Bulletin of the *Transilvania* University of Braşov • Vol 10(59), No. 1 - 2017
Series III: Mathematics, Informatics, Physics, 91-102

SOME APPLICATIONS OF CERTAIN NEW TYPES OF SETS IN GTS VIA HEREDITARY CLASSES

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Abstract

In this paper we introduce certain new types of sets in a generalized topological space via hereditary classes and investigate their several properties. In the process we achieve some nice applications of these newly defined sets to study a few lower separation properties viz μ^*-R_0, μ^*-R_1 and μ^*-$T_{\frac{1}{2}}$ spaces.

2000 *Mathematics Subject Classification:* 54A05, 54A10, 54D10.
Key words: $\wedge_\mu^*$-set, g.$\wedge_\mu^*$-set, μ^*-g-closed sets, μ^*-R_0, μ^*-$T_{\frac{1}{2}}$ space.

1 Introduction

The idea of generalized topology [2] was introduced by A. Császár in 2002 and since then there has been a growing trend to study this concept in different perspectives. In 2007, A. Császár [5] introduced the notion of hereditary class in generalized topological space and subsequently many papers (e.g. see [7, 8, 10, 11, 12, 13, 15]) appeared in the recent literature. In this article, there is another attempt to introduce and investigate some new kind of sets in a generalized topological space with a hereditary class. Also, we give some applications of these sets by characterizing certain separation axioms viz. μ^*-R_0, μ^*-R_1 and μ^*-$T_{\frac{1}{2}}$.

A collection μ of subsets of a set X is called a generalized topology [2] on X if $\phi \in \mu$ and μ is closed under arbitrary union; the pair (X, μ) is called a generalized topological space (GTS, in short). The members of μ are called μ-open sets and their complements are called μ-closed sets in (X, μ). According to [1], for $A \subseteq X$, the union of all μ-open subsets of X, each contained in A is called μ-interior of A and is denoted by $i_\mu(A)$; the map $i_\mu : \exp X \to \exp X$ is monotone (i.e., $A \subseteq B \Rightarrow i_\mu(A) \subseteq i_\mu(B)$), restricting (i.e., $i_\mu(A) \subseteq A$ for $A \subseteq X$) and

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idempotent (i.e., $i_\mu(i_\mu(A)) = i_\mu(A)$), where $\exp X$ denotes the set of all subsets of X. The generalized closure of a subset A of X, denoted by $c_\mu(A)$, is the intersection of all μ-closed subsets of X each containing A; the map $c_\mu : \exp X \to \exp X$ is monotone, idempotent and enlarging (i.e., $A \subseteq c_\mu(A)$ for $A \subseteq X$). Moreover $c_\mu(X \setminus A) = X \setminus i_\mu(A)$ [4].

A family $\mathcal{H}$ of subsets of X is said to be a hereditary class [5] on X if $A \in \mathcal{H}$ and $B \subseteq A$ implies $B \in \mathcal{H}$. For a GTS (X, μ) with a hereditary class $\mathcal{H}$, a subset $A^*(\mu, \mathcal{H})$ or simply A^* of X is defined by $A^* = \{x \in X : U \cap A \notin \mathcal{H}$ for every $U \in \mu$ containing $x\}$ [5], for each $A \subseteq X$. In [5], it was also shown that for $A \subseteq X$ if $c^*_\mu(A) = A \cup A^*$, then $\mu^*(\mu, \mathcal{H})$ (or simply μ^*)$= \{A \subseteq X : c^*_\mu(X \setminus A) = X \setminus A\}$ is a generalized topology on X with $\mu \subseteq \mu^*$. Moreover, the map c^*_μ is monotone, enlarging and idempotent. The elements of μ^* are called μ^*-open sets. The complements of μ^*-open sets are called μ^*-closed sets and equivalently A is a μ^*-closed set iff $A^* \subseteq A$ [5].

In Section 2 of this paper, we introduce two types of sets viz. $\wedge^*_\mu$-set and $\vee^*_\mu$-set in a GTS with a hereditary class and study some of their properties. In [9], we investigated μ^*-R_0, μ^*-R_1 and μ^*-$T_{\frac{1}{2}}$ spaces in a GTS with a hereditary class. In the last section of this article, we investigate some lower separation axioms viz. μ^*-R_0, μ^*-R_1 and μ^*-$T_{\frac{1}{2}}$ with the help of different types of sets introduced in Sections 2 and 3.

Definition 1. [6] *Let (X, μ) be a GTS and $A \subseteq X$. The subsets $\wedge_\mu(A)$ and $\vee_\mu(A)$ are defined by*

$$\wedge_\mu(A) = \begin{cases} \cap\{U : A \subseteq U, U \text{ is } \mu\text{-open sets}\}, & \text{if } \exists\, U \in \mu \text{ such that } A \subseteq U; \\ X, & \text{otherwise} \end{cases}$$
$$\vee_\mu(A) = \begin{cases} \cup\{F : F \subseteq A, F \text{ is } \mu\text{-closed}\}, & \text{if } \exists\, \mu\text{-closed } F \text{ such that } F \subseteq A; \\ \phi, & \text{otherwise} \end{cases}$$

Definition 2. [6] *A subset A of a GTS (X, μ) is called a $\wedge_\mu$-set ($\vee_\mu$-set) if $A = \wedge_\mu(A)$ (respectively, if $A = \vee_\mu(A)$).*

Theorem 1. *[6] Let (X, μ) be a GTS and A be any subset of X. Then $\wedge_\mu(A) = \{x \in X : c_\mu(\{x\}) \cap A \neq \phi\}$.*

2 $\wedge^*_\mu$ and $\vee^*_\mu$-sets

The intent of this section is to introduce two types of sets viz. $\wedge^*_\mu$-sets and $\vee^*_\mu$-sets, and characterize μ^*-g-closed sets with the help of these types of sets. Before we begin this section, we observe that $\mathcal{M}_\mu = \cup\{M \mid M \in \mu\}$ is the largest μ-open set of X, and certainly if B is a μ-closed set then $X \setminus \mathcal{M}_\mu \subseteq B \subseteq X$.

