Bulletin of the *Transilvania* University of Braşov • Vol 10(59), No. 1 - 2017
Series III: Mathematics, Informatics, Physics, 165-178

ON A CHLODOVSKY VARIANT OF $\alpha-$ BERNSTEIN OPERATOR

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Abstract

In this paper, we introduce the Chlodovsky variant of α Bernstein operators which are generalizations of α Bernstein operators. We investigate some elementary properties of this operator and then we study its approximation properties, including a Voronovskaja type asymptotic estimate formula for the operators.
2017 *Mathematics Subject Classification:* 41A36, 41A25.
Key words: Bernstein operator, α Bernstein operators, Chlodovsky operators, modulus of continuity.

1 Introduction

In 1912, S. N. Bernstein [2] introduced the following sequence of Operators $B_n : C[0,1] \to C[0,1]$ defined for any $n \in \mathbb{N}$ and $f \in C[0,1]$

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1] \tag{1}$$

In the last years we have several generalizations of this classical Bernstein Polynomials. It is a powerful tool for numerical analysis, solutions of differential equations and computer aided geometric design. These operators are the prototype of all the positive linear operators used in approximation and a great number of generalizations of these operators was given. We mention a recent generalization was given in [4], named α Bernstein operators.
In this paper we are especially interested in a Chlodovsky variant of it. This new variant will be presented in the next section. In 1932, Chlodowsky [9] introduced a generalization of Bernstein polynomials on an unbounded set, known as Bernstein - Chlodovsky polynomials

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)^{n-k} f\left(\frac{k}{n} b_n\right), \quad 0 \le x \le b_n \tag{2}$$

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where b_n is an increasing sequence of positive terms with the properties $b_n \to \infty$ and $\frac{b_n}{n} \to 0$ as $n \to \infty$.

Among the results obtained for the Bernstein-Chlodovsky operators we mention a Voronoskaya-type theorem for the derivative of the Kantorovich variant of the Bernstein–Chlodovsky operators, presented by Butzer and Karsli in [3]. Also, Karsli introduced a variant of the Chlodovsky–Kantorovich operators and a variant of the Chlodovsky-Durrmeyer operators in [11] and [12]. A Chlodovsky variant of Szasz operators was introduced in [17]. On the other hand, the q-modification of the Bernstein-Chlodovsky operators were studied in [10].

The aim of Chlodovsky–modification is to obtain operators for approximation on a unbounded interval. In the next section we define α–Bernstein–Chlodovsky operators. First we give definition of this new type of operators and certain elementary properties which play an important role in the theory of uniform approximation of functions. The main purpose is to study some results concerning uniform convergence and estimates of the degree of approximation.

2 The α–Chlodovsky–Bernstein operators. Basic Properties

In [4], the authors introduced a new family of operators as follows

Definition A[4] *Given a function f on $[0,1]$, for each positive integer n and any fixed real α, we define α-Bernstein operator for f as*

$$T_{n,\alpha}(f;x) = \sum_{i=0}^{n} f_i p_{n,i}^{(\alpha)}(x) \tag{3}$$

where $f_i = f\left(\frac{i}{n}\right)$. For $i = 0, 1, \ldots, n$, the α-Bernstein polynomial $p_{n,i}^{\alpha}(x)$ of degree n are defined by $p_{1,0}^{\alpha}(x) = 1 - x$, $p_{1,1}^{\alpha}(x) = x$ and

$$p_{n,i}^{(\alpha)}(x) = \left[\binom{n-2}{i}(1-\alpha)x + \binom{n-2}{i-2}(1-\alpha)(1-x) + \binom{n}{i}\alpha x(1-x)\right] \cdot x^{i-1}(1-x)^{n-i-1}, \tag{4}$$

where $n \geq 2$, $0 \leq i \leq n$, $x \in [0,1]$ and the binomial coefficients $\binom{k}{l}$ are given by

$$\binom{k}{l} = \begin{cases} \frac{k!}{l!(k-l)!}, & if\ 0 \leq l \leq k, \\ 0, & else \end{cases}$$

Calculating some terms given above we easily observe the followings formulas

$$\begin{aligned} p_{n,0}^{\alpha}(x) &= (1-\alpha x)(1-x)^{n-1} \\ p_{n,n}^{\alpha}(x) &= (1-\alpha+\alpha x)x^{n-1} \end{aligned} \tag{5}$$

The α-Bernstein operator maps a function f, defined on $[0,1]$, to $T_{n,\alpha}(f;x)$. When $\alpha = 1$, the α-Bernstein polynomial reduces to the classical Bernstein polynomial. Our main definition is the following.

