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CONTACT CONFORMAL CONNECTION ON A GEOMETRY OF HYPERSURFACES WITH CERTAIN CONNECTION IN A QUASI-SASAKIAN MANIFOLD

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Abstract

Goldberg, Rosca introduced the notion of Sasakian manifold and conformal connections and studied its several properties. The purpose of the paper is to relate with the notion of CR-submanifold and the existence of some contact conformal structures on a hypersurface of a semi symmetric non metric connection in a quasi-Sasakian manifold. Next, we study the existence of a Kahler structure on M and the existence of a globally metric frame f -structure in sence of Goldberg S.I., Yano K. [10]. Integrability of distributions on M and geometry of their leaves are also studies.

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1 Indroduction

In [2], A. Bejancu introduced and studied CR-submanifold of a Kahlerian manifold. The notion of semi-invariant submanifold of a Sasakian manifold was introduced and studied by A. Bejancu and N. Papaghiue in [4]. It is proved that in a Kahlerian manifold of CR-submanifolds closely relates a submanifolds. Some properties of existence of the structure vector field are also proved.
Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are respectively given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

The connection ∇ is symmetric if the torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric if there is a Riemannian metric g in

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M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. In ([8], [15]) Friedmann A, and Schouten J.A. introduced the idea of a semi-symmetric linear connection. A linear connection ∇ is said to be semi-symmetric if its torsion tensor T is of the form

$$T(X,Y) = \phi(Y)X - \phi(X)Y,$$

where η is a 1-form.
The paper is organized as follows: In the first section, we recall some results and formulae for the later use. In second section, we prove contact conformal connection on a hypersurface of a semi symmetric non metric connection in a quasi-Sasakian manifold M. In the third section, we prove the existence of a Kahler structure on M and the existence of a globally metric frame f-structure on contact conformal connection in sence of S.I. Goldberg-K. Yano. The third section is concerned with integrability of distributions on contact conformal connection M and geometry of their leaves.

2 Preliminaries

A differentiable manifold of dimension $2n+1$ is called an almost contact metric structure if it admits a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Riemannian metric g which satisfy

$$(a) \quad \phi^2 = -I + \eta \otimes \xi \quad (b) \quad \eta(\xi) = 1 \quad (c) \quad \eta \circ \phi = 0 \quad (d) \quad \phi(\xi) = 0 \qquad (1)$$

$$(e) \quad \eta(X) = g(X,\xi) \qquad (f) \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for any vector field X, Y tangent to $\bar{M}$, where I is the identity on the tangent bundle $\Gamma\bar{M}$ of $\bar{M}$. Throughout the paper, all manifolds and maps are differentiable of class C^∞.We denote by $F(\bar{M})$ the algebra of the differentiable functions on $\bar{M}$ and by $\Gamma(E)$ the $F(\bar{M})$ module of the sections of a vector bundle E over $\bar{M}$.

The almost contact manifold $\bar{M}$ (ϕ,ξ,η) is said to be normal if

$$N_\phi(X,Y) + 2d\eta(X,Y)\xi = 0, \quad \forall X, Y \epsilon \Gamma(T\bar{M})$$

where

$$N_\phi(X,Y) = [\phi X, \phi Y] + \phi^2[X,Y] - \phi[\phi X, \phi Y] + \phi[X, \phi Y], \quad \forall X, Y \epsilon \Gamma(T\bar{M})$$

is the Nijenbus tensor field corresponding the tensor field ϕ.

Or equivalently (cf. [7])

$$(\bar{\nabla}_{\phi X}\phi)Y = \phi(\bar{\nabla}_X \phi)Y - g(\bar{\nabla}_X \xi, Y) \quad \forall X, Y \epsilon \Gamma(T\bar{M})$$

And the fundamental 2-form Φ is defined by

$$\Phi(X,Y) = g(X, \phi Y) \quad \forall X, Y \epsilon \Gamma(T\bar{M}) \qquad (2)$$

The normal almost contact metric manifold $\bar{M}$ is called cosympletic if $d\Phi = d\eta = 0$.
Let $\bar{M}$ be an almost contact metric manifold $\bar{M}$. According to [7] we say that $\bar{M}$ is a quasi-Sasakian manifold if and only if ξ is a Killing vector field and

$$(\bar{\nabla}_X\phi)Y = g(\bar{\nabla}_{\phi X}\xi, Y)\xi - \eta(Y)\bar{\nabla}_{\phi X}\xi \quad \forall X, Y\epsilon\Gamma(T\bar{M}) \tag{3}$$

