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ON THE CONVERGENCE OF SOME MODIFIED NEWTON METHODS THROUGH COMPUTER ALGEBRA

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Abstract

For some modified Newton methods to solve a non-linear equation the convergence is established and the convergence order is computed using a Computer Algebra Software.

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1 Introduction

We present the possibility to establish the convergence and compute the convergence order of a method to solve a non-linear equation using a Computer Algebra Software (CAS).

The applied procedure is based on a well known convergence result, i.e. [**?**].

Several modified Newton methods are known. Some of them are derived from different quadrature formulas [11], [2], [7], [5]. We shall apply the convergence result to these methods, but the computations are made using a CAS. The same argumentation for convergence was used in [4] and [3], too. It implies that the convergence occurs only when the initial approximation is properly chosen and that the convergence order is 3.

In [9] we used the same approach for some methods to simultaneously compute all the roots of a polynomial.

We give a unitary simplified presentation of the convergence results for several modified Newton methods with the usage of *Mathematica* CAS [12].

The note is organized as follows. In Section 2 we recall the convergence result that will be used. In Section 3 the convergence conditions are verified for some modified Newton methods using *Mathematica.*

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2 A convergence framework

Let $\Omega \in \mathbb{C}^n$ be an open convex subset, $T : \Omega \to \mathbb{C}^n$, $T(z) = (T_1(z), \ldots, T_n(z))^T$ an m times differentiable operator such that $T^{(m)}(z)$ is continuous and the sequence $(z^{(k)})_{k\in\mathbb{N}}$ defined by

$$z^{(k+1)} = T(z^{(k)}), \quad z^{(k)} = (z_1^{(k)}, z_2^{(k)}, \ldots, z_n^{(k)})^T \qquad \Leftrightarrow \tag{1}$$

$$\Leftrightarrow \qquad z_i^{(k+1)} = T_i(z^{(k)}), \forall \quad i \in \{1, 2, \ldots, n\},\ k \in \mathbb{N}.$$

In $\mathbb{C}^n$ we shall use the *max* norm $\|z\| = \max\{|z_1|, |z_2|, \ldots, |z_n|\}$.

We remind a result enabling to establish the convergence of such methods and a lower bound of their convergence order [**?**].

The main ingredient of the convergence theorem is the following well known result, but for completeness we shall give the proof of the result that we shall use.

Theorem 1. *[1] Let X, Y be normed spaces, D an open convex subset of X and $T : D \to Y$ an m times Frèchet differentiable operator. Then, for any $x, y \in D$*

$$\|T(y) - T(x) - \sum_{j=1}^{m-1} \frac{1}{j!} T^{(j)}(x) \underbrace{(y-x)\ldots(y-x)}_{j\ times} \| \le \frac{\|y-x\|^m}{m!} \sup_{\zeta\in[x,y]} \|T^{(m)}(\zeta)\|. \tag{2}$$

Using this result, we have

Theorem 2. *Let $\alpha \in \Omega$. If*

1. $T(\alpha) = \alpha$,
2. $T'(\alpha) = T''(\alpha) = \ldots = T^{(m-1)}(\alpha) = 0$

then there exists $r > 0$ such that for any $z^{(0)} \in \mathbb{C}^n$, $\|z^{(0)} - \alpha\| < r$, the sequence $z^{(k+1)} = T(z^{(k)})$, $k \in \mathbb{N}$, (1) converges to α.

Proof. Let $r_0 > 0$ be such that $V_0 = \{z \in \mathbb{C}^n : \|z - \alpha\| \le r_0\} \subset \Omega$ and $C_0 = \max_{z\in V_0} \|T^{(m)}(z)\|$.

There exists $0 < r \le r_0$ such that

$$\frac{C_0 r^m}{m!} < r \quad \Leftrightarrow \quad \left(\frac{C_0}{m!}\right)^{\frac{1}{m-1}} r < 1.$$

We denote $V = \{z \in \mathbb{C}^n : \|z - \alpha\| \le r\}$. If $z \in V$, then (2) and the present hypothesis implies

$$\|T(z) - \alpha\| = \|T(z) - T(\alpha) - \sum_{j=1}^{m-1} \frac{1}{j!} T^{(j)}(\alpha) \underbrace{(z-\alpha)\ldots(z-\alpha)}_{j \text{ times}} \| \le$$

$$\le \frac{1}{m!}\|z-\alpha\|^m \sup_{\zeta\in[\alpha,z]} \|T^{(m)}(\zeta)\| \le \frac{C_0 r^m}{m!} < r,$$

thus $T(z) \in V$.