Definition 1. *Let $CT_{n,\alpha} : C[0,\infty) \to C[0,b_n]$ be the α-Chlodovsky-Bernstein operators, defined by*

$$CT_{n,\alpha}(f;x) := \sum_{i=0}^{n} f\left(\frac{b_n}{n}i\right) p_{n,i}^{\alpha}\left(\frac{x}{b_n}\right) \tag{6}$$

where $\alpha \in \mathbb{R}$, $p_{n,i}^{\alpha}$ are defined defined in (4), $f \in C[0,\infty)$, $x \in [0,b_n]$ and $(b_n)_{n=1}^{\infty}$ is a positive increasing sequence of reals with the properties

$$\lim_{n\to\infty} b_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{b_n}{n} = 0 \tag{7}$$

Remark 1. *For $\alpha \in [0,1]$ and $n \in \mathbb{N}$, the operator $CT_{n,\alpha}(\,\cdot\,;x)$ is positive.*

For this family of operators, we give here some of their properties and results.

Lemma 1. *For all $n \geq 1$, independent of α:*

(i) (End point interpolation) The α–Chlodovsky–Bernstein operator for f interpoletes f at both endpoints of the interval $[0,b_n]$, namely

$$CT_{n,\alpha}(f;0) = f(0) \quad \text{and} \quad CT_{n,\alpha}(f;b_n) = f(b_n) \tag{8}$$

(ii) (Linearity) The α–Chlodovsky–Bernstein operator is linear, that is,

$$CT_{n,\alpha}(\lambda f + \mu g) = \lambda CT_{n,\alpha}(f) + \mu CT_{n,\alpha}(g) \tag{9}$$

for all function $f(x)$ and $g(x)$ defined on $[0,\infty)$, and all real λ and μ.

Proof. Point i) follows easily from (4) and (5) and point ii) follows from (6) □

By using Theorem 2.1. from [4] the α–Chlodovsky–Bernstein operator can be expressed as

Theorem 1. *For $n \in \mathbb{N}$, $x \in [0,b_n]$, $\alpha \in [0,1]$, $f \in C[0,\infty)$ we have*

$$CT_{n,\alpha}(f;x) = (1-\alpha)\sum_{i=0}^{n-1} g_i \binom{n-1}{i} \left(\frac{x}{b_n}\right)^i \left(1-\frac{x}{b_n}\right)^{n-i-1} + \alpha \sum_{i=0}^{n} f_i \binom{n}{i} \left(\frac{x}{b_n}\right)^i \left(1-\frac{x}{b_n}\right)^{n-i} \tag{10}$$

where

$$g_i = \left(1-\frac{i}{n-1}\right) f\left(\frac{x}{b_n}i\right) + \frac{i}{n-1} f\left(\frac{x}{b_n}(i+1)\right) \tag{11}$$

To obtain our results in an easy and elegant way, we introduce the notion of a forward difference.

