},x_n,x_n)}\varphi(t)dt\\ &\leq k^2\int_0^{G(x_{n-2},x_{n-1},x_{n-1})}\varphi(t)dt\\ &\vdots\\ &\leq k^n\int_0^{G(x_0,x_1,x_1)}\varphi(t)dt.\end{aligned}$$

By taking the limit as $n \to \infty$ to both side of the above inequality we have

$$\lim_{n\to\infty}\int_0^{G(x_n,x_{n+1},x_{n+1})}\varphi(t)dt = 0,$$

as $k \in [0,1)$, which from (2) implies that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{4}$$

We now show that $\{x_n\}$ is a G-Cauchy sequence. Suppose the contrary. Then there exists an $\epsilon > 0$ and subsequence $\{m(p)\}$ and $\{n(p)\}$ such that $m(p) < n(p) < m(p+1)$ with

$$G(x_{m(p)}, x_{n(p)}, x_{n(p)}) \geq \epsilon,\ \ G(x_{m(p)}, x_{n(p)-1}, x_{n(p)-1}) < \epsilon. \tag{5}$$

Now

$$\begin{aligned}G(x_{m(p)-1}, x_{n(p)-1}, x_{n(p)-1}) &\leq G(x_{m(p)-1}, x_{m(p)}, x_{m(p)}) + G(x_{m(p)}, x_{n(p)-1}, x_{n(p)-1})\\ &< G(x_{m(p)-1}, x_{m(p)}, x_{m(p)}) + \epsilon.\end{aligned} \tag{6}$$

So by (4) and (6) we get

$$\lim_{p\to+\infty}\int_0^{G(x_{m(p)-1},x_{n(p)-1},x_{n(p)-1})}\varphi(t)dt \leq \int_0^{\epsilon}\varphi(t)dt. \tag{7}$$

Using (3),(5) and (7) we get

$$\begin{aligned}&\int_0^{\epsilon}\varphi(t)dt \leq \int_0^{G(x_{m(p)},x_{n(p)},x_{n(p)})}\varphi(t)dt \leq k\int_0^{G(x_{m(p)-1},x_{n(p)-1},x_{n(p)-1})}\varphi(t)dt\\ \leq\ \ & k\int_0^{\epsilon}\varphi(t)dt,\end{aligned}$$

which is a contradiction, since $k \in [0,1)$. Therefore, $\{x_n\}$ is G-Cauchy sequence in X. Since x is G-complete, we obtain that $\{x_n\}$ is G-convergent to some $z \in X$. From (1) we get

$$\begin{aligned}&\int_0^{G(Tz,x_{n+1},x_{n+1})}\varphi(t)dt = \int_0^{G(Tz,Tx_n,Tx_n)}\varphi(t)dt\\ \leq\ \ &\alpha\Big(\int_0^{G(z,x_n,x_n)}\varphi(t)dt, \int_0^{G(z,Tz,Tz)}\varphi(t)dt, \int_0^{G(x_n,x_{n+1},x_{n+1})}\varphi(t)dt\Big).\end{aligned}$$

Letting $n \to +\infty$, we get

$$\int_0^{G(Tz,z,z)} \varphi(t)dt \leq \alpha\Big(0, \int_0^{G(z,Tz,Tz)} \varphi(t)dt, 0\Big).$$

So by the axiom (A2) of function α,

$$\int_0^{G(Tz,z,z)} \varphi(t)dt = k \times 0 = 0,$$

which, from (2) implies that $G(Tz, z, z) = 0$ or $Tz = z$. Next suppose that $w(\neq z)$ is another fixed point of T. By (1), we have

$$\begin{aligned}
\int_0^{G(z,w,w)} \varphi(t)dt &= \int_0^{G(Tz,Tw,Tw)} \varphi(t)dt \\
&\leq \alpha\Big(\int_0^{G(z,w,w)} \varphi(t)dt, \int_0^{G(z,Tz,Tz)} \varphi(t)dt, \int_0^{G(w,Tw,Tw)} \varphi(t)dt\Big) \\
&= \alpha\Big(\int_0^{G(z,w,w)} \varphi(t)dt, \int_0^{G(z,z,z)} \varphi(t)dt, \int_0^{G(w,w,w)} \varphi(t)dt\Big) \\
&= \alpha\Big(\int_0^{G(z,w,w)} \varphi(t)dt, 0, 0\Big).
\end{aligned}$$

So by the axiom (A2) of function α,

$$\int_0^{G(z,w,w)} \varphi(t)dt = 0$$

which, from (2), implies that $G(z, w, w) = 0$ or $z = w$ and so the fixed point is unique. □

Next theorem describes the common fixed point of two self-maps on X having two related metrics in integral setting.