Definition 3. *Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$ and $A \subseteq X$. We define*

$$\wedge_\mu^*(A) = \begin{cases} \cap\{U : A \subseteq U, U \text{ is } \mu^*\text{-open sets }\}, & \text{for } A \subseteq \mathcal{M}_\mu; \\ X, & \text{otherwise} \end{cases}$$
$$\vee_\mu^*(A) = \begin{cases} \cup\{F : F \subseteq A, F \text{ is } \mu^*\text{-closed sets }\}, & \text{for } X\backslash\mathcal{M}_\mu \subseteq A \subseteq X; \\ \phi, & \text{otherwise} \end{cases}$$

Theorem 2. *Let (X,μ) be a GTS with a hereditary class $\mathcal{H}$. Then $\wedge_\mu^*(A) = \{x \in X : c_\mu^*(\{x\}) \cap A \neq \phi\}$ for each $A \subseteq X$.*

Proof. Let $x \in \wedge_\mu^*(A)$ be such that $c_\mu^*(\{x\}) \cap A = \phi$. Then $A \subseteq X\backslash c_\mu^*(\{x\})$, where $X\backslash c_\mu^*(\{x\})$ is a μ^*-open set not containing x and hence $x \notin \wedge_\mu^*(A)$, a contradiction. Conversely, let $c_\mu^*(\{x\}) \cap A \neq \phi$. If $x \notin \wedge_\mu^*(A)$, then by definition of $\wedge_\mu^*(A)$, there exists a μ^*-open set U with $x \notin U$ such that $A \subseteq U$. Let $y \in c_\mu^*(\{x\}) \cap A$. Then $y \in c_\mu^*(\{x\})$ and $y \in U$. Thus $x \in U$, a contradiction. □

Theorem 3. *For subsets $A, B, A_\alpha (\alpha \in \Delta)$ of a GTS (X,μ) with a hereditary class $\mathcal{H}$, the following properties hold:*
(i) $A \subseteq \wedge_\mu^*(A)$.
(ii) If A is μ^*-open, then $A = \wedge_\mu^*(A)$.
(iii) If $A \subseteq B$, then $\wedge_\mu^*(A) \subseteq \wedge_\mu^*(B)$.
(iv) $\wedge_\mu^*(\wedge_\mu^*(A)) = \wedge_\mu^*(A)$.
(v) $\wedge_\mu^*(\cap\{A_\alpha : \alpha \in \Delta\}) \subseteq \cap\{\wedge_\mu^*(A_\alpha) : \alpha \in \Delta\}$.
(vi) $\wedge_\mu^*(\cup\{A_\alpha : \alpha \in \Delta\}) = \cup\{\wedge_\mu^*(A_\alpha) : \alpha \in \Delta\}$.

Proof. (i) and (ii) follow from the definition.
(iii) Let $A \subseteq B$. If $x \notin \wedge_\mu^*(B)$, then there exists a μ^*-open set U such that $B \subseteq U$ and $x \notin U$. Since $A \subseteq B \subseteq U$, then from the definition of $\wedge_\mu^*(A)$, we have $x \notin \wedge_\mu^*(A)$ and hence $\wedge_\mu^*(A) \subseteq \wedge_\mu^*(B)$.
(iv) By (i), we have $\wedge_\mu^*(\wedge_\mu^*(A)) \supseteq \wedge_\mu^*(A)$. Suppose that $x \notin \wedge_\mu^*(A)$. Then there exists a μ^*-open set U such that $A \subseteq U$ and $x \notin U$. Since $A \subseteq \wedge_\mu^*(A) \subseteq U$, from the definition of $\wedge_\mu^*(\wedge_\mu^*(A))$, we have $\wedge_\mu^*(\wedge_\mu^*(A)) \subseteq U$, and hence $x \notin \wedge_\mu^*(\wedge_\mu^*(A))$, that is $\wedge_\mu^*(\wedge_\mu^*(A)) \subseteq \wedge_\mu^*(A)$. Thus $\wedge_\mu^*(\wedge_\mu^*(A)) = \wedge_\mu^*(A)$.
(v) It follows from (i) and (iii).
(vi) We have $\wedge_\mu^*(A_\alpha) \subseteq \wedge_\mu^*(\bigcup_{\alpha\in\Delta} A_\alpha)$ and hence $\bigcup_{\alpha\in\Delta} \wedge_\mu^*(A_\alpha) \subseteq \wedge_\mu^*(\bigcup_{\alpha\in\Delta} A_\alpha)$. Next, let $x \notin \bigcup_{\alpha\in\Delta} \wedge_\mu^*(A_\alpha)$. Then $x \notin \wedge_\mu^*(A_\alpha)$ for each $\alpha \in \Delta$ and so there exists a μ^*-open set U_α such that $A_\alpha \subseteq U_\alpha$ and $x \notin U_\alpha$. Let $U = \cup U_\alpha$. Then $U \in \mu^*$ such that $\cup A_\alpha \subseteq U$ and $x \notin U$, and hence $x \notin \wedge_\mu^*(\cup A_\alpha)$. □

Remark 1. *In (v) of Theorem 3, the equality does not hold in general, even if Δ is a finite index set. See the following example.*