Next we define a tensor field F of type $(1,1)$ by

$$\bar{\nabla}_X\xi = -FX \quad \forall X\epsilon\Gamma(T\bar{M}) \tag{4}$$

From [7] we recall

Lemma 1. *Let $\bar{M}$ be a quasi-Sasakian manifold. Then we have*

$$(a) \quad (\bar{\nabla}_\xi\phi)X = 0 \quad \forall X\epsilon\Gamma(T\bar{M}) \quad (b) \quad \phi \circ F = F \circ \phi \quad (c) \quad F\xi = 0 \tag{5}$$

$$(d) \quad g(FX, Y) + g(X, FY) = 0 \quad \forall X, Y\epsilon\Gamma(T\bar{M})$$

$$(e) \quad \eta \circ F = 0 \quad (f) \quad (\bar{\nabla}_X F)Y = \bar{R}(\xi, X)Y \quad \forall X, Y\epsilon\Gamma(T\bar{M})$$

The tersor field ϕ defined on $\bar{M}$ an ϕ-structure in sense of K. Yano that is

$$\phi^3 + \phi = 0.$$

A semi symmetric non metric connection ∇ on M is defined by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X \tag{6}$$

such that $(\bar{\nabla}_X g)(Y, Z) = -\eta(Y)g(X, Z) - \eta(Z)g(X, Y)$ for any X and $Y\epsilon TM$.
Using (6) in (3), we have

$$(\bar{\nabla}_X\phi)Y = g(\bar{\nabla}_{\phi X}\xi, Y)\xi - \eta(Y)\bar{\nabla}_{\phi X}\xi - \eta(Y)\phi X \tag{7}$$

$$\bar{\nabla}_X\xi = -FX + X \tag{8}$$

3 Contact conformal connection on a hypersurface of a semi symmetric non metric connection in a quasi-Sasakian manifold M

Let us consider an affine connection satisfies

$$\bar{\nabla}_X\bar{g}(Y, Z) = \nabla_X\{e^{2p}g(Y, Z)\} = e^{2p}p(X)\eta(Y)\eta(Z) \tag{9}$$

where p ia a scalor point function and $\bar{g}(X, Y) = e^{2p}g(X, Y)$ a new metric tensor.
The torsion tensor of the connection $\bar{\nabla}$ is of the form

$$T(X, Y) = -2g(\phi X, Y)U = S(X, Y) - S(Y, X) \tag{10}$$

where U is a vector field. Let

$$\bar{\nabla}_X Y = \nabla_X Y + S(X,Y) \tag{11}$$

where S is a tensor of type $(1,2)$. Using (9), (10), (11), we have

$$\bar{\nabla}_X Y = \nabla_X Y + p(X)\{Y - \eta(Y)\xi\} + p(Y)\{X - \eta(X)\xi\}$$
$$-g(\phi X, \phi Y)P + u(X)\phi Y + u(Y)\phi X - g(\phi X, Y)U \tag{12}$$

where $g(P,X) = p(X), g(QX,P) = p(\phi X) = -q(X), g(Q,X) = q(X)$, $g(U,X) = u(X)$.

$$(\bar{\nabla}_X \phi)(Y) = 0 = (\nabla_X \phi)(Y) + \{X - \eta(X)\xi\}p(\phi Y) - p(Y)\phi X$$
$$+g(\phi X, Y)p + g(\phi X, \phi Y)\phi P + u(\phi Y)\phi X + u(Y)\{X - \eta(X)\xi\}$$
$$-g(\phi X, \phi Y)U + g(\phi X, Y)\phi U = 0 \tag{13}$$

Using (7), the relation becomes

$$g(\bar{\nabla}_{\phi X}\xi, Y)\xi - \eta(Y)\bar{\nabla}_{\phi X}\xi - \eta(Y)\phi X - p(Y)\phi X$$
$$+\{X - \eta(X)\xi\}p(\phi Y) + g(\phi X, Y)p + g(\phi X, \phi Y)\phi P + u(\phi Y)\phi X$$
$$+u(Y)\{X - \eta(X)\xi\} - g(\phi X, \phi Y)U + g(\phi X, Y)\phi U = 0$$

contracting with respect to X,

$$2mp(\phi Y) + 2p(\phi Y) + 2mu(Y) - 2u(Y) + 2\eta(U)\eta(Y) = 0$$
$$2(m-1)p(\phi Y) + 2(m-1)u(Y) + 2\eta(Y)\eta(U) = 0 \tag{14}$$