Definition 2. *Let $I \subset \mathbb{R}$ be a compact interval and $f : I \to \mathbb{R}$ and $h > 0$. We define the forward difference as*

$$\Delta_h f(x) = f(x+h) - f(x). \tag{12}$$

Let us define also the higher-order forward difference as

$$\Delta_h^k f(x) = \Delta_h^{k-1} \Delta_h f(x). \tag{13}$$

This last formula leads us to

$$\Delta_h^k f(x) = \sum_{i=0}^{k} (-1)^{k+i} \binom{i}{k} f(x+ih). \tag{14}$$

We will use this higher-order forward differences to rewrite the form of the operator and to simplify our calculation for the moments of the operator. We need only the fourth higher-order forward difference to calculate the fourth moment of the operator. So, let us take $h = \frac{b_n}{n}$ and the polynomial function of degree k $f(x) = x^k$, where $n \geq k$. Then we have

$$\Delta^r f(0) = 0 \ for \ r > k \tag{15}$$

and

$$\Delta^k f\left(\frac{b_n}{n} i\right) = \frac{b_n^k}{n^k} f^{(k)}\left(\xi_i\right) = \frac{b_n^k \cdot k!}{n^k}, \quad \xi_i \in \left(\frac{b_n \cdot i}{n}, \frac{b_n \cdot (i+k)}{n}\right) \tag{16}$$

We consider now the following result

Lemma 2. *The higher-order forward difference of g_i, adapted to our case, can be expressed in the form*

$$\Delta^r g_i = \left(1 - \frac{i}{n-1}\right) \Delta^r f_i + \frac{i+r}{n-1} \Delta^r f_{i+1} \tag{17}$$

where $g_i = \left(1 - \frac{i}{n-1}\right) f\left(\frac{b_n}{n} i\right) + \frac{i}{n-1} f\left(\frac{b_n}{n}(i+1)\right)$.

Now from Theorem 3.1 ([4]), pag. 6, we have

Theorem 2. *The α-Chlodovsky Bernstein Operator has the following representation in terms of the difference operators*

$$CT_{n,\alpha}(f;x) = \sum_{k=0}^{n} \left[(1-\alpha) \binom{n-1}{k} \Delta^k g_0 + \alpha \binom{n}{k} \Delta^k f_0 \right] x^k \tag{18}$$

Theorem 2 and Lemma 2 show that the α-Chlodovsky Bernstein Operator has the degree-preserving property. In particular case, for $f(x) = x^k$ and $n \geq k+1$ it follows that

$$CT_{n,\alpha}(t^k;x) = a_k \left(\frac{x}{b_n}\right)^k + a_{k-1} \left(\frac{x}{b_n}\right)^{k-1} + \ldots + a_1 \left(\frac{x}{b_n}\right) + a_0 \tag{19}$$

where

$$a_k = (1 - \alpha \binom{n-1}{k} \Delta^k g_0 + \alpha \binom{n}{k} \Delta^k f(0) \tag{20}$$

Lemma 3. *Let $CT_{n,\alpha}(f;x)$ be given by (6). The first few moments of the operators are*

(i) $CT_{n,\alpha}(1;x) = 1$

(ii) $CT_{n,\alpha}(t;x) = x$

(iii) $CT_{n,\alpha}(t^2;x) = x^2 + \frac{n+2(1-\alpha)}{n^2}x(b_n - x)$

(iv) $CT_{n,\alpha}(t^3;x) = x^3 + \frac{3[n+2(1-\alpha)]}{n^2}x^2(b_n - x) + \frac{n+6(1-\alpha)}{n^3}x(b_n - x)(b_n - 2x)$

(v) $CT_{n,\alpha}(t^4;x) = x^4 + \frac{6[n+2(1-\alpha)]}{n^2}x^3(b_n - x) + \frac{4[n+6(1-\alpha)]}{n^3}x^2(b_n - x)(b_n - 2x) + \frac{[3n(n-2)+12(n-6)(1-\alpha)]x(b_n-x)+[n+14(1-\alpha)]}{n^4}x(b_n - x)$

(vi) $CT_{n,\alpha}(t - x;x) = 0;$

(vii) $CT_{n,\alpha}\left((t-x)^2;x\right) = \frac{n+2(1-\alpha)}{n^2}(b_n - x)x$

(viii) $CT_{n,\alpha}\left((t-x)^4;x\right) = \frac{[3n(n-2)+12(n-6)(1-\alpha)]x(b_n-x)+[n+14(1-\alpha)]}{n^4}x(b_n - x)$

where $n \in \mathbb{N},\ x \in [0, b_n]$.