For $z = z^{(k)}$ from the above relations we obtain

$$\|z^{(k+1)} - \alpha\| = \|T(z^{(k)}) - \alpha\| \leq \frac{C_0}{m!}\|z^{(k)} - \alpha\|^m. \tag{3}$$

Using recursively the inequality (3), we find

$$\|z^{(k)} - \alpha\| \leq \frac{C_0}{m!}\|z^{(k-1)} - \alpha\|^m \leq \frac{C_0}{m!}\left(\frac{C_0}{m!}\|z^{(k-2)} - \alpha\|^m\right)^m =$$

$$= \left(\frac{C_0}{m!}\right)^{1+m}\|z^{(k-2)} - \alpha\|^{m^2} \leq \ldots \leq \left(\frac{C_0}{m!}\right)^{1+m+\ldots+m^{k-1}}\|z^{(0)} - \alpha\|^{m^k} \leq$$

$$\leq \left(\frac{C_0}{m!}\right)^{\frac{m^k-1}{m-1}} r^{m^k} = \left(\left(\frac{C_0}{m!}\right)^{\frac{1}{m-1}} r\right)^{m^k-1} r \to 0,$$

for $k \to \infty$. ■

Let $\lim_{k\to\infty} x_k = x_*$. If $\lim_{k\to\infty} \frac{\|x_{k+1}-x_*\|}{\|x_k - x_*\|^r} = \rho$, with $0 < \rho < \infty$, then r is the convergence order of the sequence $(x_k)_{k\in\mathbb{N}}$.

From the inequality (3) it results that the convergence order of the sequence $(z^{(k)})_{k\in\mathbb{N}}$ is at least m.

3 Modified Newton methods

Let there be a differentiable function $F : \Omega \subseteq \mathbb{R} \to \mathbb{R}$ and the non-linear equation

$$F(x) = 0, \tag{4}$$

such that $F(a) = 0, F'(a) \neq 0$.

The iteration formula of a modified Newton methods is

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \cdot G(x_n), \qquad n \in \mathbb{N}. \tag{5}$$

Denoting

$$T(x) = x - \frac{F(x)}{F'(x)} \cdot G(x),$$

in order to prove the third order convergence of the method (5) the following relations must be verified

$$T(a) = 1 \qquad T'(a) = 0, \qquad T''(a) = 0 \qquad T^{(3)}(a) \neq 0. \tag{6}$$

The convergence occurs when the initial approximation x_0 is properly chosen.

The Halley's method [13] is defined by

$$x_{n+1} = x_n - \frac{2F(x_n)F'(x_n)}{2F'^2(x_n) - F(x_n)F''(x_n)}, \qquad G(x) = \frac{2F'^2(x)}{2F'^2(x) - F(x)F''(x)}.$$

If

$$T1(x) = x - \frac{2F(x)F'(x0}{2F'^2(x) - F(x)F''(x)}$$

then the computation in *Mathematica*[2] is performed by the code below:

$F[a] = 0$

0

$T1[\text{x\_}]{:=}x - (2F[x]F'[x])/(2F'[x]\text{^}2 - F[x]F\text{''}[x])$

$T1[x]/.x \to a$

a

$D[T1[x], x]/.x \to a$

0

$D[T1[x], \{x, 2\}]/.x \to a$

0

$\text{Simplify}[D[T1[x], \{x, 3\}]/.x \to a]$

$\frac{3F''[a]^2 - 2F'[a]F^{(3)}[a]}{2F'[a]^2}$

The same scheme will be applied to the following methods.

