Bulletin of the *Transilvania* University of Braşov • Vol 10(59), No. 1 - 2017
Series III: Mathematics, Informatics, Physics, 83-90

GENERALIZED MICROPOLAR THERMOELASTICITY WITH FRACTIONAL ORDER STRAIN

Adina CHIRILĂ[1]

Abstract

This article presents the basic equations, the initial and boundary conditions of the mixed problem of micropolar thermoelasticity, then introduces the Caputo fractional derivative and applies it in this context. The theorems of this paper give some formulations for the constitutive equations in the fractional case and for Cattaneo's heat conduction equations.

2000 *Mathematics Subject Classification:* 74A35, 74A20, 74F05, 74E10.
Key words: micropolar thermoelasticity, fractional derivative, Cattaneo equations.

1 Introduction

According to [8], experiments have shown that some deformation processes cannot be described properly with the help of the classical theories, in which the number of degrees of freedom of each material point is three. This happens since materials have a granular and molecular nature [3]. The Cosserat brothers introduced the idea of generalized continua in [4]. In the framework of continuum mechanics, they consider extra degrees of freedom for material points in order to be able to better model materials with microstructure. In this theory, the number of degrees of freedom of a rigid body of each material point is six. The orientation of a given point of such a medium can be described mathematically by means of the values of three directors that are mutually orthogonal unit vectors [3]. According to Eringen [6], possible substances that can be modelled by Cosserat or micropolar continua are liquid crystals with rigid molecules, rigid suspensions, animal blood with rigid cells, chopped fiber composites, bones, magnetic fluids, clouds with dusts, concrete with sand, muddy fluids.

Many developments have been reported since the seminal work of the Cosserat brothers. Thermal effects were added. For example, micropolar thermoelasticity

---

[1] Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania, e-mail: adina.chirila@unitbv.ro

is discussed in [12], [10] and more recently in [2]. Similar studies are described in [9], [11] and [14].

In [15], the author constructs a new model of thermoelasticity theory by considering heat conduction with fractional order. In this context, the paper [17] presents the theory of fractional order generalized thermoelasticity with microstructure modelling for porous elastic bodies.

This article follows [16] and constructs a new model of micropolar thermoelasticity by considering the strain with fractional order.

2 Basic equations and conditions

This paper is about an anisotropic thermoelastic micropolar material that occupies a bounded domain Ω of the three-dimensional Euclidean space $\mathbb{R}^3$ with a $C^{1,1}$ boundary $\partial\Omega$. The closure of Ω is denoted by $\bar{\Omega}$.

The motion of the body corresponds to a fixed system of rectangular Cartesian axes $Ox_i, i = 1, 2, 3$. Cartesian tensor notation is used in the paper. A comma followed by a subscript means partial differentiation with respect to the spatial coordinates. A superposed dot represents the material time derivative. The Einstein summation convention is adopted over repeated indices. The spatial and the time arguments of a function are omitted when there is no likelihood of confusion.

The deformation of the medium is described by the displacement vector field $u = (u_i)_{i=1}^3$ and the microrotation vector field $\varphi = (\varphi_j)_{j=1}^3$.

The equations of motion [8] are defined for $(x, t) \in \Omega \times [0, \infty)$

$$\begin{aligned} &t_{ji,j} + \rho_0 F_i = \rho_0 \ddot{u}_i \\ &m_{ji,j} + e_{ijk} t_{jk} + \rho_0 M_i = I_{ij} \ddot{\varphi}_j \end{aligned} \tag{1}$$

Above t_{ij} are the components of the stress tensor, m_{ij} are the components of the couple stress tensor, ρ_0 is the constant mass density, I_{ij} are the components of the inertia, F_i are the components of the external body force vector, M_i are the components of the external body couple vector and e_{ijk} is the alternating symbol.

The deformation tensors ε_{ij} and γ_{ij} are defined in $\bar{\Omega} \times [0, \infty)$ by means of the geometric equations

$$\varepsilon_{ij} = u_{j,i} + e_{jik}\varphi_k, \quad \gamma_{ij} = \varphi_{j,i} \tag{2}$$

Below T is the temperature, q_i are the components of the heat conduction vector and S is the heat supply per unit volume, $S = \rho_0 Q$.