Proof. (i) By taking $f(x) \equiv 1$ from (10) and (11) we have $f\left(\frac{x}{b_n}i\right) = g_i = 1$ and

$$CT_{n,\alpha}(1;x) = (1-\alpha)\sum_{i=0}^{n-1}\binom{n-1}{i}\left(\frac{x}{b_n}\right)^i\left(1-\frac{x}{b_n}\right)^{n-i-1} + \alpha\sum_{i=0}^{n-1}\binom{n}{i}\left(\frac{x}{b_n}\right)^i\left(1-\frac{x}{b_n}\right)^{n-i}$$

So, the α-Chlodovsky Bernstein operator for the constant function 1 is

$$CT_{n,\alpha}(1;x) = (1-\alpha)\sum_{i=0}^{n-1}p_{n-1,i}\left(\frac{x}{b_n}\right) + \alpha\sum_{i=0}^{n}p_{n,i}\left(\frac{x}{b_n}\right) = 1.$$

(ii) By taking $f(x) = x$ from (10) and (11) we have $f\left(\frac{x}{b_n}i\right) = \frac{b_n}{n}i$ and

$$g_i = \left(1 - \frac{i}{n-1}\right)\cdot\frac{b_n}{n}i + \frac{i}{n-1}\cdot\frac{b_n}{n}(i+1) = \frac{b_n}{n-1}i$$

and

$$CT_{n,\alpha}(t;x) = (1-\alpha)\sum_{i=0}^{n-1} \frac{b_n}{n-1} i \binom{n-1}{i} \left(\frac{x}{b_n}\right)^i \left(1-\frac{x}{b_n}\right)^{n-i-1}$$
$$+ \alpha \sum_{i=0}^{n} \frac{b_n}{n} i \binom{n}{i} \left(\frac{x}{b_n}\right)^i \left(1-\frac{x}{b_n}\right)^{n-i}$$
$$= (1-\alpha)x \sum_{i=0}^{n-1} p_{n-2,i}(\frac{x}{b_n}) + \alpha x \sum_{i=0}^{n} p_{n-1,i}(\frac{x}{b_n})$$
$$= (1-\alpha)x + \alpha x = x$$

$$f(0) = 0 \qquad \Delta f(0) = \left(\frac{b_n}{n}\right)^2 \quad \Delta^2 f(0) = 2\cdot\left(\frac{b_n}{n}\right)^2$$
$$f\left(\frac{b_n}{n}\right) = \left(\frac{b_n}{n}\right)^2 \qquad \Delta f\left(\frac{b_n}{n}\right) = 3\cdot\left(\frac{b_n}{n}\right)^2 \quad \Delta^2 f\left(\frac{b_n}{n}\right) = 2\cdot\left(\frac{b_n}{n}\right)^2$$
$$f\left(\frac{2b_n}{n}\cdot\right) = 4\cdot\left(\frac{b_n}{n}\right)^2 \qquad \Delta f\left(\frac{2b_n}{n}\right) = 5\cdot\left(\frac{b_n}{n}\right)^2$$
$$f\left(\frac{3b_n}{n}\right) = 9\cdot\left(\frac{b_n}{n}\right)^2$$

and

$$g_0 = 0$$
$$\Delta g_0 = \Delta f(0) + \frac{1}{n-1}\Delta f\left(\frac{b_n}{n}\right) = \frac{n+2}{n-1}\left(\frac{b_n}{n}\right)^2$$
$$\Delta^2 g_0 = \Delta^2 f(0) + \frac{2}{n-1}\Delta^2 f\left(\frac{b_n}{n}\right) = 2\cdot\frac{n+1}{n-1}\left(\frac{b_n}{n}\right)^2$$