In [11] Weerakoon and Fernando had introduced the third order convergence method

$$x_{n+1} = x_n - \frac{2F(x_n)}{F'(x_n) + F'\left(x_n - \frac{F(x_n)}{F'(x_n)}\right)}, \quad G(x) = \frac{2F'(x)}{F'(x) + F'\left(x - \frac{F(x)}{F'(x)}\right)}. \quad (7)$$

For

$$T2(x) = x - \frac{2F(x)}{F'(x) + F'\left(x - \frac{F(x)}{F'(x)}\right)}$$

we found

$F[a] = 0$

0

$\text{T2}[\text{x\_1}]{:=}x - 2F[x]/(F'[x] + F'[x - F[x]/F'[x]])$

$\text{T2}[x]/.x \to a$

a

$D[\text{T2}[x], x]/.x \to a$

0

$D[\text{T2}[x], \{x, 2\}]/.x \to a$

0

$\text{Simplify}[D[\text{N1}[x], \{x, 3\}]/.x \to a]$

$\frac{3T''[a]^2 + T'[a]T^{(3)}[a]}{2T'[a]^2}$

Frontini and Sormani [2] considered the method

$$x_{n+1} = x_n - \frac{F(x_n)}{F'\left(x_n - \frac{F(x_n)}{2F'(x_n)}\right)}, \qquad G(x) = \frac{F'(x)}{F'\left(x - \frac{F(x)}{2F'(x)}\right)}. \qquad (8)$$

[2]The settings and given commands are printed with bold characters.

So, for

$$T3(x) = x - \frac{F(x)}{F'\left(x - \frac{F(x)}{2F'(x)}\right)}.$$

the *Mathematica* code is

$\boldsymbol{F[a] = 0}$

0

T3[x_]:= $\boldsymbol{x - F[x]/F'[x - F[x]/(2F'[x])]}$

T3[$\boldsymbol{x}$ **]/.** $\boldsymbol{x \to a}$

a

$\boldsymbol{D}$**[T3[** $\boldsymbol{x}$ **],** $\boldsymbol{x}$ **]/.** $\boldsymbol{x \to a}$

0

$\boldsymbol{D}$**[T3[** $\boldsymbol{x}$ **], {** $\boldsymbol{x}$ **, 2}]/.** $\boldsymbol{x \to a}$

0

$\boldsymbol{D}$**[T3[** $\boldsymbol{x}$ **], {** $\boldsymbol{x}$ **, 3}]/.** $\boldsymbol{x \to a}$

$\frac{3F''[a]^2}{2F'[a]^2} - \frac{F^{(3)}[a]}{4F'[a]}$

In [4], [10] the following method is defined

$$x_{n+1} = x_n - \frac{F(x_n)}{2}\left(\frac{1}{F'(x_n)} + \frac{1}{F'\left(x_n - \frac{F(x_n)}{F'(x_n)}\right)}\right), \tag{9}$$

$$G(x) = \frac{F'(x)}{2}\left(\frac{1}{F'(x)} + \frac{1}{F'\left(x - \frac{F(x)}{F'(x)}\right)}\right).$$

The corresponding *Mathematica* code for this method is

$\boldsymbol{F[a] = 0}$

0

T4[x_]:= $\boldsymbol{x - F[x]/2(1/F'[x] + 1/F'[x - F[x]/F'[x]])}$

T4[$\boldsymbol{x}$ **]/.** $\boldsymbol{x \to a}$

a

$\boldsymbol{D}$**[T4[** $\boldsymbol{x}$ **],** $\boldsymbol{x}$ **]/.** $\boldsymbol{x \to a}$

0

$\boldsymbol{D}$**[T4[** $\boldsymbol{x}$ **], {** $\boldsymbol{x}$ **, 2}]/.** $\boldsymbol{x \to a}$

0

$\boldsymbol{D}$**[T4[** $\boldsymbol{x}$ **], {** $\boldsymbol{x}$ **, 3}]/.** $\boldsymbol{x \to a}$

$\frac{3F''[a]^2}{2F'[a]^2} - \frac{F^{(3)}[a]}{F'[a]} - \frac{3}{2}F'[a]\left(\frac{F''[a]^2}{F'[a]^3} - \frac{F^{(3)}[a]}{F'[a]^2}\right)$