Example 1. *Let $X = \{a,b,c,d\}$. Consider a GT μ on X, where $\mu = \{\phi, \{a\}, \{d\}, \{a,d\}, \{a,c\}, \{a,c,d\}\}$ and a hereditary class $\mathcal{H} = \{\phi, \{b\}, \{c\}\}$. Then $\mu^* = \{\phi, \{a\}, \{d\}, \{a,d\}, \{a,c\}, \{a,c,d\}\}$. Let us consider $A = \{a,d\}$ and $B = \{c,d\}$. Then $\wedge_\mu^*(A) = \{a,d\}$, $\wedge_\mu^*(B) = \{a,c,d\}$ and $\wedge_\mu^*(A \cap B) = \{d\}$. Thus $\wedge_\mu^*(A \cap B) \neq \wedge_\mu^*(A) \cap \wedge_\mu^*(B)$.*

Lemma 1. *Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$. Then $\wedge_\mu^*(X \backslash A) = X \backslash \vee_\mu^*(A)$ for every $A \subseteq X$.*

Proof. We have $X \backslash \vee_\mu^*(A) = X \backslash (\cup \{F : F \subseteq A$ and F is a μ^*-closed set$\}) = \cap\{X \backslash F : X \backslash A \subseteq X \backslash F$ and $X \backslash F$ is a μ^*-open set $\} = \wedge_\mu^*(X \backslash A)$. □

Using the above lemma and Theorem 3, we have the following result:

Theorem 4. *For subsets $A, B, A_\alpha (\alpha \in \Delta)$ of a GTS (X, μ) with a hereditary class $\mathcal{H}$, the following properties hold:*
(i) $\vee_\mu^*(A) \subseteq A$.
(ii) If A is μ^*-closed, then $A = \vee_\mu^*(A)$.
(iii) If $A \subseteq B$, then $\vee_\mu^*(A) \subseteq \vee_\mu^*(B)$.
(iv) $\vee_\mu^*(\vee_\mu^*(A)) = \vee_\mu^*(A)$.
(v) $\vee_\mu^*(\cap\{A_\alpha : \alpha \in \Delta\}) = \cap\{\vee_\mu^*(A_\alpha) : \alpha \in \Delta\}$.
(vi) $\cup\{\vee_\mu^*(A_\alpha) : \alpha \in \Delta\} \subseteq \vee_\mu^*(\cup\{A_\alpha : \alpha \in \Delta\})$.

Definition 4. *Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$. A subset A of X is said to be a*
(i) $\wedge_\mu^*$-set if $A = \wedge_\mu^*(A)$,
(ii) $\vee_\mu^*$-set if $A = \vee_\mu^*(A)$.
Therefore a subset A of X is a $\wedge_\mu^*$-set if and only if $X \backslash A$ is a $\vee_\mu^*$-set.

Theorem 5. *Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$. Then the following statements hold:*
(a) ϕ is a $\wedge_\mu^*$-set and X is a $\vee_\mu^*$-set.
(b) Arbitrary union of $\wedge_\mu^*$-sets is a $\wedge_\mu^*$-set.
(c) Arbitrary intersection of $\vee_\mu^*$-sets is a $\vee_\mu^*$-set.

Proof. (a) Clear.
(b) Let $\{A_\alpha : \alpha \in \Delta\}$ be an arbitrary family of $\wedge_\mu^*$-sets. Then $A_\alpha = \wedge_\mu^*(A_\alpha)$, for each $\alpha \in \Delta$. Let $A = \cup\{A_\alpha : \alpha \in \Delta\}$. Then by (vi) of Theorem 3, we have $\wedge_\mu^*(A) = A$ and hence A is a $\wedge_\mu^*$-set.
(c) It follows from Lemma 1 and (b) above. □

Definition 5. [9] *A subset A of a GTS (X, μ) with a hereditary class $\mathcal{H}$ is said to be μ^*-g-closed if $c_\mu(A) \subseteq U$ whenever $A \subseteq U$ and U is μ^*-open.*
The complement of a μ^*-g-closed set is μ^*-g-open.

Theorem 6. *Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$ and $A \subseteq X$. Then A is μ^*-g-closed if and only if $c_\mu(A) \subseteq \wedge_\mu^*(A)$.*

Proof. Let A be a μ^*-g-closed set and $x \in c_\mu(A)$. If $x \notin \wedge_\mu^*(A)$, then there exists a μ^*-open set U containing A such that $x \notin U$. Now since A is μ^*-g-closed and $A \subseteq U$, where U is μ^*-open, it follows that $c_\mu(A) \subseteq U$ and thus $x \notin c_\mu(A)$, a contradiction. Therefore $c_\mu(A) \subseteq \wedge_\mu^*(A)$.
Conversely suppose that $c_\mu(A) \subseteq \wedge_\mu^*(A)$. Let $A \subseteq U$, where U is μ^*-open. Then $\wedge_\mu^*(A) \subseteq U$ and hence $c_\mu(A) \subseteq U$. Therefore A is μ^*-g-closed. □

Corollary 1. *Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$ and $A \subseteq X$. Then A is μ^*-g-open if and only if $\vee_\mu^*(A) \subseteq i_\mu(A)$.*

Corollary 2. *Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$ and A be a $\wedge_\mu^*$-set. Then A is μ^*-g-closed if and only if A is μ-closed in (X, μ).*

Proof. Suppose that A is μ^*-g-closed. Then by using Theorem 6, we have $c_\mu(A) \subseteq \wedge_\mu^*(A) = A$. Thus A is μ-closed.
The converse is obvious. □

Corollary 3. *Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$ and A be a $\vee_\mu^*$-set. Then A is μ^*-g-open if and only if A is μ-open in (X, μ).*

Theorem 7. *Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$ and $A \subseteq X$. Then A is μ^*-g-closed if $\wedge_\mu^*(A)$ is μ^*-g-closed.*

Proof. Let $\wedge_\mu^*(A)$ be μ^*-g-closed. Suppose that $A \subseteq U$, where U is μ^*-open. Then $\wedge_\mu^*(A) \subseteq U$. Since $\wedge_\mu^*(A)$ is μ^*-g-closed, it follows that $c_\mu(\wedge_\mu^*(A)) \subseteq U$. Since $A \subseteq \wedge_\mu^*(A) \subseteq U$, we have $c_\mu(A) \subseteq c_\mu(\wedge_\mu^*(A)) \subseteq U$ i.e., $c_\mu(A) \subseteq U$ and thus A is a μ^*-g-closed set. □