To the above system of equations one should add the following initial conditions [5]

$$\begin{aligned} &u_i(x, 0) = u_i^0(x), \quad \dot{u}_i(x, 0) = u_i^1(x) \\ &\varphi_i(x, 0) = \varphi_i^0(x), \quad \dot{\varphi}_i(x, 0) = \varphi_i^1(x) \\ &T(x, 0) = T^0(x), \quad \dot{T}(x, 0) = T^1(x) \end{aligned} \tag{3}$$

where $u_i^0, u_i^1, \varphi_i^0, \varphi_i^1, T^0, T^1$ are prescribed functions of x on Ω. Furthermore, the boundary conditions on $\partial\Omega \times [0,\infty)$ are [5]

$$\begin{aligned}
&u_i(x,t) = \tilde{u}_i(x,t) \text{ on } \bar{\Sigma}_1 \times [0,\infty) \\
&t_i(x,t) = \tilde{t}_i(x,t) \text{ on } \Sigma_2 \times [0,\infty) \\
&T(x,t) = \tilde{T}(x,t) \text{ on } \bar{\Sigma}_3 \times [0,\infty) \\
&q(x,t) = \tilde{q}(x,t) \text{ on } \Sigma_4 \times [0,\infty) \\
&\varphi_j(x,t) = \tilde{\varphi}_j(x,t) \text{ on } \bar{\Sigma}_5 \times [0,\infty) \\
&m_i(x,t) = \tilde{m}_i(x,t) \text{ on } \Sigma_6 \times [0,\infty)
\end{aligned} \tag{4}$$

with

$$\begin{aligned}
t_i(x,s) &:= t_{ji}(x,s)n_j(x) \\
q(x,s) &:= q_i(x,s)n_i(x) \\
m_i(x,s) &:= m_{ji}(x,s)n_j(x)
\end{aligned} \tag{5}$$

where n_i are the components of the outward unit normal vector to the boundary surface and $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5$ and Σ_6 are subsurfaces of $\partial\Omega$ such that $\bar{\Sigma}_1 \cup \Sigma_2 = \bar{\Sigma}_3 \cup \Sigma_4 = \bar{\Sigma}_5 \cup \Sigma_6 = \partial\Omega$ and $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \Sigma_5 \cap \Sigma_6 = \emptyset$. Moreover, $\tilde{u}_i, \tilde{t}_i, \tilde{T}, \tilde{q}, \tilde{\varphi}_j$ and $\tilde{m}_i$ are prescribed functions on the corresponding subsurfaces.

According to [16], the law of conservation of mass can be written as

$$\frac{d}{dt}\int_\Omega \rho_0 \mathrm{d}V = 0 \tag{6}$$

which leads to the continuity condition in the form

$$\frac{d}{dt}(\rho_0 dV) = 0. \tag{7}$$

The internal energy per unit mass is denoted by e. According to [5], the principle of conservation of energy is

$$\begin{aligned}
&\int_\Omega (\rho_0 \dot{e} + \rho_0 \dot{u}_i \ddot{u}_i + I_{ij}\dot{\varphi}_i\ddot{\varphi}_j)\,\mathrm{d}V = \int_\Omega \rho_0 F_i \dot{u}_i \mathrm{d}V + \int_\Omega \rho_0 M_i \dot{\varphi}_i \mathrm{d}V + \\
&+ \int_{\partial\Omega} t_{ji} n_j \dot{u}_i \mathrm{d}A + \int_{\partial\Omega} m_{ji} n_j \dot{\varphi}_i \mathrm{d}A + \int_\Omega S \mathrm{d}V + \int_{\partial\Omega} q_i n_i \mathrm{d}A
\end{aligned} \tag{8}$$

According to [7], Cattaneo's heat conduction equations take the form

$$q_i + \tau_0 \dot{q}_i = -K_{ij} T_{,j}, \quad i,j = 1,2,3 \tag{9}$$

where K_{ij} is the thermal conductivity tensor and τ_0 is the relaxation time. These replace the conventional Fourier's law.

Caputo introduced the fractional derivative with respect to time

$$D_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\beta} \mathrm{d}\tau, \quad 0 \le \beta \le 1 \tag{10}$$

as defined in [13]. Other fractional derivatives are presented in [1].

