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ON THE EXISTENCE AND MULTIPLICITY RESULTS FOR A CLASS OF ELLIPTIC PROBLEMS WITH SINGULAR WEIGHTS AND FAILING ZEROES

S. H. RASOULI[1]

Abstract

In this paper we consider the existence of positive solutions of singular elliptic problems of the form

$$\begin{cases} -div(|x|^{-ap}\,|\nabla u|^{p-2}\,\nabla u) = \lambda\,|x|^{-(a+1)p+b}\,f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of R^N with $0 \in \Omega$, $1 < p < N$, $0 \leq a < \frac{N-p}{p}$, and b, λ are positive parameters. Here $f : [0, \infty) \to R$ is continuous function. We discuss the existence of positive solution when f satisfies certain additional conditions. We use the method of sub-super solutions to establish our results.

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1 Introduction

We study the existence of positive solutions to the singular elliptic problem

$$\begin{cases} -div(|x|^{-ap}\,|\nabla u|^{p-2}\,\nabla u) = \lambda\,|x|^{-(a+1)p+b}\,f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded smooth domain of R^N with $0 \in \Omega$, $1 < p < N$, $0 \leq a < \frac{N-p}{p}$, and b, λ are positive parameters. Here $f : [0, \infty) \to R$ is continuous function.

Elliptic problems involving more general operator, such as the degenerate quasilinear elliptic operator given by $-div(|x|^{-ap}\,|\nabla u|^{p-2}\,\nabla u)$, were motivated by the following Caaffarelli, Kohn and Nirenberg's inequality (see [6], [15]). The study of this type of problem is motivated by its various applications, for example, in fluid mechanics, in newtonian fluids, in flow through porous media and

[1]Faculty of Basic Science, *Babol* Noshirvani University of Technology, Babol, Iran, e-mail: s.h.rasouli@nit.ac.ir

in glaciology (see [3, 9]). So, the study of positive solutions of singular elliptic problems has more practical meanings. We refer to [1], [2], [5], [11] for additional results on elliptic problems.

For the regular case, that is, when $a = 0$ and $b = p$ and the quasilinear elliptic equation has been studied by several authors (see [12, 4]). See [8] where the authors discussed the problem (1) when $a = 0$, $b = p = 2$. In [14], the authors extended the study of [8], to the case when $p > 1$. Here we focus on further extending the study in [12] for the quasilinear elliptic problem involving singularity. Due to this singularity in the weights, the extensions are challenging and nontrivial. Our approach is based on the method of sub-super solutions, see [7, 10].

2 Preliminaries

In this paper, we denote $W_0^{1,p}(\Omega, |x|^{-ap})$, the completion of $C_0^\infty(\Omega)$, with respect to the norm $\|u\| = (\int_\Omega |x|^{-ap}|\nabla u|^p dx)^{\frac{1}{p}}$. To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} -div(|x|^{-ap}\,|\nabla\phi|^{p-2}\,\nabla\phi) = \lambda|x|^{-(a+1)p+b}\,|\phi|^{p-2}\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

Let $\phi_{1,p}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,p}$ of (2) such that $\phi_{1,p}(x) > 0$ in Ω, and $||\phi_{1,p}||_\infty = 1$ (see [13, 16]). It can be shown that $\frac{\partial\phi_{1,p}}{\partial n} < 0$ on $\partial\Omega$. Here n is the outward normal. This result is well known and hence, depending on Ω, there exist positive constants $\epsilon, \delta, \sigma_p$ such that

$$\lambda_{1,p}\,|x|^{-(a+1)p+b}\,\phi_{1,p}^p - |x|^{-ap}\,|\nabla\phi_{1,p}|^p \,\leq\, -\epsilon, \qquad x \in \bar{\Omega}_\delta, \tag{3}$$

$$\phi_{1,p} \geq \sigma_p, \qquad x \in \Omega_0 = \Omega \setminus \bar{\Omega}_\delta, \tag{4}$$

where $\bar{\Omega}_\delta = \{x \in \Omega \mid d(x, \partial\Omega) \leq \delta\}$ (see [13]).