If we put $\eta(U) = -1 = u(\xi)$ then $U = Q + m'\xi$ where $m' = 1/(m-1)$. Thus (12) takes the form

$$\bar{\nabla}_X Y = \nabla_X Y + \{Y - \eta(Y)\xi\}p(X) + \{X - \eta(X)\xi\}p(Y)$$
$$-g(\phi X, \phi Y)P + \{q(X) + m'\eta(X)\}\phi Y + \{q(Y) + m'\eta(Y)\}\phi X$$
$$-g(\phi X, Y)(Q + m'\xi) \tag{15}$$

Then

$$\bar{\nabla}_X \xi = 0 = \nabla_X \xi + \{X - \eta(X)\xi\}p(\xi) + m'\phi X$$

Using (8) in this equation, we have

$$-FX + X + \nabla_X \xi + \{X - \eta(X)\xi\}p(\xi) + m'\phi X = 0$$

which implies that

$$FX = X + \{X - \eta(X)\xi\}p(\xi) + m'\phi X$$

Proposition 1. *On a hypersurface of a semi symmetric non metric connection M in a quasi- Sasakian manifold $\bar{M}$ the affine connection $\bar{\nabla}$ which satisfies (10), is given by (15) with the conditions $\eta(U) = -1 = u(\xi), FX = X+\{X-\eta(X)\xi\}p(\xi)+ m'\phi X$ Using (15) in (7), we have*

$$(\bar{\nabla}_X\phi)Y = g(\bar{\nabla}_{\phi X}\xi, Y)\xi - \eta(Y)\bar{\nabla}_{\phi X}\xi - \eta(Y)\phi X$$

$$+g(\phi X, Y)P + \{q(\phi Y) - p(Y)\}\phi X - g(\phi X, \phi Y) + 2m'\eta(Y)X \tag{16}$$

and

$$\bar{\nabla}_X\xi = -FX + X + \{X - \eta(X)\xi\}p(\xi) + m'\phi X \tag{17}$$

Let M be a contact conformal connection on a hypersurface of a semi symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$ and denote by N the unit vector field normal to M. Denote by the same symbol g the induced tensor metric on M, by ∇ the induced Levi-Civita connection on M and by $TM^{\perp}$ the normal vector bundle to M.The Gauss and Weingarten formulae of contact conformal connection on a hypersurfaces of a semi symmetric non metric connections are

$$(a) \quad \bar{\nabla}_XY = \nabla_XY + B(X,Y)N, \quad (b) \quad \bar{\nabla}_XN = -AX \tag{18}$$

where A is the shape operator with respect to the section N . It is known that

$$B(X,Y) = g(AX,Y) \quad \forall X, Y \epsilon \Gamma(TM) \tag{19}$$

Because the position of the structure vector field with respect to M is very important we prove the following results.

Theorem 1. *Let M be a contact conformal connection on a hypersurface of a semi symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. If the structure vector field ξ is normal to M then $\bar{M}$ is cosympletic manifold and M is totally geodesic immersed in $\bar{M}$.*

Proof. Because $\bar{M}$ is quasi-Sasakian manifold , then it is normal and $d\Phi = 0$ ([4]). By direct calculation using (18) (b), we infer

$$d\eta(X,Y) = \frac{1}{2}\{(\bar{\nabla}_X\eta)(Y) - (\bar{\nabla}_Y\eta)(X)\} = \frac{1}{2}\{g(\bar{\nabla}_X\xi, Y) - g(\bar{\nabla}_Y\xi, X)\}$$

$$2d\eta(X,Y) = g(AY,X) - g(AX,Y) = 0 \quad \forall X, Y \epsilon \Gamma(T\bar{M}) \tag{20}$$