Remark 2. *The converse of the above theorem is false as shown in the following example.*

Example 2. *Consider a GT μ and a hereditary class $\mathcal{H}$ on $X = \{a, b, c\}$, where $\mu = \{\phi, \{a, b\}, \{b, c\}, X\}$ and $\mathcal{H} = \{\phi, \{a\}, \{c\}\}$. Then $A = \{a\}$ is a μ^*-g-closed but $\wedge_\mu^*(A) = \{a, b\}$ which is not a μ^*-g-closed set, since $c_\mu(\{a, b\}) = X \not\subseteq \wedge_\mu^*(\{a, b\}) = \{a, b\}$(refer to Theorem 6).*

3 Generalized $\wedge_\mu^*$ and $\vee_\mu^*$-sets

In this section, we introduce and study two other types of sets viz. g.$\wedge_\mu^*$-sets, g.$\vee_\mu^*$-sets. We discuss several properties of these sets, a few of which involve sets introduced in the previous section. We start this section by recalling the following definition from [16]:

Definition 6. *A subset A of a GTS (X, μ) is said to be a generalized $\wedge_\mu$-set (g.$\wedge_\mu$-set, in short) if $\wedge_\mu(A) \subseteq F$, whenever $A \subseteq F$ and F is μ-closed in X.
A subset A of X is said to be a g.$\vee_\mu$-set if $X \backslash A$ is a g.$\wedge_\mu$-set.*

In an analogous way we define generalized $\wedge_\mu^*$-sets in our setting as follows:

Definition 7. *Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$. A subset A of X is said to be a generalized $\wedge_\mu^*$-set (g.$\wedge_\mu^*$-set, in short) if $\wedge_\mu^*(A) \subseteq F$, whenever F is μ-closed in X and $A \subseteq F$.
A subset A of X is said to be a g.$\vee_\mu^*$-set if $X \backslash A$ is a g.$\wedge_\mu^*$-set.*

Remark 3. *(i) Every g.$\wedge_\mu$-set (g.$\vee_\mu$-set) is a g.$\wedge^*_\mu$-set (resp. g.$\vee^*_\mu$-set). But the converse is false (see Example 3(a)).*
(ii) Every $\wedge^*_\mu$-set ($\vee^*_\mu$-set) is a g.$\wedge^*_\mu$-set (resp. g.$\vee^*_\mu$-set). But the converse is false (see Example 3(b)).

Example 3. *(a) Let $X=\{a,b,c\}, \mu=\{\phi,\{c\},\{a,b\},\{b,c\},X\}$ and $\mathcal{H}=\{\phi,\{b\},\{c\}\}$. Consider $A=\{a\}$. Then $\wedge_\mu(A)=\{a,b\}$ and $\wedge^*_\mu(A)=\{a\}$. Thus, it follows that A is a g.$\wedge^*_\mu$-set but not a g.$\wedge_\mu$-set.*
(b) Consider $X=\{a,b,c,d\}, \mu=\{\phi,\{a,b\},\{b,c\},\{a,b,c\}\}$ and $\mathcal{H}=\{\phi,\{a\},\{c\}\}$. Let $A=\{a,c\}$. Then $\wedge^*_\mu(A)=\{a,b,c\}$. Thus A is a g.$\wedge^*_\mu(A)$-set, but not a $\wedge^*_\mu(A)$-set.

Theorem 8. *Let (X,μ) be a GTS with a hereditary class $\mathcal{H}$ and $A\subseteq X$. Then A is a g.$\vee^*_\mu$-set if and only if $U\subseteq \vee^*_\mu(A)$, whenever $U\subseteq A$ and U is μ-open in X.*

Proof. Let A be a g.$\vee^*_\mu$-set and $U\subseteq A$, where U is μ-open in X. Then $X\backslash A\subseteq X\backslash U$, where $X\backslash U$ is μ-closed in X. Since $X\backslash A$ is a g.$\wedge^*_\mu$-set, $\wedge^*_\mu(X\backslash A)\subseteq X\backslash U$ which implies by Lemma 1 that $X\backslash \vee^*_\mu(A)\subseteq X\backslash U$. Thus $U\subseteq \vee^*_\mu(A)$.
Conversely, let the condition hold. Let A be a subset of X such that $X\backslash A\subseteq F$, where F is μ-closed in X. Then $X\backslash F\subseteq A$ and $X\backslash F$ is μ-open in X and so by given condition, $X\backslash F\subseteq \vee^*_\mu(A)$. Thus $X\backslash \vee^*_\mu(A)\subseteq F$ and hence by Lemma 1, $\wedge^*_\mu(X\backslash A)\subseteq F$. Then $X\backslash A$ is a g.$\wedge^*_\mu$-set. Hence A is a g.$\vee^*_\mu$-set. □

Theorem 9. *Let (X,μ) be a GTS with a hereditary class $\mathcal{H}$ and $A\subseteq X$. If A is a g.$\vee^*_\mu$-set, then $F=X$ whenever $\vee^*_\mu(A)\cup(X\backslash A)\subseteq F$ and F is μ-closed in X.*

Proof. Let A be a g.$\vee^*_\mu$-set and $\vee^*_\mu(A)\cup(X\backslash A)\subseteq F$, where F is μ-closed in X. Then we have $X\backslash F\subseteq X\backslash(\vee^*_\mu(A)\cup(X\backslash A))=(X\backslash \vee^*_\mu(A))\cap A$. Thus $X\backslash F\subseteq (X\backslash \vee^*_\mu(A))$ and $X\backslash F\subseteq A$. It follows that $X\backslash F\subseteq \vee^*_\mu(A)$ (by Theorem 8). Thus $X\backslash F\subseteq (X\backslash \vee^*_\mu(A))\cap \vee^*_\mu(A)=\phi$ and hence $F=X$. □

Theorem 10. *Let (X,μ) be a GTS with a hereditary class $\mathcal{H}$. Then a g.$\vee^*_\mu$-set is a $\vee^*_\mu$-set if and only if $\vee^*_\mu(A)\cup(X\backslash A)$ is a μ-closed set.*

Proof. Suppose that a g.$\vee^*_\mu$-set A is a $\vee^*_\mu$-set. Then $\vee^*_\mu(A)=A$. Thus $\vee^*_\mu(A)\cup(X\backslash A)=A\cup(X\backslash A)=X$ which is μ-closed.