One knows that ϕ, the Helmholtz free energy, is a combination of the internal energy and the entropy function η defined as follows

$$\phi = e - T\eta. \tag{11}$$

One adopts the equation of the state of a thermoelastic material in the following form

$$\begin{aligned} \phi &= \phi(\tilde{\varepsilon}_{ij}, \gamma_{ij}, T, T_{,i}) \\ e &= e(\tilde{\varepsilon}_{ij}, \gamma_{ij}, T, T_{,i}) \\ \eta &= \eta(\tilde{\varepsilon}_{ij}, \gamma_{ij}, T, T_{,i}) \\ q &= q(\tilde{\varepsilon}_{ij}, \gamma_{ij}, T, T_{,i}) \end{aligned} \tag{12}$$

where

$$\tilde{\varepsilon}_{ij} = (1 + \tau^{\beta} D_t^{\beta})\varepsilon_{ij} \tag{13}$$

and τ is the mechanical relaxation time parameter and is constant. It has significant effects on the stress, the displacement and the strain distribution.

The free energy can be written in the following form [5]

$$\begin{aligned} &\rho_0 \phi(\tilde{\varepsilon}_{ij}, \gamma_{ij}, \theta) = \frac{1}{2} A_{ijmn} \tilde{\varepsilon}_{ij} \tilde{\varepsilon}_{mn} + B_{ijmn} \tilde{\varepsilon}_{ij} \gamma_{mn} + \frac{1}{2} C_{ijmn} \gamma_{ij} \gamma_{mn} - \\ &- D_{ij} \tilde{\varepsilon}_{ij} \theta - E_{ij} \gamma_{ij} \theta - \frac{1}{2} c \theta^2 \end{aligned} \tag{14}$$

where $T = T^0 + \theta$.

In the expression of the free energy, the constitutive coefficients are assumed to satisfy the following symmetry relations

$$A_{ijmn} = A_{mnij}, \quad C_{ijmn} = C_{mnij}, \quad I_{ij} = I_{ji}. \tag{15}$$

3 Main results

In this section the time derivative of the free energy is written in two ways by means of the equations of motion and of the principle of conservation of energy and by means of the equations of the state of a thermoelastic material. Thus one obtains the constitutive equations and the equation of energy for the generalized micropolar thermoelasticity with fractional order strain. Furthermore, the equation of energy, the expressions of the entropy and of the free energy lead to a formulation of the non-Fourier heat equations.

First, the usual approach for deriving the equation of energy is given. For similar examples see [8]. This is used in obtaining a formula with fractional order strain.

Lemma 1. *The derivative of the free energy with respect to time is given by the following formula*

$$\rho_0 \dot{\phi} = t_{ij} \dot{\tilde{\varepsilon}}_{ij} + m_{ij} \dot{\gamma}_{ij} + \rho_0 Q + q_{i,i} - \rho_0 \dot{T} \eta - \rho_0 T \dot{\eta}. \tag{16}$$

Proof. The first equation from (1) is multiplied by $\dot{u}_i$ and is integrated over Ω to obtain

$$\int_\Omega \rho_0 \ddot{u}_i \dot{u}_i \mathrm{d}V = \int_\Omega t_{ji,j} \dot{u}_i \mathrm{d}V + \int_\Omega \rho_0 F_i \dot{u}_i \mathrm{d}V \tag{17}$$

The second equation from (1) is multiplied by $\dot{\varphi}_i$. Then one should sum up over i and integrate over Ω to obtain

$$\int_\Omega I_{ij} \ddot{\varphi}_j \dot{\varphi}_i \mathrm{d}V = \int_\Omega m_{ji,j} \dot{\varphi}_i \mathrm{d}V + \int_\Omega e_{ijk} t_{jk} \dot{\varphi}_i \mathrm{d}V + \int_\Omega \rho_0 M_i \dot{\varphi}_i \mathrm{d}V \tag{18}$$