From (18) (b) and (20) we deduce

$$0 = d\eta(X,Y) = \frac{1}{2}\{(\bar{\nabla}_X\eta)(Y) - (\bar{\nabla}_Y\eta)(X)\} = \frac{1}{2}\{g(\bar{\nabla}_X\xi, Y) - g(\bar{\nabla}_Y\xi, X)\} \tag{21}$$

$$= g(Y, \bar{\nabla}_X\xi) = -g(AX,Y) = 0 \quad \forall X, Y \epsilon \Gamma(T\bar{M})$$

which proves that M is totally geodesic. From (21) we obtain $\bar{\nabla}_X\xi = 0 \quad \forall X \epsilon \Gamma(T\bar{M})$ By using (17),(5)(b)and (1) (d) from the above relation we state

$$-\phi(\bar{\nabla}_{\phi X}\xi) + X + 2m'\phi X = \bar{\nabla}_X\xi \quad \forall X \epsilon \Gamma(T\bar{M}) \tag{22}$$

because $\phi X \epsilon \Gamma(T\bar{M}) \quad \forall X \epsilon \Gamma(T\bar{M})$. Using (22) and the fact that ξ is a not Killing vector field, we deduce $d\eta \neq 0$.

Next we consider only the hypersurface which are tangent to ξ. Denote by $U = \phi N$ and from (1) (f), we deduce $g(U,U) = 1$. Moreover, it is easy to see that $U \epsilon \Gamma(TM)$ .Denote by $D^{\perp} = Span(U)$ the 1-dimensional distribution generated by U, and by D the orthogonal complement of $D^{\perp} \oplus \{\xi\}$ in TM. It is easy to see that

$$\phi D = D, \quad D^{\perp} \subseteq TM^{\perp}; \quad TM = D \oplus D^{\perp} \oplus \{\xi\}. \tag{23}$$

where $\oplus$ denote the orthogonal direct sum. According with [1] from (16) we deduce that M is a CR-submanifold of $\bar{M}$. $\square$

Definition 1. *A CR-submanifold M of a quasi-Sasakian manifold $\bar{M}$ is called CR-product if both distributions $D \oplus \{\xi\}$ and $D^{\perp}$ are integrable and their leaves are totally geodesic submanifold of M.*

Denote by P the projection morphism of TM to D and using the decomposion in (21) we deduce

$$X = PX + a(X)U + \eta(X)\xi \quad \forall X \epsilon \Gamma(T\bar{M}) \tag{24}$$

$$\phi X = \phi PX + a(X)\phi U + \eta(\phi X)\xi \qquad \therefore \quad \phi X = \phi PX - a(X)\phi U$$

Since $\quad U = \phi N, \quad \phi U = \phi^2 N = -N + \eta(N)\xi = -N + g(N,\xi)\xi = -N$ where a is a 1-form on M defined by $a(X) = g(X,U), \quad X \epsilon \Gamma(TM)$. From (23) using (1) (a) we infer

$$\phi X = tX - a(X)N \quad \forall X \epsilon \Gamma(TM) \tag{25}$$

where t is a tensor field defined by $tX = fPX, \quad X \epsilon \Gamma(TM)$
It is easy to see that

$$(a) \quad t\xi = 0 \qquad (b) \quad tU = 0 \tag{26}$$

4 Induced structures on contact conformal connection on a hypersurface of a semi symmetric non metric connection in a quasi-Sasakian manifold

The purpose of this section is to study the existence of some induced structure on contact conformal connection on a hypersurface of a semi symmetric non metric connection in a quasi-Sasakian manifold. Let M be a contact conformal connection on a hypersurface of a semi symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. From (1) (a), (25) and (26) we obtain $t^3 + t = 0$, that is the tensor field t defines an f-structure on M in sense of Yano K. [16]. Moreover, from (1) (a), (25), (26) we infer

$$t^2 X = -X + a(X)U + \eta(X)\xi \quad \forall X \epsilon \Gamma(TM) \tag{27}$$

Lemma 2. *The following statement holds: On a contact conformal connection on a hypersurface of a semi symmetric non metric connection M in a quasi-Sasakian manifold $\bar{M}$ the tensor field t satisfies*

$$(a) \quad g(tX, tY) = g(X, Y) - \eta(X)\eta(Y) - a(X)a(Y), \tag{28}$$

$$(b) \quad g(tX, Y) + g(X, tY) = 0 \quad \forall X, Y \epsilon \Gamma(TM).$$

Proof. From (1) (f), and (25) we deduce

$$g(X, Y) - \eta(X)\eta(Y) = g(\phi X, \phi Y) = g(tX - a(X)N, tY - a(Y)N)$$

$$= g(tX, tY) - a(Y)g(tX, N) - a(X)g(N, tY)$$

$$+ a(X)a(Y)g(N, N) \quad \forall X, Y \epsilon \Gamma(TM)$$

$$= g(tX, tY) + a(X)a(Y)$$

$$g(tX, tY) = g(X, Y) - \eta(X)\eta(Y) - a(X)a(Y)$$

$$(b) \quad g(tX, Y) + g(X, tY) = g(\phi X + a(X)N, Y) + g(X, \phi Y + a(Y)N)$$

$$= g(\phi X, Y) + a(X)g(N, Y) + g(X, \phi Y) + a(Y)g(X, N)$$

$$= g(\phi X, Y) + g(X, \phi Y) = 0.$$

□

Lemma 3. *Let M be a contact conformal connection on a hypersurface of a semi symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. Then we have*