Conversely, let A be a g.$\vee^*_\mu$-set such that $\vee^*_\mu(A)\cup(X\backslash A)$ is μ-closed in X. Then by Theorem 9, we have $\vee^*_\mu(A)\cup(X\backslash A)=X$ and hence $A\subseteq \vee^*_\mu(A)$. Again by Theorem 4(i), $A\supseteq \vee^*_\mu(A)$. Thus $\vee^*_\mu(A)=A$ and hence A is a $\vee^*_\mu$-set. □

Corollary 4. *Let (X,μ) be a GTS with a hereditary class $\mathcal{H}$. Then a g.$\wedge^*_\mu$-set is $\wedge^*_\mu$-set if and only if $\wedge^*_\mu(A)\cup(X\backslash A)$ is μ-open.*

Theorem 11. *Let (X,μ) be a GTS with a hereditary class $\mathcal{H}$. Then for each $x\in X$, either $\{x\}$ is a μ-open set in X or a g.$\vee^*_\mu$-set.*

Proof. Let $x\in X$. Suppose $\{x\}$ is not μ-open in X, then X is the only μ-closed set containing $X\backslash\{x\}$ and hence $X\backslash\{x\}$ is a g.$\wedge^*_\mu$-set. Thus $\{x\}$ is a g.$\vee^*_\mu$-set. □

Theorem 12. *Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$. Then every singleton of X is a g.$\wedge_\mu^*$-set if and only if $U = \vee_\mu^*(U)$ for every μ-open set U in X.*

Proof. Let every singleton set of X be a g.$\wedge_\mu^*$-set. Let U be a μ-open set in X and $x \in X \backslash U$. Since $\{x\}$ is a g.$\wedge_\mu^*$-set, we have $\wedge_\mu^*\{x\} \subseteq X \backslash U$. It follows that $\bigcup\{\wedge_\mu^*\{x\} : x \in X \backslash U\} \subseteq X \backslash U$ and thus using Theorem 3(vi), we get $\wedge_\mu^*(\bigcup\{\{x\} : x \in X \backslash U\}) \subseteq X \backslash U$. Therefore $\wedge_\mu^*(X \backslash U) \subseteq X \backslash U$ and hence by Lemma 1, we have $X \backslash U = \wedge_\mu^*(X \backslash U) = X \backslash \vee_\mu^*(U)$. Thus $U = \vee_\mu^*(U)$.
Conversely, let $x \in X$ and $\{x\} \subseteq F$, where F is a μ-closed subset of X. Then $X \backslash F$ is μ-open in X and so by hypothesis, $X \backslash F = \vee_\mu^*(X \backslash F) = X \backslash \wedge_\mu^*(F)$ (by Lemma 1). It follows that $F = \wedge_\mu^*(F)$. Thus $\wedge_\mu^*\{x\} \subseteq \wedge_\mu^*(F) = F$ and hence $\{x\}$ is a g.$\wedge_\mu^*$-set. □

4 Applications

In [9], we introduced and studied the concept of μ^*-R_0, μ^*-R_1 and μ^*-$T_{\frac{1}{2}}$ spaces. Here we deduce some characterizations of the above lower separation axioms in terms of $\wedge_\mu^*$ and $\vee_\mu^*$ sets.

Definition 8. *[9] A GTS (X, μ) with a hereditary class $\mathcal{H}$ is said to be a μ^*-R_0-space if for every μ^*-open set U and each $x \in U$, one has $c_\mu(\{x\}) \subseteq U$ that is, every singleton is μ^*-g-closed.*

Definition 9. *[9] A GTS (X, μ) with a hereditary class $\mathcal{H}$ is said to be μ^*-R_1 if for each $x, y \in X$ with $c_\mu(\{x\}) \neq c_\mu^*(\{y\})$, there exist two disjoint μ^*-open sets U and V such that $c_\mu(\{x\}) \subseteq U$ and $c_\mu^*(\{y\}) \subseteq V$.*

Proposition 1. [9] *Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$. If it is μ^*-R_1, then it is also μ^*-R_0.*

Theorem 13. *(Theorem 3.2 of [9]) Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$. Then the following are equivalent:*
(a) A GTS (X, μ) with a hereditary class $\mathcal{H}$ is μ^*-R_0-space.
(b) $x \in c_\mu^*(\{y\})$ if and only if $y \in c_\mu(\{x\})$, where x and y are any two distinct points of X.

Theorem 14. *(Theorem 3.3 of [9]) For a GTS (X, μ) with a hereditary class $\mathcal{H}$, the following are equivalent:*
(a) A GTS (X, μ) with a hereditary class $\mathcal{H}$ is μ^*-R_0-space.
(b) If x and y are two distinct points of X then $x \notin c_\mu^*(\{y\}) \Rightarrow c_\mu(\{x\}) \cap c_\mu^*(\{y\}) = \phi$.
(c) Every μ^*-closed set F can be written as, $F = \cap\{U : U$ is μ-open and $F \subseteq U\}$.