One should substitute relations (17) and (18) into the principle of conservation of energy (8) to obtain

$$\begin{aligned} &\int_\Omega \rho_0 \dot{e} \mathrm{d}V + \int_\Omega t_{ji,j} \dot{u}_i \mathrm{d}V + \int_\Omega \rho_0 F_i \dot{u}_i \mathrm{d}V + \int_\Omega m_{ji,j} \dot{\varphi}_i \mathrm{d}V + \int_\Omega e_{ijk} t_{jk} \dot{\varphi}_i \mathrm{d}V + \\ &+ \int_\Omega \rho_0 M_i \dot{\varphi}_i \mathrm{d}V = \int_\Omega \rho_0 F_i \dot{u}_i \mathrm{d}V + \int_\Omega \rho_0 M_i \dot{\varphi}_i \mathrm{d}V + \int_{\partial\Omega} t_{ji} n_j \dot{u}_i \mathrm{d}A + \\ &+ \int_{\partial\Omega} m_{ji} n_j \dot{\varphi}_i \mathrm{d}A + \int_\Omega S \mathrm{d}V + \int_{\partial\Omega} q_i n_i \mathrm{d}A \end{aligned} \tag{19}$$

By doing integration by parts

$$\int_\Omega t_{ji,j} \dot{u}_i \mathrm{d}V = \int_{\partial\Omega} t_{ji} n_j \dot{u}_i \mathrm{d}A - \int_\Omega t_{ji} \dot{u}_{i,j} \mathrm{d}V \tag{20}$$

$$\int_\Omega m_{ji,j} \dot{\varphi}_i \mathrm{d}V = \int_{\partial\Omega} m_{ji} n_j \dot{\varphi}_i \mathrm{d}A - \int_\Omega m_{ji} \dot{\varphi}_{i,j} \mathrm{d}V \tag{21}$$

and by substituting (20) and (21) into (19) one obtains

$$\begin{aligned} &\int_\Omega \rho_0 \dot{e} \mathrm{d}V - \int_\Omega t_{ji} \dot{u}_{i,j} \mathrm{d}V - \int_\Omega m_{ji} \dot{\varphi}_{i,j} \mathrm{d}V + \int_\Omega e_{ijk} t_{jk} \dot{\varphi}_i \mathrm{d}V = \\ &= \int_\Omega S \mathrm{d}V + \int_{\partial\Omega} q_i n_i \mathrm{d}A \end{aligned} \tag{22}$$

which can be rewritten as

$$\int_\Omega \left(\rho_0 \dot{e} - t_{ji} \dot{u}_{i,j} - m_{ji} \dot{\varphi}_{i,j} + e_{ijk} t_{jk} \dot{\varphi}_i - \rho_0 Q - q_{i,i} \right) \mathrm{d}V = 0 \tag{23}$$

or in the pointwise form as

$$\rho_0 \dot{e} = t_{ji} \dot{u}_{i,j} + m_{ji} \dot{\varphi}_{i,j} - e_{ijk} t_{jk} \dot{\varphi}_i + \rho_0 Q + q_{i,i} \tag{24}$$

because the integrand is a continuous function and the domain of integration is arbitrary. Since by (11) $\phi = e - T\eta$ and $\dot{\phi} = \dot{e} - \dot{T}\eta - T\dot{\eta}$ one obtains by (24)

$$\rho_0 \dot{\phi} = t_{ji} \dot{u}_{i,j} + m_{ji} \dot{\varphi}_{i,j} - e_{ijk} t_{jk} \dot{\varphi}_i + \rho_0 Q + q_{i,i} - \rho_0 \dot{T} \eta - \rho_0 T \dot{\eta} \tag{25}$$

The final result is obtained by formulae (2) and by considering the strain with fractional order. □

Furthermore, the previous result is compared with another expression of the time derivative of the free energy that results from the equations of the state of a thermoelastic material. Hence one obtains the constitutive equations for the stress tensor, the couple stress tensor and the entropy.