3 Our results

A nonnegative function ψ is called a subsolution of (1) if it satisfy $\psi \leq 0$ on $\partial\Omega$ and

$$\int_\Omega |x|^{-ap}\,|\nabla\psi|^{p-2}\,|\nabla\psi|\cdot\,\nabla w\,dx \leq \lambda\int_\Omega |x|^{-(a+1)p+b}\,f(\psi)\,w\,dx,$$

$$\int_\Omega |x|^{-ap}\,|\nabla\,z|^{p-2}\,|\nabla\,z|\cdot\,\nabla w\,dx \geq \lambda\int_\Omega |x|^{-(a+1)p+b}\,f(z)\,w\,dx,$$

for all $w \in W = \{w \in C_0^\infty(\Omega)|w \geq 0, x \in \Omega\}$. Then the following result holds:

Lemma 3.1. (See [13]) Suppose there exist sub and super- solutions ψ and z

respectively of (1) such that $\psi \leq z$. Then (1) has a solution u such that $\psi \leq u \leq z$.

We make the following assumptions:

(H1) There exists $\mu > 0$ such that $f(y)(\mu - y) > 0$; $y \neq \mu_1$.

(H2)

$$\lim_{y \to 0^+} \frac{f(y)}{y^{p-1}} = 0.$$

We establish:

Theorem 3.2. Assume (H1) holds. Then the problem (1) admits a positive large solution provided λ is large.

Theorem 3.3. Assume (H1) and (H2) hold. Then the problem (1) has at least two positive solutions provided λ is large.

4 Proof of Theorems 3.2-3.3

Proof of Theorem 3.2

For fixed $\gamma \in (0, \mu)$, we shall verify that $\psi = (\frac{\gamma}{2})^{\frac{1}{p-1}} (\frac{p-1}{p}) \phi_{1,p}^{\frac{p}{p-1}}$, is a sub-solution of (1). Let $w \in W$. Then a calculation shows that

$$\begin{aligned} & \int_\Omega |x|^{-ap} \, |\nabla \psi|^{p-2} \, \nabla \psi \, \nabla w \, dx \\ = \; & (\frac{\gamma}{2}) \int_\Omega |x|^{-ap} \, \phi_{1,p} \, |\nabla \phi_{1,p}|^{p-2} \, \nabla \phi_{1,p} \, \nabla w \, dx \\ = \; & (\frac{\gamma}{2}) \int_\Omega |x|^{-ap} \, |\nabla \phi_{1,p}|^{p-2} \, \nabla \phi_{1,p} \, [\nabla(\phi_{1,p} w) - |\nabla \phi_{1,p}|^p \, w] \, dx \\ = \; & (\frac{\gamma}{2}) \int_\Omega [\lambda_{1,p} \, |x|^{-(a+1)p+b} \, \phi_{1,p}^p - |x|^{-ap} \, |\nabla \phi_{1,p}|^p] w \, dx. \end{aligned}$$

First we consider the case when $x \in \bar{\Omega}_\delta$. We have $\lambda_{1,p} \, |x|^{-(a+1)p+c_1} \, \phi_{1,p}^p - |x|^{-ap} \, |\nabla \phi_{1,p}|^p \leq -\epsilon$ on $\bar{\Omega}_\delta$. Since $f(\psi) \geq 0$, it follows that

$$\begin{aligned} & (\frac{\gamma}{2}) \int_{\bar{\Omega}_\delta} [\lambda_{1,p} \, |x|^{-(a+1)p+c_1} \, \phi_{1,p}^p - |x|^{-ap} \, |\nabla \phi_{1,p}|^p] w \, dx \\ \leq \; & -(\frac{\gamma}{2}) \epsilon \int_{\bar{\Omega}_\delta} w \, dx \\ \leq \; & \lambda \int_{\bar{\Omega}_\delta} |x|^{-(a+1)p+p} \, f(\psi) \, w \, dx. \end{aligned}$$