for $n \geq 2$. Thus it follows from (19) that

$$CT_{n,\alpha}(t^2;x) = \left[(1-\alpha)\binom{n-1}{1}\left(\frac{b_n}{n}\right)^2 \frac{n+2}{n-1} + \alpha\binom{n}{1}\left(\frac{b_n}{n}\right)^2\right]\frac{x}{b_n}$$
$$+ \left[(1-\alpha)\binom{n-1}{2} 2\cdot\left(\frac{b_n}{n}\right)^2 \cdot \frac{n+1}{n-1} + \alpha\binom{n}{2} 2\cdot\left(\frac{b_n}{n}\right)^2\right]\left(\frac{x}{b_n}\right)^2$$
$$= \left(\frac{b_n}{n}\right)^2 \cdot \frac{x}{b_n} \cdot (n + 2(1-\alpha))$$
$$+ \left(\frac{b_n}{n}\right)^2 \cdot \left(\frac{x}{b_n}\right)^2 \cdot (n^2 - n - 2(1-\alpha))$$

Grouping this terms we obtain

$$CT_{n,\alpha}(t^2;x) = x^2 + \frac{n+2(1-\alpha)}{n^2} x(b_n - x)$$

Similarly, we can prove (iv) and (v).
To prove (vi), (vii) and $(viii)$ we will use the first five relations (i), (ii), (iii), (iv) and (v), and the linearity of the Operator α-Chlodovsky Bernstein. □

Corollary 1. *The α Bernstein-Chlodovsky operator reproduces linear functions, so*

$$CT_{n,\alpha}(ax+b;x) = ax+b \tag{21}$$

for all real numbers a and b.

Proof. It immediately follows from Lemma 3 (i), (ii) and the linear property. □

3 Convergence Properties

Theorem A [[1], Th. 3.2. page 98][Altomare's Theorem] *Let (X,d) be a metric space and consider a lattice subspacce E of $F(X)$ containing the constant functions and all function $d_x^2 (x \in X)$ where $F(X)$ is the space of real functions on X. Let $(L_n)_{n\geq 1}$ be a sequence of positive linear operators from E into $F(x)$ and let Y be a subset of X such that*

(i) $\lim_{n\to\infty} L_n(1) = 1$ $uniformly$ on Y

(i) $\lim_{n\to\infty} L_n(d_x^2;x) = 0$ $uniformly$ $with$ $respect$ to $x \in Y$.

Then for every $f \in E \cap UC_b(X)$

$$\lim_{n\to\infty} L_n(f) = f \ uniformly \ on \ Y$$

where $UC_b(X)$ is the space of all continuous bounded functions on X.
Thus, we have the following result

Theorem 3. *Let $f \in UC_b[0,\infty)$ and $(b_n)_{n\geq 1}$ defined in (7) then for any $K \subset [0,\infty)$ we have*

$$\lim_{n\to\infty} CT_{n,\alpha}(f;x) = f(x) \tag{22}$$

uniformly with regard to $x \in K$, where $CT_{n,\alpha}(f;x)$ is considered for n sufficiently large .

Proof. We apply Theorem A, by considering $X = [0,\infty)$, $d(x,y) = |x-y|$ and $Y = K$. So

$$CT_{n,\alpha}(1;x) = 1$$

and

$$CT_{n,\alpha}\left((t-x)^2;x\right) = \frac{n+2(1-\alpha)}{n^2}(b_n - x)x$$

When n tends to ∞ the last relation is

$$\lim_{n\to\infty}\frac{n+2(1-\alpha)}{n^2}(b_n-x)x=\lim_{n\to\infty}\left(1+\frac{2(1-\alpha)}{n}\right)\left(\frac{b_n}{n}-\frac{x}{n}\right)x=0$$

Condition (i) and (ii) of Theorem A are satisfied for $f\in UC[0,\infty)$, so from this we have

$$\lim_{n\to\infty}CT_{n,\alpha}(f;x)=f(x).$$

□

Let be an interval. In order to determinate the degree of approximation, we use the moduli of continuity of order 1 and 2 given by

$$\omega_1(f;h)=\sup\{|f(x)-f(y)|,\ x,y\in I,\ |x-y|\le h,\ I\subset\mathbb{R}\} \tag{23}$$

$$\omega_2(f;h)=\sup\left\{|f(x)-2f\left(\frac{x+y}{2}\right)+f(y)|,\ x,y\in I,\ |x-y|\le 2h,\ I\subset\mathbb{R}\right\} \tag{24}$$