Theorem 15. *In a μ^*-R_0-space, for any two points x, y in X, the following are equivalent:*
(a) $c_\mu(\{x\}) \neq c_\mu(\{y\})$
(b) $c_\mu(\{x\}) \neq c_\mu^*(\{y\})$
(c) either $c_\mu(\{x\}) \cap c_\mu^*(\{y\}) = \phi$ or $c_\mu^*(\{x\}) \cap c_\mu(\{y\}) = \phi$.

Proof. $(\mathbf{a}) \Rightarrow (\mathbf{b})$: Let $x, y \in X$ be such that $c_\mu(\{x\}) \neq c_\mu(\{y\})$. Then either $x \notin c_\mu(\{y\})$ or $y \notin c_\mu(\{x\})$. If $x \notin c_\mu(\{y\})$ then $x \notin c^*_\mu(\{y\})$ and hence $c_\mu(\{x\}) \neq c^*_\mu(\{y\})$. If $y \notin c_\mu(\{x\})$ then $c_\mu(\{x\}) \neq c^*_\mu(\{y\})$.

$(\mathbf{b}) \Rightarrow (\mathbf{c})$: Let $x, y \in X$ be such that $c_\mu(\{x\}) \neq c^*_\mu(\{y\})$. Then either $x \notin c^*_\mu(\{y\})$ or $y \notin c_\mu(\{x\})$. If $x \notin c^*_\mu(\{y\})$ then by Theorem 14($(a) \Leftrightarrow (b)$), $c_\mu(\{x\}) \cap c^*_\mu(\{y\}) = \phi$. Next, suppose $y \notin c_\mu(\{x\})$ which implies that $y \notin c^*_\mu(\{x\})$ and hence again by Theorem 14($(a) \Leftrightarrow (b)$), $c_\mu(\{y\}) \cap c^*_\mu(\{x\}) = \phi$.

$(\mathbf{c}) \Rightarrow (\mathbf{a})$: Let $x, y \in X$ be such that either $c_\mu(\{x\}) \cap c^*_\mu(\{y\}) = \phi$ or $c^*_\mu(\{x\}) \cap c_\mu(\{y\}) = \phi$. If $c_\mu(\{x\}) \cap c^*_\mu(\{y\}) = \phi$ then $y \notin c_\mu(\{x\})$ and hence $c_\mu(\{x\}) \neq c_\mu(\{y\})$. Next if $c^*_\mu(\{x\}) \cap c_\mu(\{y\}) = \phi$ then $x \notin c_\mu(\{y\})$ and thus $c_\mu(\{x\}) \neq c_\mu(\{y\})$. □

Theorem 16. *For a GTS (X, μ) with a hereditary class $\mathcal{H}$ the following are equivalent:*
(a) A GTS (X, μ) with a hereditary class $\mathcal{H}$ is a μ^*-R_0 space.
(b) If F is μ^*-closed and $x \in F$, then $\wedge^*_\mu(\{x\}) \subseteq F$ and $F = \wedge_\mu(F) = \wedge^*_\mu(F)$.
(c) If $x \in X$, then $\wedge^*_\mu(\{x\}) = c_\mu(\{x\})$.

Proof. $(\mathbf{a}) \Rightarrow (\mathbf{b})$: Let F be μ^*-closed and $x \in F$. We first prove that $F = \wedge_\mu(F) = \wedge^*_\mu(F)$. By $((a) \Rightarrow (c))$ of Theorem 14 and Definition 1, we have $F = \wedge_\mu(F)$. Again obviously $\wedge^*_\mu(F) \subseteq \wedge_\mu(F)$ and by Theorem 3, $F \subseteq \wedge^*_\mu(F)$. Thus $F = \wedge^*_\mu(F)$ and hence $F = \wedge_\mu(F) = \wedge^*_\mu(F)$. Now $x \in F$, we get $\wedge^*_\mu(\{x\}) \subseteq \wedge^*_\mu(F) = F$.
$(\mathbf{b}) \Rightarrow (\mathbf{c})$: Let $x \in X$. Then $x \in c^*_\mu(\{x\})$ where $c^*_\mu(\{x\})$ is a μ^*-closed set and so by (b), $\wedge^*_\mu(\{x\}) \subseteq c^*_\mu(\{x\})$ and thus $\wedge^*_\mu(\{x\}) \subseteq c_\mu(\{x\})$ ...(1). Next we show that $c_\mu(\{x\}) \subseteq \wedge^*_\mu(\{x\})$. For that, let $y \notin \wedge^*_\mu(\{x\})$. Then there exists $V \in \mu^*$ such that $x \in V$ and $y \notin V$. So $c^*_\mu(\{y\}) \cap V = \phi$. By (b), we have $c^*_\mu(\{y\}) = \wedge_\mu(c^*_\mu(\{y\})) = \cap\{G \in \mu : c^*_\mu(\{y\}) \subseteq G\}$ which implies that $\cap\{G \in \mu : c^*_\mu(\{y\}) \subseteq G\} \cap V = \phi$. Thus, there exists $G \in \mu$ such that $x \notin G$ with $c^*_\mu(\{y\}) \subseteq G$. Since $x \notin G$, $c_\mu(\{x\}) \cap G = \phi$ and hence $y \notin c_\mu(\{x\})$. Thus $c^*_\mu(\{x\}) \subseteq c_\mu(\{x\}) \subseteq \wedge^*_\mu(\{x\})$. ...(2). Therefore, from (1) and (2), we have $\wedge^*_\mu(\{x\}) = c_\mu(\{x\})$.
$(\mathbf{c}) \Rightarrow (\mathbf{a})$: Let x, $y \in X$ with $x \neq y$. Then $x \in c^*_\mu(\{y\})$ if and only if $y \in \wedge^*_\mu(\{x\})$ (by using Theorem 2) i.e., $x \in c^*_\mu(\{y\})$ if and only if $y \in c_\mu(\{x\})$ (by using (c)). Hence by Theorem 13, (X, μ) with a hereditary class $\mathcal{H}$ is a μ^*-R_0 space. □

Theorem 17. *The following are equivalent for a GTS (X, μ) with a hereditary class $\mathcal{H}$:*
(a) (X, μ) with a hereditary class $\mathcal{H}$ is a μ^*-R_0 space.
(b) $x \in \wedge^*_\mu(\{y\})$ if and only if $y \in \wedge_\mu(\{x\})$, for any two distinct points $x, y \in X$.