On the other hand, on $\Omega \setminus \bar{\Omega}_\delta$, we have $\phi_{1,p} \geq \sigma_p$, for some $0 < \sigma_p < 1$. We can find λ_* sufficiently large such that

$$(\frac{\gamma}{2})\,\lambda_{1,p} < \lambda \min_{s\in[\frac{\gamma\sigma_p}{2},\gamma]} f(s),$$

for all $x \in \Omega \setminus \bar{\Omega}_\delta$ and for all $\lambda \geq \lambda_*$. Hence

$$\begin{aligned}
& (\frac{\gamma}{2}) \int_{\Omega\setminus\bar{\Omega}_\delta} [\lambda_{1,p}\,|x|^{-(a+1)p+b}\,\phi_{1,p}^p - |x|^{-ap}\,|\nabla\phi_{1,p}|^p] w\,dx \\
\leq\ & (\frac{\gamma}{2}) \int_{\Omega\setminus\bar{\Omega}_\delta} |x|^{-(a+1)p+b}\,\lambda_{1,p}\,w\,dx \\
\leq\ & \lambda \int_{\Omega\setminus\bar{\Omega}_\delta} |x|^{-(a+1)p+b} \min_{s\in[\frac{\gamma\sigma_p}{2},\gamma]} f(s)\,w\,dx \\
\leq\ & \lambda \int_{\Omega\setminus\bar{\Omega}_\delta} |x|^{-(a+1)p+b} f(\psi)\,w\,dx.
\end{aligned}$$

Hence

$$\int_\Omega |x|^{-ap}\,|\nabla\psi_1|^{p-2}\,|\nabla\psi_1|\cdot\,\nabla w\,dx \leq \int_\Omega |x|^{-(a+1)p+c_1}\,f(\psi_1)\,h(\psi_2)\,w\,dx,$$

i.e., ψ is a sub-solution of (1).
Next it is easy to see that constant function $z = \mu$ is a super-solution of (1) with $z \geq \psi$. Thus, by [13] there exists a positive solution u of (1) such that $\psi \leq u \leq z$. This completes the proof of Theorem 3.2. □

Proof of Theorem 3.3
To prove Theorem 3.3, we will construct a subsolution ψ, a strict supersolution ξ, a strict subsolution w_1, and a supersolution z_1 for (1) such that $\psi \leq \xi \leq z$, $\psi \leq w \leq z$, and $w \not\leq \xi$. Then (1) has at least three distinct solutions $u_i,\ i = 1, 2, 3$, such that $u_1 \in [\psi, \xi], u_2 \in [w, z]$, and

$$u_3 \in [\psi, z] \setminus \big([\psi, \xi] \cup [w, z]\big).$$

We first note that $\psi = 0$ is a solution (hence a subsolution). In the proof of Theorem 3.3 we saw that for λ large, $w = (\frac{\gamma}{2})^{\frac{1}{p-1}}(\frac{p-1}{p})\,\phi_{1,p}^{\frac{p}{p-1}}$, is a positive strict subsolution. And also we know that $z = \mu$ is a super-solution of (1) with $z \geq w$. Now we will show that there is a positive and strict supersolution ξ such that $\xi \leq z$ and $w \not\leq \xi$. From (H2) we can choose $\alpha \in (0, (\frac{\gamma}{2})^{\frac{1}{p-1}}(\frac{p-1}{p}))$ such that for $0 < y < \alpha$,

$$\lambda\,f(y) < \lambda_{1,p}\,y^{p-1}.$$

Let $\xi = \alpha\,\phi_{1,p}$. Then,

$$\begin{aligned}\int_\Omega |x|^{-ap}\,|\nabla\xi_1|^{p-2}\,\nabla\xi_1\,\nabla w\,dx &= \alpha^{p-1}\int_\Omega |x|^{-ap}\,|\nabla\phi_{1,p}|^{p-2}\,\nabla\phi_{1,p}\,\nabla w\,dx\\ &= \lambda_{1,p}\int_\Omega |x|^{-(a+1)p+b}\,|\alpha\,\phi_{1,p}|^{p-2}\,w\,dx\\ &> \lambda\int_\Omega |x|^{-(a+1)p+b}\,f(\alpha\,\phi_{1,p})\,w\,dx\\ &\geq \lambda\int_\Omega |x|^{-(a+1)p+b}\,f(\xi)\,w\,dx.\end{aligned}$$

Thus ξ is a strict supersolution and $w \not\leq \xi$. Hence there exists solutions $u_2 \in [\psi, \xi]$, $u_3 \in [w, z]$, and $u \in [\psi, z] \setminus \big([\psi, \xi] \cup [w, z]\big)$. Thus we have two positive solutions u_2 and u_3. Hence Theorem 3.3 holds. □

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