Theorem 3.2. *Let X be a set with two G-metrics G_1, G_2, satisfying the following conditions:*

(i) $\int_0^{G_1(x,y,y)} \varphi(t)dt \leq \int_0^{G_2(x,y,y)} \varphi(t)dt$ *for all* $x, y \in X$;

(ii) *X is G-complete with respect to G_1;*

(iii) (S, T) *are self-maps on X such that T is continuous with respect to G_1 and*

$$\begin{aligned}
&\int_0^{G_2(Tx,Sy,Sy)} \varphi(t)dt \\
\leq\ & \alpha\Big(\int_0^{G_2(x,y,y)} \varphi(t)dt, \int_0^{G_2(x,Tx,Tx)} \varphi(t)dt, \int_0^{G_2(y,Sy,Sy)} \varphi(t)dt\Big) \qquad (8)
\end{aligned}$$

for each $x, y \in X$ with some $\alpha \in A$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lesbesgue-integrable mapping which is summable on each compact subset of $[0, +\infty)$, non-negative,and such that for $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0. \tag{9}$$

Then S and T have a unique common fixed point $z \in X$.

Proof. For each integer $n \geq 0$, we define

$$x_{2n+1} = Tx_{2n}, \;\; x_{2n+2} = Sx_{2n+1}.$$

Then from (8) we get,

$$\begin{aligned}
&\int_0^{G_2(x_1,x_2,x_2)} \varphi(t)dt = \int_0^{G_2(Tx_0,Sx_1,Sx_1)} \varphi(t)dt \\
\leq\;& \alpha\Big(\int_0^{G_2(x_0,x_1,x_1)} \varphi(t)dt, \int_0^{G_2(x_0,Tx_0,Tx_0)} \varphi(t)dt, \int_0^{G_2(x_1,Sx_1,Sx_1)} \varphi(t)dt\Big) \\
\leq\;& \alpha\Big(\int_0^{G_2(x_0,x_1,x_1)} \varphi(t)dt, \int_0^{G_2(x_0,x_1,x_1)} \varphi(t)dt, \int_0^{G_2(x_1,x_2,x_2)} \varphi(t)dt\Big).
\end{aligned}$$

Then by the axiom $(A2)$ of function α,

$$\int_0^{G_2(x_1,x_2,x_2)} \varphi(t)dt \leq k \int_0^{G_2(x_0,x_1,x_1)} \varphi(t)dt, \tag{10}$$

for some $k \in [0, 1)$. Similarly one can show that

$$\int_0^{G_2(x_2,x_3,x_3)} \varphi(t)dt \leq k \int_0^{G_2(x_1,x_2,x_2)} \varphi(t)dt \tag{11}$$

for some $k \in [0, 1)$. In general, for any $r \in N$ odd or even,

$$\int_0^{G_2(x_r,x_{r+1},x_{r+1})} \varphi(t)dt \leq k \int_0^{G_2(x_{r-1},x_r,x_r)} \varphi(t)dt \tag{12}$$

and so for any $n \in N$ odd or even,one can easily obtain that

$$\int_0^{G_2(x_n,x_{n+1},x_{n+1})} \varphi(t)dt \leq k^n \int_0^{G_2(x_0,x_1,x_1)} \varphi(t)dt. \tag{13}$$

Then by the condition (i) one obtains

$$\int_0^{G_1(x_n,x_{n+1},x_{n+1})} \varphi(t)dt \leq \int_0^{G_2(x_n,x_{n+1},x_{n+1})} \varphi(t)dt \leq k^n \int_0^{G_2(x_0,x_1,x_1)} \varphi(t)dt.$$

Taking limit as $n \to +\infty$, we get

$$\int_0^{G_1(x_n,x_{n+1},x_{n+1})} \varphi(t)dt = 0,$$

which, from (9) implies that

$$\lim_{n\to+\infty} G_1(x_n, x_{n+1}, x_{n+1}) = 0 \tag{14}$$

We show that $\{x_n\}$ is a G-Cauchy sequence with respect to (X, G_1). So for any integer $p > 0$,