The general estimates with these moduli of continuity by positive linear operators are given by the following theorems:

Theorem B[13] *Let $L:V\to F(I)$ be a linear positive operator, where $F(I)$ is the space of real functions on compact interval I and V is a linear subspace of $F(I)$ such that $e_j\in V,\ e_j=t^j,\ j\in 0,1,2$ and $g\in V$. For all $y\in I$ and $h>0$ one has*

$$\begin{aligned}|L(g,y)-g(y)|\le&|g(y)||L(e_0,y)-1|\\&+\left(L(e_0,y)+\frac{1}{h^2}L((e_1-ye_0)^2,y)\right)\cdot\omega_1(g,h).\end{aligned} \tag{25}$$

Theorem C[14] *Let $L:V\to F(I)$ be a linear positive operator, where V is linear subspace of $C(I)$ such that $e_0,\ e_1,\ e_2\in V$ and $g\in V$. Let $y\in I$ and $h>0$, such that $h\le\frac{1}{2}length(I)$. Then*

$$\begin{aligned}|L(g,y)-g(y)|]\le&|g(y)||L(e_0,y)-1|+\frac{1}{h}\cdot|L(e_1-ye_0,y)|\cdot\omega_1(g,h)\\&+\left(L(e_0,y)+\frac{1}{2h^2}L((e_1-ye_0)^2,y)\right)\cdot\omega_2(g,h)\end{aligned} \tag{26}$$

We obtain

Theorem 4. *If $0\le\alpha\le 1,\ n\ge 1,\ f\in C[0,\infty)$ and $x\in[0,b_n]$ then*

$$|CT_{n,\alpha}(f;x)-f(x)|\le 2\omega_1\left(f;\sqrt{\frac{n+2(1-\alpha)}{n^2}(b_n-x)x}\right) \tag{27}$$

Proof. We apply Theorem B and Lemma 3. □

Remark 2. *From Theorem 4 we can obtain an improvement of Theorem 3 since if $K \subset [0, \infty)$ is an compact interval, then $\max_{x\in K} \sqrt{\frac{n+2(1-\alpha)}{n^2}(b_n - x)x} \longrightarrow 0$ when $n \to \infty$. Consequently we have*

$$\lim_{n\to\infty} CT_{n,\alpha}(f, x) = f(x) \quad uniformly\ on\ K\ for\ any\ function\ f \in UC[0, \infty).$$

Theorem 5. *If $0 \le \alpha \le 1$, $n \ge 1$, $f \in C[0, \infty)$ and $x \in [0, b_n]$ then*

$$|CT_{n,\alpha}(f; x) - f(x)| \le \frac{3}{2}\omega_2 \left(f; \sqrt{\frac{n + 2(1-\alpha)}{n^2}(b_n - x)x} \right) \tag{28}$$

Proof. We apply Theorem C and Lemma 3. □

4 A Voronovskaja type Theorem for α–Bernstein–Chlodovsky Operators

Theorem 6. *Let $f \in C^2[0, \infty)$ and $x \in [0, b_n]$. If f'' is uniform continuous on $[0, \infty)$ and $\lim_{n\to\infty} \omega(f''; \sqrt{\frac{CT_{n,\alpha}((t-x)^4;x)}{CT_{n,\alpha}((t-x)^2;x)}}) = 0$, then*

$$\lim_{n\to\infty} \frac{n}{b_n}[CT_{n,\alpha}(f; x) - f(x)] = \frac{1}{2}xf''(x). \tag{29}$$

uniformly on any compact set.