$$(a) \quad FU = \phi A\xi + U + m'\phi U \quad (b) \quad FN = A\xi + N + m'U \quad (c) \quad [U, \xi] \neq 0 \tag{29}$$

Proof. We take $X = U$, and $Y = \xi$ in (16)

$$\phi(\bar{\nabla}_U \xi) = -\bar{\nabla}_N \xi - p(\xi)N - 2m'U$$

Then using (1) (a), (17), (18)(b), we deduce the assertion (a). The assertion (b) follows from (1) (a), (5) (b) and (18) (b) we derive

$$\bar{\nabla}_\xi U = (\bar{\nabla}_\xi \phi)N + \phi\bar{\nabla}_\xi N = -\phi A\xi = -FU - U - m'\phi U = \bar{\nabla}_U \xi,$$

$$[U, \xi] = \bar{\nabla}_U \xi - \bar{\nabla}_\xi U = \bar{\nabla}_U \xi - \bar{\nabla}_U \xi \neq 0$$

which prove assertion (c).By using the decomposition $T\bar{M} = TM \oplus TM^\perp$, we deduce

$$FX = \alpha X - \eta(AX)N, \quad \forall X \epsilon \Gamma(T\bar{M}) \tag{30}$$

where α is a tensor field of type (1, 1) on M, since $g(FX, N) = -g(X, FN) = -g(X, A\xi + N + m'U) = -\eta(AX) - m'a(X) \quad \forall X \epsilon \Gamma(T\bar{M})$. By using (16), (17), (18), (25) and (27), we obtain

□

Theorem 2. *Let M be a contact conformal connection on a hypersurface of a semi symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. Then the covariant derivative of a tensors t, a, η and α are given by*

$$(a) \quad (\nabla_X t)Y = g(FX, \phi Y)\xi + g(\phi X, Y)\xi + g(\phi X, Y)p(\xi)\xi - m'g(X,Y)\xi \quad (31)$$

$$+m'\eta(X)\eta(Y)\xi + \eta(Y)[\alpha tX - \eta(AX)U - m'a(X)U]$$

$$-\eta(Y)\phi X p(\xi) - 2\eta(Y)\phi X + 3m'\eta(Y)X - m'\eta(X)\eta(Y)\xi$$

$$+g(\phi X, Y)P + [q(\phi Y) - p(Y)]\phi X$$

$$-g(\phi X, \phi Y)Q - a(Y)AX + B(X,Y)U$$

$$(b) \quad (\nabla_X a)Y = B(X, tY) + a(X)\eta(Y) + \eta(Y)\eta(AtX)$$

$$(c) \ (\nabla_X \eta)(Y) = 0$$

$$(d) \quad (\nabla_X \alpha)Y = R(\xi, X)Y + B(X,Y)A\xi - \eta(AY)AX - m'B(X,Y)U \quad \forall X, Y \epsilon \Gamma(TM)$$

respectively, where R is the curvature tensor field of M.

From (16), (17), (26) and (31) (a) we get

Proposition 2. *On a contact conformal connection on a hypersurface of a semi symmetric non metric connection M in a quasi-Sasakian manifold $\bar{M}$, we have*

$$(a) \quad \nabla_X U = -tAX + \eta(AtX)\xi + m'a(tX)\xi + 2a(X)\xi + a(X)p(\xi)\xi + a(X)P \quad (32)$$

$$(b) \quad B(X,U) = a(AX) \quad \forall X \epsilon \Gamma(TM)$$

Next we state

Theorem 3. *Let M be a contact conformal connection on a hypersurface of a semi symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. The tensor field t is a parallel with respect to the Levi Civita connection ∇ on M iff*