$$\begin{aligned}
\int_0^{G_1(x_n,x_{n+p},x_{n+p})} \varphi(t)dt &\leq \int_0^{G_2(x_n,x_{n+p},x_{n+p})} \varphi(t)dt \\
&\leq \int_0^{G_2(x_n,x_{n+1},x_{n+1})} \varphi(t)dt + \int_0^{G_2(x_{n+1},x_{n+2},x_{n+2})} \varphi(t)dt \\
&+ ... + \int_0^{G_2(x_{n+p-1},x_{n+p},x_{n+p})} \varphi(t)dt \\
&\leq k^n \int_0^{G_2(x_0,x_1,x_1)} \varphi(t)dt + k^{n+1} \int_0^{G_2(x_0,x_1,x_1)} \varphi(t)dt \\
&+ ... + k^{n+p-1} \int_0^{G_2(x_0,x_1,x_1)} \varphi(t)dt \\
&\leq \frac{k^n}{1-k} \int_0^{G_2(x_0,x_1,x_1)} \varphi(t)dt \to 0, \quad as, \quad n \to \infty,
\end{aligned}$$

since $k \in [0, 1)$. Therefore, $\{x_n\}$ is G-Cauchy. Hence by completeness of X, $\{x_n\}$ convergent to some $z \in X$, i.e. $G_1(x_n, z, z) \to 0$ as $n \to +\infty$ for some $z \in X$. Since T is given to be continuous with the respect to G_1 we have

$$\begin{aligned}
0 &= \lim_{n\to+\infty} \int_0^{G_1(x_{2n+1},z,z)} \varphi(t)dt = \lim_{n\to+\infty} \int_0^{G_1(Tx_{2n},z,z)} \varphi(t)dt \\
&= \lim_{n\to+\infty} \int_0^{G_1(Tz,z,z)} \varphi(t)dt
\end{aligned}$$

So by (9) $G_1(Tz, z, z) = 0$ i.e. $Tz = z$. Now by (8)

$$\begin{aligned}
\int_0^{G_2(z,Sz,Sz)} \varphi(t)dt &= \int_0^{G_2(Tz,Sz,Sz)} \varphi(t)dt \\
&\leq \alpha\Big(\int_0^{G_2(z,z,z)} \varphi(t)dt, \int_0^{G_2(z,Tz,Tz)} \varphi(t)dt, \int_0^{G_2(z,Sz,Sz)} \varphi(t)dt\Big) \\
&\alpha\Big(0, 0, \int_0^{G_2(z,Sz,Sz)} \varphi(t)dt\Big).
\end{aligned}$$

Then by the axiom (A2) of function α,

$$\int_0^{G_2(z,Sz,Sz)} \varphi(t)dt \leq k.0 = 0 \tag{15}$$

and so by (9) $Sz = z$. Thus z is a common fixed point of T and S. For the uniqueness, let $w(\neq z)$ be another common fixed point of T and S in X. Then by

(8)

$$\begin{aligned}
\int_0^{G_2(z,w,w)}\varphi(t)dt &= \int_0^{G_2(Tz,Sw,Sw)}\varphi(t)dt\\
&\le \alpha\Big(\int_0^{G_2(z,w,w)}\varphi(t)dt, \int_0^{G_2(z,Tz,Tz)}\varphi(t)dt, \int_0^{G_2(w,Tw,Tw)}\varphi(t)dt\Big)\\
&\le \alpha\Big(\int_0^{G_2(z,w,w)}\varphi(t)dt, 0, 0\Big)\\
&\le k.0, \quad as, \quad \alpha \in A
\end{aligned}$$

Then by (9) we have $G_2(z, w, w) = 0$ and so $z = w$. □

If $S = T$, then the Theorem 3.2 gives as follows.

Corollary 3.3. *Let X be a set with two G-metrics G_1 and G_2 satisfying conditions:*

(i) $\int_0^{G_1(x,y,y)}\varphi(t)dt \le \int_0^{G_2(x,y,y)}\varphi(t)dt$ *for all* $x, y \in X$;

(ii) *X is G-complete with respect to G_1;*

(iii) *T is a self-map on X such that T is continuous with respect to G_1 and*

$$\begin{aligned}
&\int_0^{G_2(Tx,Ty,Ty)}\varphi(t)dt\\
&\le \alpha\Big(\int_0^{G_2(x,y,y)}\varphi(t)dt, \int_0^{G_2(x,Tx,Tx)}\varphi(t)dt, \int_0^{G_2(y,Ty,Ty)}\varphi(t)dt\Big) \qquad (16)
\end{aligned}$$

for each $x, y \in X$ with some $\alpha \in A$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lesbesgue-integrable mapping which is summable on each compact subset of $[0, +\infty)$, non-negative, and such that for each$\epsilon > 0$,

$$\int_0^{\epsilon}\varphi(t)dt > 0 \qquad (17)$$

then T has a unique fixed point $z \in X$.

We have another similar result if we omit the condition (ii) of corollary 2-3 and the continuity of T with respect to G_1 is replaced by assuming the continuity at α point.
Then we get the same conclusion under a much less restricted condition.