Proof. We have by Taylor's formula

$$f(t) = f(x) + (t - x)f'(x) + \frac{1}{2}(t - x)^2 f''(x) + \eta_x(t)(t - x)^2$$

where $\lim_{t\to x} \eta_x(t) = 0.$. Thus we have

$$\frac{n}{b_n}[CT_{n,\alpha}(f; x) - f(x)] = \frac{n}{b_n}(f'(x)CT_{n,\alpha}(t - x; x) + \frac{1}{2}f''(x)CT_{n,\alpha}((t - x)^2; x) + CT_{n,\alpha}(\eta_x(t)(t - x)^2; x)).$$

From Lemma (3) we obtain

$$\frac{n}{b_n}[CT_{n,\alpha}(f; x) - f(x)] = \frac{n}{b_n}(\frac{1}{2}f''(x)\frac{n + 2(1-\alpha)}{n^2}x(b_n - x) + CT_{n,\alpha}(\eta_x(t)(t - x)^2; x)).$$

Now by taking limit as $n \to \infty$ for the first term, we obtain that

$$\begin{aligned} & \lim_{n\to\infty} \frac{n}{b_n}\frac{1}{2}f''(x)\frac{n+2(1-\alpha)}{n^2}x(b_n-x) \\ = & \frac{1}{2}xf''(x)\lim_{n\to\infty}\frac{n}{b_n}\frac{n\left(1+\frac{2(1-\alpha)}{n}\right)}{n^2}b_n\left(1-\frac{x}{b_n}\right) \\ = & \frac{1}{2}xf''(x). \end{aligned}$$

Now we have to show that the rest of Taylor's formula tends to zero also. So

$$\begin{aligned} |\eta_x(t)| &= \left|\int_x^t (t-u)[f''(u)-f''(x)]du\right| \\ &\le \int_x^t |(t-u)|\left|f''(u)-f''(x)\right| du \\ &\le \int_x^t (t-u)\omega(f'',h)\left(1+\frac{(u-x)^2}{h^2}\right)du \\ &\le \omega(f'',h)\left[\frac{(t-x)^2}{2}+\frac{1}{h^2}\int_x^t (u-x)^2(t-u)du\right] \\ &\le \omega(f'',h)\left[\frac{(t-x)^2}{2}+\frac{(t-x)^4}{12h^2}\right]. \end{aligned}$$

Applying the operator to the above inequality we obtain

$$CT_{n,\alpha}\left(|\eta_x(t)|;x\right) \le \omega_1(f'',h)\left[\frac{CT_{n,\alpha}\left((t-x)^2;x\right)}{2}+\frac{CT_{n,\alpha}\left((t-x)^4;x\right)}{12h^2}\right].$$

From Lemma 3 we have that

$$CT_{n,\alpha}((t-x)^4;x) = o\left(CT_{n,\alpha}((t-x)^2;x)\right).$$

And by taking $h = h_n = \sqrt{\frac{CT_{n,\alpha}((t-x)^4;x)}{CT_{n,\alpha}((t-x)^2;x)}} = \sqrt{\frac{[3n(n-2)+12(n-6)(1-\alpha)]x(b_n-x)+[n+14(1-\alpha)]}{n^2[n+2(1-\alpha)]}}$ we have

$$\lim_{n\to\infty}\omega(f'';h_n) = 0$$

All this leads us to the assertion in Theorem 6. □

For a quantitative estimations of the above result we consider the least concave majorant $\tilde{\omega}_1(f;\epsilon)$

$$\tilde{\omega}_1(f;\epsilon) = \begin{cases} \sup\left\{\frac{(\epsilon-x)\omega_1(f;y)+(y-\epsilon)\omega(f;x)}{y-x},\ 0\le x\le\epsilon\le y\le b-a,\ x\ne y\right\} \\ \omega_1(f;b-a),\ \epsilon > b-a \end{cases}$$

where $f \in C[a,b],\ \epsilon > 0$.