Proof. $(\mathbf{a}) \Rightarrow (\mathbf{b})$: Suppose that (X, μ) with a hereditary class $\mathcal{H}$ is a μ^*-R_0 space. Let $x, y \in X$ with $x \neq y$. First let $x \in \wedge^*_\mu(\{y\})$. Then by Theorem 2, $y \in c^*_\mu(\{x\})$. We now show that $y \in \wedge_\mu(\{x\})$. Indeed, if $y \notin \wedge_\mu(\{x\})$ then there exists $V \in \mu$ such that $x \in V$ and $y \notin V$ and so $c_\mu(\{y\}) \cap V = \phi$. Thus $\cap\{G \in \mu : c_\mu(\{y\}) \subseteq G\} \cap V = \phi$ (refer to Theorem 14($(a) \Rightarrow (c)$) and hence there exists $G \in \mu$ such that $x \notin G$ with $c_\mu(\{y\}) \subseteq G$ and hence $c_\mu(\{x\}) \cap G = \phi$. This implies that

$y \notin c_\mu(\{x\})$ and hence $y \notin c^*_\mu(\{x\})$, a contradiction. Next, let $y \in \wedge_\mu(\{x\})$. Then by Theorem 1, $x \in c_\mu(\{y\})$. Since (X, μ) with a hereditary class $\mathcal{H}$ is μ^*-R_0, by $((a) \Rightarrow (c))$ of Theorem 16, we have $\wedge^*_\mu(\{y\}) = c_\mu(\{y\})$ and hence $x \in \wedge^*_\mu(\{y\})$.
$(\mathbf{b}) \Rightarrow (\mathbf{a})$: Let the condition (b) hold. Let U be any μ^*-open set and $x \in U$. Claim: $c_\mu(\{x\}) \subseteq U$. In fact, let $y \notin U$. Then $x \notin c^*_\mu(\{y\})$ and so by Theorem 2, $y \notin \wedge^*_\mu(\{x\})$. Therefore by hypothesis, we have $x \notin \wedge_\mu(\{y\})$ and consequently $y \notin c_\mu(\{x\})$ (by Theorem 1). Hence (X, μ) with a hereditary class $\mathcal{H}$ is a μ^*-R_0 space. □

Theorem 18. *A GTS (X, μ) with a hereditary class $\mathcal{H}$ is μ^*-R_1 if and only if for any two distinct points $x, y \in X$ with $\wedge^*_\mu(\{x\}) \neq \wedge_\mu(\{y\})$, there exist two disjoint μ^*-open sets U and V such that $c_\mu(\{x\}) \subseteq U$ and $c^*_\mu(\{y\}) \subseteq V$.*

Proof. Suppose that (X, μ) with a hereditary class $\mathcal{H}$ is μ^*-R_1. Let x, y be any two distinct points in X such that $\wedge^*_\mu(\{x\}) \neq \wedge_\mu(\{y\})$. Then we have either $x \notin \wedge_\mu(\{y\})$ or $y \notin \wedge^*_\mu(\{x\})$. If not then $x \in \wedge_\mu(\{y\})$ and $y \in \wedge^*_\mu(\{x\})$ and so $\wedge_\mu(\{x\}) \subseteq \wedge_\mu(\wedge_\mu(\{y\})) = \wedge_\mu(\{y\})$ and thus $\wedge^*_\mu(\{x\}) \subseteq \wedge_\mu(\{y\})$. By hypothesis, we have $\wedge^*_\mu(\{x\}) \subsetneqq \wedge_\mu(\{y\})$. Since (X, μ) with the hereditary class $\mathcal{H}$ is μ^*-R_1, by Proposition 1, it is a μ^*-R_0 space. Now by using $((a) \Rightarrow (c))$ of Theorem 16, we have $c_\mu(\{x\}) \subsetneqq \wedge_\mu(\{y\})$...(1). Let $z \in \wedge_\mu(\{y\})$. Then by Theorem 1, we have $y \in c_\mu(\{z\})$. It follows that $c^*_\mu(\{y\}) \subseteq c_\mu(\{z\})$ and hence by Theorem 15 $((b) \Rightarrow (c))$, we must have $c^*_\mu(\{y\}) = c_\mu(\{z\})$. Thus $z \in c^*_\mu(\{y\})$ and hence $\wedge_\mu(\{y\} \subseteq c^*_\mu(\{y\})$...(2). From (1) and (2), we get $c_\mu(\{x\}) \subsetneqq c^*_\mu(\{y\})$ and hence there are no two disjoint μ^*-open sets U and V such that $c_\mu(\{x\}) \subseteq U$ and $c^*_\mu(\{y\}) \subseteq V$ which contradicts the fact that the space is μ^*-R_1.
Now we show that $c_\mu(\{x\}) \neq c^*_\mu(\{y\})$.
Case I: If $x \notin \wedge_\mu(\{y\})$ then $y \notin c_\mu(\{x\})$ which implies $c_\mu(\{x\}) \neq c^*_\mu(\{y\})$.
Case II: If $y \notin \wedge^*_\mu(\{x\})$ then $x \notin c^*_\mu(\{y\})$ which implies $c_\mu(\{x\}) \neq c^*_\mu(\{y\})$.
Thus, in both the cases, we have $c_\mu(\{x\}) \neq c^*_\mu(\{y\})$. Then by definition of μ^*-R_1 space, there exist two disjoint μ^*-open sets U and V such that $c_\mu(\{x\}) \subseteq U$ and $c^*_\mu(\{y\}) \subseteq V$.
Conversely, let the condition hold. Let x and y be two distinct points of X such that $c_\mu(\{x\}) \neq c^*_\mu(\{y\})$. Then either $y \notin c_\mu(\{x\})$ or $x \notin c^*_\mu(\{y\})$.
Case I: If $y \notin c_\mu(\{x\})$ then there exists a μ-open set G containing y such that $x \notin G$ and hence from the definition $x \notin \wedge_\mu(\{y\})$ which follows that $\wedge_\mu(\{y\}) \neq \wedge^*_\mu(\{x\})$.
Case II: If $x \notin c^*_\mu(\{y\})$ then by Theorem 2, $y \notin \wedge^*_\mu(\{x\})$ which shows that $\wedge^*_\mu(\{x\}) \neq \wedge_\mu(\{y\})$.
Thus, in both the cases, we have $\wedge^*_\mu(\{x\}) \neq \wedge_\mu(\{y\})$ and hence by hypothesis, there exist two disjoint μ^*-open sets U and V such that $c_\mu(\{x\}) \subseteq U$ and $c^*_\mu(\{y\}) \subseteq V$. This shows that (X, μ) with a hereditary class $\mathcal{H}$ is μ^*-R_1. □