$$(a) \quad AX = \eta(AX)\xi - p(U)\phi X + a(AX)U \quad (33)$$

$$(b) \quad FX = \phi X - \eta(AX)N + a(X)N \quad \forall X \epsilon \Gamma(TM)$$

$$if \quad g(X,Z)P = g(X,Z)p(U)U - g(\phi X, Z)Q + 2g(Q,Z)\phi X$$

Proof. Suppose that the tensor field t is parallel with respect to ∇, that is $\nabla t = 0$. By using (31) (a), we deduce

$$g(FX, \phi Y)\xi + g(\phi X, Y)\xi + g(\phi X, Y)p(\xi)\xi - m'g(X,Y)\xi \quad (34)$$

$$+m'\eta(X)\eta(Y)\xi + \eta(Y)[\alpha tX - \eta(AX)U - m'a(X)U]$$

$$-\eta(Y)\phi X p(\xi) - 2\eta(Y)\phi X + 3m'\eta(Y)X - m'\eta(X)\eta(Y)\xi$$

$$+g(\phi X, Y)P + [q(\phi Y) - p(Y)]\phi X$$
$$-g(\phi X, \phi Y)Q - a(Y)AX + B(X, Y)U = 0$$

Take $Y = U$ in (34) and using (18) (b), (19), (32) (b) we infer

$$\eta(U)[\alpha tX + X - \eta(AX)U] - a(U)AX + g(FX, fU)\xi - g(X, U)\xi + B(X, U)U = 0$$
$$\eta(U) = 0, \quad a(U) = 1, \quad g(X, N) = 0$$
$$-AX + g(FX, fU)\xi - g(X, U)\xi + a(AX)U = 0$$
$$AX = g(FX, -N)\xi - m'a(X)\xi - p(U)\phi X + a(AX)U$$
$$= g(X, FN)\xi - m'a(X)\xi - p(U)\phi X + a(AX)U$$
$$AX = \eta(AX)\xi - p(U)\phi X + a(AX)U$$

And the assertion (32) (a) is proved. Next let $Y = \phi Z, \quad Z\epsilon\Gamma(D)$ in (34) and using (1) (f), (18) (b), (29) (a) (b), (33) (a), we deduce

$$g(X, FZ) = 0 \quad if \quad g(X, Z)P = g(X, Z)p(U)U - g(\phi X, Z)Q + 2g(Q, Z)\phi X$$
$$\Rightarrow FX = \phi X - \eta(AX)N + a(X)N \quad \forall X\epsilon\Gamma(TM)$$

The proof is complete. □

Proposition 3. *Let M be a contact conformal connection on a hypersurface of a semi symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. Then we have the assertions*

$$(a) \quad (\nabla_X a)Y = 0 \Leftrightarrow \nabla_X U = 0$$
$$(b) \quad (\nabla_X \eta)Y = 0 \Leftrightarrow \nabla_X \xi = 0 \qquad \forall X, Y\epsilon\Gamma(TM)$$

Proof. Let $\forall X, Y\epsilon\Gamma(TM)$ and using (19), (28) (b), (31) (b) and (32) (a) we obtain

$$g(\nabla_X U, Y) = g(-tAX + \eta(AtX)\xi, Y)$$
$$= g(-tAX, Y) + \eta(AtX)g(\xi, Y)$$
$$= g(AX, tY) + \eta(AtX)\eta(Y)$$
$$= (\nabla_X a)Y$$

which proves assertion (a). The assertion (b) is consequence of the fact that ξ is not a killing vector field.
According to Theorem 2 in [17], the tensor field

$$\bar{\phi} = t + \eta \otimes U - a \otimes \xi,$$

Defines an almost complex structure on M. Moreover, from Proposition 2 we deduce

□

Theorem 4. *Let M be a contact conformal connection on a hypersurface of a semi symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. If the tensor fields t, a, η are parallel with respect to the connection ∇, then $\bar{f}$ defines a Kahler structure on M.*

5 Integrability of distributions on a contact conformal connection on a hypersurface of a semi symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$

In this section we established conditions for the Integrability of all distributions on a contact conformal connection on a hypersurface of a semi symmetric non metric connection M in a quasi-Sasakian manifold $\bar{M}$. From Lemma 3 we obtain

Corollary 1. *On a contact conformal connection on a hypersurface of a semi symmetric non metric connection M of a quasi-Sasakian manifold $\bar{M}$ there exists a 2-dimensional foliation determined by the integral distribution $D^{\perp} \oplus (\xi)$*