Theorem D.[6] *Let $q \in \mathbb{N}_0$, $f \in C^q[0,1]$ and $L : C[0,1] \to C[0,1]$ be a positive linear*

$$\begin{aligned} &\left| L(f;x) - \sum_{r=0}^{q} L((e_1 - x)^r; x) \cdot \frac{f^{(r)}(x)}{r!} \right| \\ \leq\ & \frac{L(|e_1 - x|^q; x)}{q!} \cdot \tilde{\omega}_1 \left(f^{(q)}; \frac{1}{q+1} \cdot \frac{L(|e_1 - x|^{q+1}; x)}{L(|e_1 - x|^q; x)} \right) \end{aligned}$$

Remark 3. *Theorem D can be extended also for a linear and positive operator $L : C[a,b] \to C[a,b]$ without any change.*

Theorem 7. *Let $f \in C^2[0,\infty)$, $x \in [0, b_n]$ and $(b_n)_{n=1}^{\infty}$ defined in (7). Then*

$$\begin{aligned} \left| \frac{n}{b_n} \left(CT_{n,\alpha}(f;x) - f(x) \right) - \frac{1}{2} x f''(x) \right| \leq & |f''(x)| x \left| -\frac{x}{b_n} + \frac{2(1-\alpha)}{n} - \frac{2(1-\alpha)}{n b_n} x \right| \\ & + \frac{x}{2} \left(1 + \frac{2(1-\alpha)}{n} \right) \tilde{\omega}_1 \left(f''; \frac{1}{3} \sqrt{\frac{M_2(x)}{M_4(x)}} \right) \end{aligned}$$

where $M_i(x) = CT_{n,\alpha}\left(|t-x|^i; x \right)$, $i \in \mathbb{N}$

Proof. From Theorem D by taking $q = 2$, we obtain

$$\begin{aligned} &\left| CT_{n,\alpha}(f;x) - f(x) - \frac{1}{2} \frac{n + 2(1-\alpha)}{n^2} (b_n - x) x \cdot f''(x) \right| \\ \leq\ & \frac{M_2(x)}{2} \tilde{\omega}_1 \left(f''; \frac{1}{3} \frac{M_3(x)}{M_2(x)} \right). \end{aligned} \tag{30}$$

Applying the Cauchy-Schwartz-Buniakowski inequality we have

$$\frac{M_3(x)}{M_2(x)} \leq \sqrt{\frac{M_4(x)}{M_2(x)}} \tag{31}$$

From inequalities (30) and (31) it results

$$\begin{aligned} \left| \frac{n}{b_n} \left(CT_{n,\alpha}(f;x) - f(x) \right) - \frac{1}{2} x f''(x) \right| & \leq |\tfrac{n}{b_n} [CT_{n,\alpha}(f;x) - f(x) \\ & -\tfrac{1}{2} f''(x) \cdot \tfrac{n+2(1-\alpha)}{n^2} (b_n - x) x]| \\ & + \tfrac{1}{2} |f''(x) \cdot x| \left| \tfrac{n}{b_n} \cdot \tfrac{n+2(1-\alpha)}{n^2} (b_n - x) - 1 \right| \\ & \leq \tfrac{x}{2} \left(1 + \tfrac{2(1-\alpha)}{n} \right) \tilde{\omega}_1 \left(f''; \tfrac{1}{3} \sqrt{\tfrac{M_2(x)}{M_4(x)}} \right) \\ & + \tfrac{1}{2} |f''(x)| x \left| -\tfrac{x}{b_n} + \tfrac{2(1-\alpha)}{n} - \tfrac{2(1-\alpha)}{n b_n} x \right| \end{aligned}$$

□

Remark 4. *From Theorem 7 we can obtain Theorem 4 since if $K \subset [0,\infty)$ is an compact interval, then* $\max_{x\in K}\sqrt{\frac{M_2(x)}{M_4(x)}} \longrightarrow 0$ *when* $n\to\infty$ *and* $\lim_{n\to\infty} -\frac{x}{b_n} + \frac{2(1-\alpha)}{n} - \frac{2(1-\alpha)}{nb_n}x = 0$. *Consequently we have*

$$\lim_{n\to\infty}\frac{n}{b_n}[CT_{n,\alpha}(f;x)-f(x)] = \frac{1}{2}xf''(x). \quad uniformly\ on\ K\ for\ any\ function$$

$f\in UC[0,\infty)$.

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