Definition 10. *[9] A GTS (X, μ) with a hereditary class $\mathcal{H}$ is said to be μ^*-$T_{\frac{1}{2}}$ if every μ^*-g-closed set is μ-closed in X.*

Theorem 19. *[9] A GTS (X, μ) with a hereditary class $\mathcal{H}$ is a μ^*-$T_{\frac{1}{2}}$ space if and only if for each $x \in X$, either $\{x\}$ is μ^*-closed or μ-open in X.*

Theorem 20. *Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$. Then it is a μ^*-$T_{\frac{1}{2}}$-space if and only if every g.$\vee_\mu^*$-set is a $\vee_\mu^*$-set.*

Proof. Let a GTS (X, μ) with a hereditary class $\mathcal{H}$ be a μ^*-$T_{\frac{1}{2}}$-space. We prove by contradiction. Suppose that A is a g.$\vee_\mu^*$-set but not a $\vee_\mu^*$-set. Then there exists an element $x \in A$ such that $x \notin \vee_\mu^*(A)$. Thus by definition of $\vee_\mu^*(A)$, $\{x\}$ is not μ^*-closed. Thus by Theorem 19, we have $\{x\}$ is μ-open, that is, $X \backslash \{x\}$ is μ-closed in X. Since $x \in A$ and $x \notin \vee_\mu^*(A)$, we have $\vee_\mu^*(A) \cup (X \backslash A) \subseteq X \backslash \{x\}$. Therefore by Theorem 9, $X \backslash \{x\} = X$, a contradiction.
Conversely, let every g.$\vee_\mu^*$-set be a $\vee_\mu^*$-set. Suppose the GTS (X, μ) with a hereditary class $\mathcal{H}$ is not a μ^*-$T_{\frac{1}{2}}$-space. Then there exists a μ^*-g-closed set A which is not μ-closed in X. Thus, there exists an element $x \in X$ such that $x \in c_\mu(A)$ but $x \notin A$. Now by Theorem 11, $\{x\}$ is either a μ-open set or a g.$\vee_\mu^*$-set.
Case I: Let $\{x\}$ be μ-open. Then $x \in c_\mu(A)$ implies that $\{x\} \cap A \neq \phi$ and so $x \in A$, a contradiction.
Case II: Let $\{x\}$ be a g.$\vee_\mu^*$-set. Then by hypothesis $\{x\}$ is a $\vee_\mu^*$-set and so $\{x\} = \vee_\mu^*(\{x\})$ and hence by definition of $\vee_\mu^*$-set, we have $\{x\}$ is μ^*-closed. Since A is μ^*-g-closed and $A \subseteq X \backslash \{x\}$, $c_\mu(A) \subseteq X \backslash \{x\}$, which contradict that $x \in c_\mu(A)$.
Hence (X, μ) with a hereditary class $\mathcal{H}$ is a μ^*-$T_{\frac{1}{2}}$-space. □

Corollary 5. *Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$. Then it is a μ^*-$T_{\frac{1}{2}}$-space if and only if every g.$\wedge_\mu^*$-set is a $\wedge_\mu^*$-set.*

Corollary 6. *Let (X, μ) be a GTS with a hereditary class $\mathcal{H}$. Then it is a μ^*-$T_{\frac{1}{2}}$-space if and only if for each $x \in X$, either $\{x\}$ is a $\vee_\mu^*$-set or μ-open in X.*

Proof. Let X be a μ^*-$T_{\frac{1}{2}}$-space. Now we have from Theorem 11 that for each $x \in X$, either $\{x\}$ is a μ-open set or a g.$\vee_\mu^*$-set in X. If $\{x\}$ is μ-open in (X, μ) then we are done. So suppose that $\{x\}$ is not a μ-open set in X. Then it must be a g.$\vee_\mu^*$-set and so by Theorem 20, we get $\{x\}$ is a $\vee_\mu^*$-set.
Conversely, let $x \in X$. Then either $\{x\}$ is a $\vee_\mu^*$-set or μ-open in X.
Case I: If $\{x\}$ is a $\vee_\mu^*$-set in X, then $\{x\} = \vee_\mu^*(\{x\})$ and so by definition of $\vee_\mu^*$-set, we have $\{x\}$ is μ^*-closed. Hence by Theorem 19, X is a μ^*-$T_{\frac{1}{2}}$-space.
Case II: If $\{x\}$ is μ-open in X, then by Theorem 19, X is a μ^*-$T_{\frac{1}{2}}$-space. □

Acknowledgements: The authors are thankful to the referee for some suggestions towards the improvement of the paper.

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