Theorem 5. *Let M be a contact conformal connection on a hypersurface of a semi symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. Then we have*

(a) A leaf of $D^{\perp} \oplus \{\xi\}$ is totally geodesic submanifold of M if and only if

$$(1) \quad AU = a(AU)U + \eta(AU)\xi - p(U)\phi U \quad and \quad (2) \quad FN = a(FN)U. \tag{35}$$

(b) A leaf of $D^{\perp} \oplus \{\xi\}$ is totally geodesic submanifold of $\bar{M}$ if and only if

$$(1) \quad AU = 0 \quad and \quad (2) \quad a(FX) = a(FN) - m' = 0, \forall X \epsilon \Gamma(D)$$

Proof. (a) Let M^* be a leaf of integrable distribution $D^{\perp} \oplus \{\xi\}$ and h^* be the second fundamental form of the immersion $M^* \to M$. By using (1) (f), and (18) (b) we get

$$\begin{aligned} g(h^*(U,U),X) = g(\bar{\nabla}_U U, X) &= -g(N, (\bar{\nabla}_U \phi)X - g(\bar{\nabla}_U N, \phi X) \\ &= 0 - g(-AU, \phi X) = g(AU, \phi X) \\ &= g(AU, \phi X) \quad \forall X \epsilon \Gamma(TM), \end{aligned} \tag{36}$$

and

$$\begin{aligned} g(h^*(U,\xi),X) &= g(\bar{\nabla}_U \xi, X) \\ &= g(-FU + U + Up(\xi) - \eta(U)p(\xi)\xi + m'\phi U, X) \\ &= g(FN, \phi X) \quad \forall X \epsilon \Gamma(TM), \end{aligned} \tag{37}$$

Because $g(FU, N) = 0$ and $f\xi = 0$ the assertion (a) follows from (36) and (37).

(b) Let h_1 be the second fundamental form of the immersion $M^* \to M$. It is easy to see that

$$h_1(X,Y) = h^*(X,Y) + B(X,Y)N, \quad \forall X, Y \epsilon \Gamma(D^{\perp} \oplus (\xi)). \tag{38}$$

From (17) and (19) we deduce

$$(h_1(U,U), N) = g(\bar{\nabla}_U U, N) = a(AU), \tag{39}$$

$$g(h_1(U,\xi), N) = g(\bar{\nabla}_U \xi, N) = a(FN) - m', \tag{40}$$

The assertion (b) follows from (37)-(40). □

Theorem 6. *Let M be a contact conformal connection on a hypersurface of a semi symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. Then*
(a) the distribution $D \oplus \{\xi\}$ is integrable iff

$$g(A\phi X + \phi AX, Y) = 0, \quad \forall X, Y \epsilon \Gamma(D), \tag{41}$$

(b) the distribution D is integrable iff (41) holds and

$$FX = X + \{X - \eta(X)\}p(\xi)\xi - m'\phi X + \eta(X)P - \eta(X)p(U)U,$$

$$(equivalent \quad with \quad FD \perp D) \quad \forall X \epsilon \Gamma(D)$$

(c) The distribution $D \oplus D^{\perp}$ is integrable iff $FX = 0, \quad \forall X \epsilon \Gamma(D)$.

Proof. Let $X, Y \epsilon \Gamma(D)$. Since ∇ is a torsion free and ξ is a Killing vector field, we infer

$$g([X,\xi],U) = g(\bar{\nabla}_X \xi, U) - g(\bar{\nabla}_\xi X, U) = g(\nabla_X \xi, U) + g(\nabla_U \xi, X) = 0 \quad \forall X \epsilon \Gamma(D) \tag{42}$$

Using (1) (a), (18) (a) we deduce

$$g([X,Y],U) = g(\bar{\nabla}_X Y - \bar{\nabla}_Y X, U) = g(\bar{\nabla}_X Y - \bar{\nabla}_Y X, \phi N)$$

$$= g(\bar{\nabla}_Y \phi X - \bar{\nabla}_X \phi Y, N) = -g(A\phi X + \phi AX, Y) \quad \forall X, Y \epsilon \Gamma(D) \tag{43}$$

Next by using (5) (d) (17) and the fact that ∇ is a metric connection we get

$$g([X,Y],\xi) = g(\bar{\nabla}_X Y, \xi) - g(\bar{\nabla}_Y X, \xi) \tag{44}$$

$$= 2g(FX - X - \{X - \eta(X)\}p(\xi)\xi - m'\phi X, Y) \quad \forall X, Y \epsilon \Gamma(D)$$