Theorem 3.4 . *Let X be with two G-metric G_1 and G_2 satisfying conditions:*

(i) $\int_0^{G_1(x,y,y)}\varphi(t)dt \le \int_0^{G_2(x,y,y)}\varphi(t)dt$ *for all* $x, y \in X$;

(ii) *T is a self-map on X such that T is continuous at $z \in X$ with respect to G_1 and*

$$\begin{aligned} &\int_0^{G_2(Tx,Ty,Ty)} \varphi(t)dt \\ \leq\ & \alpha\Big(\int_0^{G_2(x,y,y)} \varphi(t)dt, \int_0^{G_2(x,Tx,Tx)} \varphi(t)dt, \int_0^{G_2(y,Ty,Ty)} \varphi(t)dt\Big) \end{aligned} \quad (18)$$

for each $x, y \in X$ with some $\alpha \in A$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lesbesgue-integrable mapping which is summable on each compact subset of $[0, +\infty)$, non-negative, and such that for each$\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0; \quad (19)$$

(iii) *There exists a point $x_0 \in X$ such that the sequence of iterates $\{T^n x_0\}$ has a subsequence $\{T^{n_i} x_0\}$ converging to z in (X, G).*

Then T has a unique fixed point $z \in X$.

Proof. Considering the sequence $\{x_n\}$ as fixed by $x_{n+1} = Tx_n$ for $n \geq 0$ i.e. $x_1 = Tx_0$,$x_2 = Tx_1 = T^2x_0$,...,$x_n = T^n x_0$ and proceeding as in the proof of Theorem 3.2 we can easily arrive at a conclusion that the sequence is Cauchy with respect to G_1. Since the subsequence $\{x_{n_i}\}$ of the Cauchy sequence $\{x_n\}$ converges to z, therefore $\{x_n\}$ converges to z in X with respect to G_1 i.e. $\lim_{n\to\infty} x_n = z$. Since T is given to be continuous at z with respect to G_1 we have

$$\begin{aligned} 0 &= \lim_{n\to+\infty} \int_0^{G_1(x_{n+1},z,z)} \varphi(t)dt = \lim_{n\to+\infty} \int_0^{G_1(Tx_n,z,z)} \varphi(t)dt \\ &= \lim_{n\to+\infty} \int_0^{G_1(Tz,z,z)} \varphi(t)dt. \end{aligned}$$

So by (9) $G_1(Tz, z, z) = 0$ i.e. $Tz = z$. Thus T has a fixed point. Uniqueness of z is also very clear. □

Remark 3.5 On setting $\varphi(t) = 1$ over $[0, +\infty)$ in each result, the contractive condition of integral type transforms into a general contractive not involving integrals.

Example 3.6 Let $X = \{0, 1, 2, 3, 4\}$, $d(x, y) = |x - y|$, and

$$G(x, y, z) = \max\{|x - y| , |y - z| , |z - x|\}.$$

Let $T : X \to X$ be given by $T(x) = 2$ if $x = 0$ and $T(x) = 1$ if $x \neq 0$. Again let $\varphi : [0, \infty) \to [0, \infty)$ be given by $\varphi(t) = 1$ for all $t \in [0, \infty)$. Then $\varphi : [0, \infty) \to [0, \infty)$ is a Lesbesgue-integrable mapping which is summable on each compact subset of $[0, +\infty)$, non-negative, and such for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$. Now as we know a self-map T satisfying $d(Tx,Ty) \leq \beta \max\{d(Tx,x)+d(Ty,y), d(Ty,y)+d(x,y), d(Tx,x)+d(x,y)\}$ for all $x,y \in X$ and some $\beta \in [0,\frac{1}{2})$,is an A-contraction. We have

$$\begin{aligned} & \int_0^{G(Tx,Ty,Ty)} \varphi(t)dt \\ \leq\ & \alpha\Big(\int_0^{G(x,y,y)} \varphi(t)dt, \int_0^{G(x,Tx,Tx)} \varphi(t)dt, \int_0^{G(y,Ty,Ty)} \varphi(t)dt\Big) \\ =\ & \beta \max\Big\{\int_0^{G(Tx,x,x)+G(x,y,y)} \varphi(t)dt, \int_0^{G(Tx,x,x)+G(Ty,y,y)} \varphi(t)dt, \\ & \int_0^{G(Ty,y,y)+G(x,y,y)} \varphi(t)dt\Big\}, \end{aligned}$$

which is satisfied for all $x,y \in X$ and some $\beta \in [0,\frac{1}{2})$. So all the axioms of Theorem 3.1 are satisfied and 1 is, of course, a unique fixed point of T.

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