Theorem 1. *The constitutive equations of the theory of generalized thermoelasticity with fractional order strain for micropolar materials are the following*

$$\begin{aligned} t_{ij} &= A_{ijmn}\tilde{\varepsilon}_{mn} + B_{ijmn}\gamma_{mn} - D_{ij}\theta \\ &= A_{ijmn}(1 + \tau^{\beta} D_t^{\beta})\varepsilon_{mn} + B_{ijmn}\gamma_{mn} - D_{ij}\theta \end{aligned} \tag{26}$$

$$\begin{aligned} m_{ij} &= B_{mnij}\tilde{\varepsilon}_{mn} + C_{ijmn}\gamma_{mn} - E_{ij}\theta \\ &= B_{mnij}(1 + \tau^{\beta} D_t^{\beta})\varepsilon_{mn} + C_{ijmn}\gamma_{mn} - E_{ij}\theta \end{aligned} \tag{27}$$

$$\begin{aligned} \rho_0\eta &= D_{ij}\tilde{\varepsilon}_{ij} + E_{ij}\gamma_{ij} + c\theta \\ &= D_{ij}(1 + \tau^{\beta} D_t^{\beta})\varepsilon_{ij} + E_{ij}\gamma_{ij} + c\theta \end{aligned} \tag{28}$$

Proof. One obtains from the first relation in (12) $\phi = \phi(\tilde{\varepsilon}_{ij}, \gamma_{ij}, T, T_{,i})$ and the chain rule

$$\rho_0\dot{\phi} = \rho_0\frac{\partial\phi}{\partial\tilde{\varepsilon}_{ij}}\dot{\tilde{\varepsilon}}_{ij} + \rho_0\frac{\partial\phi}{\partial\gamma_{ij}}\dot{\gamma}_{ij} + \rho_0\frac{\partial\phi}{\partial T}\dot{T} + \rho_0\frac{\partial\phi}{\partial T_{,i}}\dot{T}_{,i} \tag{29}$$

By comparing formulae (16) and (29) one obtains

$$\begin{aligned} &t_{ij} = \rho_0\frac{\partial\phi}{\partial\tilde{\varepsilon}_{ij}}, \quad m_{ij} = \rho_0\frac{\partial\phi}{\partial\gamma_{ij}} \\ &\eta = -\frac{\partial\phi}{\partial T}, \quad \frac{\partial\phi}{\partial T_{,i}} = 0 \end{aligned} \tag{30}$$

and

$$\rho_0 Q + q_{i,i} - \rho_0 T\dot{\eta} = 0 \tag{31}$$

The proof is complete if one computes the derivatives above. □

Finally, the equation of energy obtained in the previous theorem and the expression of the entropy in terms of the free energy are combined to obtain a new expression of Cattaneo's heat conduction equations.

Theorem 2. *In the context of the theory of generalized thermoelasticity with fractional order strain for micropolar materials the non-Fourier heat equations lead to*

$$\begin{aligned} (K_{ij}T_{,j})_{,i} = &\left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)\left[-T^0 D_{ij}(1 + \tau^{\beta} D_t^{\beta})\varepsilon_{ij} - E_{ij}T^0\gamma_{ij} - cT^0 T\right] + \\ &+ \rho_0\left(1 + \tau_0\frac{\partial}{\partial t}\right)Q \end{aligned} \tag{32}$$

Proof. By using the equality for η from (30) and (31) one obtains

$$-q_{i,i} = \rho_0 Q + \rho_0 T \left(\frac{\partial^2 \phi}{\partial \tilde{\varepsilon}_{ij} \partial T} \dot{\tilde{\varepsilon}}_{ij} + \frac{\partial^2 \phi}{\partial \gamma_{ij} \partial T} \dot{\gamma}_{ij} + \frac{\partial^2 \phi}{\partial T^2} \dot{T} \right) \tag{33}$$

By using equations (14) and (33), one obtains the gradient of the heat flux in the form

$$-q_{i,i} = \rho_0 Q - D_{ij} \dot{\tilde{\varepsilon}}_{ij} T - E_{ij} \dot{\gamma}_{ij} T - c\dot{T}T \tag{34}$$

Let $T \approx T^0$, where T^0 is the constant absolute temperature of the body in its reference state. Therefore, one gets

$$-q_{i,i} = \rho_0 Q - D_{ij} \dot{\tilde{\varepsilon}}_{ij} T^0 - E_{ij} T^0 \dot{\gamma}_{ij} - cT^0 \dot{T} \tag{35}$$