In [5] the following method is introduced

$$x_{n+1} = x_n - \frac{2MF(x_n)}{\sum_{k=1}^{2M} F'\left(x_n - \frac{F(x_n)}{F'(x_n)}\frac{k-0.5}{2M}\right)}, \tag{10}$$

where $M \in \mathbb{N}^*$. Now

$$G(x) = \frac{2MF'(x)}{\sum_{k=1}^{2M} F'\left(x - \frac{F(x)}{F'(x)}\frac{k-0.5}{2M}\right)}.$$

The required computations is given by the code

$F[a]=0$

0

T5[x_] $= x - 2MF[x]/\text{Sum}[F'[x - F[x]/F'[x](k-1/2)/(2M)], \{k, 1, 2M\}]$

$$x - \frac{2MF[x]}{\sum_{k=1}^{2M} F'\left[x - \frac{\left(-\frac{1}{2}+k\right)F[x]}{2MF'[x]}\right]}$$

T5$[x]/.x \to a$

a

$D[\text{T5}[x], x]$

$$1 + \frac{2MF[x]\sum_{k=1}^{2M}\left(1 - \frac{-\frac{1}{2}+k}{2M} + \frac{\left(-\frac{1}{2}+k\right)F[x]F''[x]}{2MF'[x]^2}\right)F''\left[x - \frac{\left(-\frac{1}{2}+k\right)F[x]}{2MF'[x]}\right]}{\left(\sum_{k=1}^{2M} F'\left[x - \frac{\left(-\frac{1}{2}+k\right)F[x]}{2MF'[x]}\right]\right)^2} - \frac{2MF'[x]}{\sum_{k=1}^{2M} F'\left[x - \frac{\left(-\frac{1}{2}+k\right)F[x]}{2MF'[x]}\right]}$$

$D[\text{T5}[x], x]/.x \to a$

0

$D[\text{T5}[x], \{x, 2\}]/.x \to a$

0

$D[\text{T5}[x], \{x, 3\}]/.x \to a$

$$-2M\left(-\frac{3F''[a]^2}{4MF'[a]^2} + \frac{F^{(3)}[a]}{2MF'[a]} + 3F'[a]\left(\frac{F''[a]^2}{4MF'[a]^3} - \frac{24M^2F''[a]^2 - F'[a]F^{(3)}[a] + 16M^2F'[a]F^{(3)}[a]}{96M^3F'[a]^3}\right)\right)$$

Even the Laguerre method to compute a root of a polynomial [8], [6] may be presented in the same manner. Let be $P(x) = \prod_{i=1}^{m}(x - x_i)$ a polynomial having only simple roots. The Laguerre method is defined by

$$x_{n+1} = x_n - \frac{P(x_n)}{P'(x_n)} \cdot \frac{1}{\frac{1}{m} + \frac{m-1}{m}\sqrt{1 - \frac{m}{m-1}\frac{P(x_n)P''(x_n)}{P'^2(x_n)}}}, \qquad n \in \mathbb{N}, \quad x_0 \in \mathbb{C}.$$

The square root of a complex number is chosen to have a non-negative real part. In this case

$$G(x) = \frac{1}{\frac{1}{m} + \frac{m-1}{m}\sqrt{1 - \frac{m}{m-1}\frac{P(x)P''(x)}{P'^2(x)}}}.$$

The *Mathematica* code is

$P[a]=0$

0

G[x_]:=

$1/(1/m + (m-1)/m\text{Sqrt}[1 - m/(m-1)P[x]D[P[x], \{x, 2\}]/D[P[x], x]\text{^}2])$

$T6$[x_]:=$x - P[x]/D[P[x], x]G[x]$

$T6[x]/.x \to a$

a

Simplify$[D[T6[x], x]/.x \to a]$

0

Simplify$[D[T6[x], \{x, 2\}]/.x \to a]$

0

Simplify$[D[T6[x], \{x, 3\}]/.x \to a]$

$$\frac{3(-2+m)P''[a]^2 - 4(-1+m)P'[a]P^{(3)}[a]}{4(-1+m)P'[a]^2}$$

As expected, we found that the convergence order of the method is 3.

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