The assertion (a) follows from (42), (43) and assertion (b) follows from (42)-(44). Using 7(d) and (17) we obtain

$$g([X,U],\xi) = g(\bar{\nabla}_X U, \xi) - g(\bar{\nabla}_U X, \xi) \quad \forall X \epsilon \Gamma(D) \tag{45}$$

$$= 2g(FX, U) - 2g(X, U) - 2g(X, U)p(\xi) \quad \forall X \epsilon \Gamma(D)$$

Taking on account of

$$g(FX, N) = g(F\phi X, \phi N) = g(F\phi X, U) \quad \forall X \epsilon \Gamma(D) \tag{46}$$

The assertion (c) follows from (44) and (45). □

Theorem 7. *Let M be a contact conformal connection on a hypersurface of a semi symmetric non metric connection in a quasi-Sasakian manifold $\bar{M}$. Then we have*
(a) the distribution D is integrable and its leaves are totally geodesic immersed in M if and only if

$$FD \perp D \quad and \quad AX = a(AX)U - p(U)\phi X + \eta(AX)\xi, \quad \forall X \epsilon \Gamma(D) \tag{47}$$

(b) the distribution $D \oplus \{\xi\}$ is integrable and its leaves are totally geodesic immersed in if and only if

$$AX = a(AX)U - p(U)\phi X, \quad X\epsilon\Gamma(D) \quad and \quad FU = 0 \tag{48}$$

(c) the distribution $D \oplus D^{\perp}$ is integrable and its leaves are totally geodesic immersed in M if and only if .

$$FX = 0, \quad and \quad g(X,Y) = g(X,Y)p(\xi) - \eta(X)\eta(Y)p(\xi) + m'g(\phi X, Y) \quad X\epsilon\Gamma(D).$$

Proof. Let M_1^* be a leaf of integrable distribution D and h_1^* the second fundamental form of immersion $M_1^* \to M$. Then by direct calculation we infer

$$g(h_1^*(X,Y),U) = g(\bar{\nabla}_X Y, U) = -g(Y, \nabla_X U) = -g(AX, tY), \tag{49}$$

and

$$g(h_1^*(X,Y),\xi) = g(\bar{\nabla}_X Y, \xi) \tag{50}$$

$$= g(FX,Y) - g(X,Y) - g(X,Y)p(\xi) + \eta(X)\eta(Y)p(\xi) - m'g(\phi X, Y) \quad \forall X, Y\epsilon\Gamma(D)$$

Now suppose M_1^* is a totally submanifold of M. Then (47) follows from (49) and (50). Conversely suppose that (47) is true. Then using the assertion (b) in Theorem 6 it is easy to see that the distribution D is integrable. Next the proof follows by using (49) and (50). Next, suppose that the distribution $D \oplus (\xi)$ is integrable and its leaves are totally geodesic submanifolds of M. Let M_1 be a leaf of $D \oplus \{\xi\}$ and h_1 the second fundamental form of immersion $M_1 \to M$. By direct calculations, using (17), (18) (b), (28) (b) and (32) (c), we deduce

$$g(h_1(X,Y),U) = g(\bar{\nabla}_X Y, U) = -g(AX, tY), \quad \forall X, Y\epsilon\Gamma(D) \tag{51}$$

and

$$g(h_1(X,\xi),U) = g(\bar{\nabla}_X \xi, U)$$

$$= -g(FX,U) + a(X) + a(X)p(\xi), \quad \forall X\epsilon\Gamma(D) \tag{52}$$

Then the assertion (b) follows from (46), (51), (52) and the assertion (a) of Theorem 6 . Next let $\bar{M_1}$ a leaf of the integrable distribution $D \oplus D^{\perp}$ and $\bar{h_1}$ the second fundamental form of the immersion $M_1 \to M$. By direct calculation we get

$$g(\bar{h_1}(X,Y),\xi) = g(FX,Y) - g(X,Y) - g(X,Y)p(\xi) \tag{53}$$

$$+\eta(X)\eta(Y)p(\xi) - m'g(\phi X, Y) \quad \forall X\epsilon\Gamma(D), Y\epsilon\Gamma(D \oplus D^{\perp}).$$

The assertion (c) follows from (5) (c), (46) and (53).

□

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