The non-Fourier heat equations lead to

$$\begin{aligned} &(K_{ij} T_{,j})_{,i} = -q_{i,i} - \tau_0 \dot{q}_{i,i} = \\ &= \rho_0 Q - D_{ij} \dot{\tilde{\varepsilon}}_{ij} T^0 - E_{ij} T^0 \dot{\gamma}_{ij} - cT^0 \dot{T} + \\ &+ \tau_0 \left[\rho_0 \dot{Q} - D_{ij} \ddot{\tilde{\varepsilon}}_{ij} T^0 - E_{ij} T^0 \ddot{\gamma}_{ij} - cT^0 \ddot{T} \right] \end{aligned} \tag{36}$$

□

4 Conclusions

This article constructs a new model of generalized micropolar thermoelasticity by considering the strain with fractional order. The computations revolve around the principle of conservation of energy, the expression of the free energy, the state of a thermoelastic material and Cattaneo's heat conduction equations. Even though the usual derivatives are replaced with fractional derivatives, the essence of the constitutive equations and of the non-Fourier heat equations does not change.

The author would like to thank professor Marin Marin for valuable comments. The author is also grateful to the referees for useful suggestions.

References

[1] Atangana, A., Baleanu, D., *Application of fixed point theorem for stability analysis of a nonlinear Schrödinger with Caputo-Liouville derivative*, Filomat **31** (2017), no. 8, 2243-2248.

[2] Chirilă, A., Agarwal, R. P., Marin, M., *Proving uniqueness for the solution of the problem of homogeneous and anisotropic micropolar thermoelasticity*, Boundary Value Problems **3** (2017), doi 10.1186/s13661-016-0734-0.

[3] Ciarletta, M., Ieşan, D., *Non-classical elastic solids*, Pitman research notes in mathematics series, Longman Scientific and Technical, 1993.

[4] Cosserat, E., Cosserat, F., *Sur la théorie des corps deformables*, Dunod, Paris, 1909.

[5] El-Karamany, A. S., and Ezzat, M. A., *On the three-phase-lag linear micropolar thermoelasticity theory*, European Journal of Mechanics A/Solids **40** (2013), 198-208.

[6] Eringen, A. C., *Microcontinuum field theories, I. Foundations and solids*, Springer-Verlag, New York, 1999.

[7] Hetnarski, R. B., *Thermal Stresses IV*, Amsterdam: Elsevier, 1996.

[8] Ieşan, D., *Mecanica generalizată a solidelor*, Universitatea "Al. I. Cuza", Centrul de multiplicare, Iaşi, 1980.

[9] Marin, M., *A domain of influence theorem for microstretch elastic materials*, Nonlinear analysis: RWA, **11** (2010), no. 5, 3446-3452.

[10] Marin, M., Baleanu, D., *On vibrations in thermoelasticity without energy dissipation for micropolar bodies*, Boundary Value Problems **111** (2016).

[11] Marin, M., Ellahi, R., Chirilă, A., *On solutions of Saint-Venant's problem for elastic dipolar bodies with voids*, Carpathian J. Math. **33** (2017), no. 2, 199-212.

[12] Marin, M., Lupu, M., *On harmonic vibrations in thermoelasticity of micropolar bodies*, Journal of Vibration and Control, Volume **4** (1998), no. 5, 507-518.

[13] Podlubny, I., *Fractional Differential Equations. An introduction to fractional derivatives, fractional differential equations, some methods of their solution and some of their applications*, Academic Press, 368 pages, 1999.

[14] Sharma, K., Marin, M., *Effect of distinct conductive and thermodynamic temperatures on the reflection of plane waves in micropolar elastic half-space*, U.P.B. Sci. Bull., Series A-Appl. Math. Phys., **75** (2013), no. 2, 121-132.

[15] Youssef, H., *Theory of fractional order generalized thermoelasticity*, Journal of Heat Transfer, **132** (2010), no. 6, 1-7.

[16] Youssef, H., *Theory of generalized thermoelasticity with fractional order strain*, Journal of Vibration and Control, **22** (2016), no. 18, 3840-3857.

[17] Yu, Y. J., Tian, X. G., Lu, T. J., *On fractional order generalized thermoelasticity with micromodeling*, Acta Mechanica, **224** (2013), no. 12, 2911-2927.