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STARLIKE AND CONVEX FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS ASSOCIATED WITH THE GENERALIZED STRUVE FUNCTION

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Abstract

In this paper we determine conditions for the family of Struve function in order to belong to the classes $S_s^*(\alpha)$ and $K_s(\alpha)$. Several corollaries follow as special cases.

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1 Introduction

Let $\mathcal{A}$ be the class of all functions f which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U. \tag{1.1}$$

We denote by $S_s^*(\alpha)$, $0 \leq \alpha < 1/2$ the class of functions defined by

$$S_s^*(\alpha) := \left\{ f \in \mathcal{A} : \mathrm{Re}\left[\frac{zf'(z)}{f(z) - f(-z)} \right] > \alpha,\, z \in U,\, 0 \leq \alpha < 1/2 \right\}.$$

Let $K_s(\alpha)$, $0 \leq \alpha < 1/2$ denote the class of functions defined by

$$K_s(\alpha) := \left\{ f \in \mathcal{A} : \mathrm{Re}\left[\frac{(zf'(z))'}{(f(z) - f(-z))'} \right] > \alpha,\, z \in U,\, 0 \leq \alpha < 1/2 \right\}.$$

It is well known that:
(i) $S_s^*(0) \equiv S_s^*$, where S_s^* is called the class of starlike functions with respect to

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symmetric points (see [7]).
(ii) $K_s(0) \equiv K_s$, where functions in the class K_s are called convex functions with respect to symmetric points (see [3]).
(iii) $f \in K_s(\alpha)$ if and only if $zf' \in S_s^*(\alpha)$.

Recently, in [4, 5, 9] various conditions were obtained for the Struve function in order to belong to the classes of starlike, convex, uniformly starlike and uniformly convex functions. Also, in [1] conditions for the univalence and the order of convexity of certain integral operators involving generalized Struve function were given. Motivated by these works we find sufficient conditions for the parameters of the normalized form of the generalized Struve function to belong to the classes $S_s^*(\alpha)$ and $K_s(\alpha)$.

The generalized Struve function of order p is defined by the infinite series

$$w_{p,b,c}(z) = \sum_{n\geq 0} \frac{(-1)^n c^n}{\Gamma(n+3/2)\Gamma(p+n+(b+2)/2)} \left(\frac{z}{2}\right)^{2n+p+1}, \quad z \in \mathbb{C}, \tag{1.2}$$

where $p, b, c \in \mathbb{C}$ and Γ stands for the Euler's gamma function (see [6, 8]). Next, we consider the function $u_{p,b,c}$ defined by this transformation:

$$u_{p,b,c}(z) = 2^p \sqrt{\pi}\Gamma\left(p + \frac{b+2}{2}\right) z^{(-p-1)/2} w_{p,b,c}(\sqrt{z}). \tag{1.3}$$

By using the Pochhammer symbol defined by

$$(\kappa)_n = \frac{\Gamma(\kappa+n)}{\Gamma(\kappa)} = \begin{cases} 1, & \text{if } n = 0, \\ \kappa(\kappa+1)\cdots(\kappa+n-1), & \text{if } n \in \{1, 2, \dots\} \end{cases} \tag{1.4}$$

we can express $u_{p,b,c}$ as

$$u_{p,b,c}(z) = \sum_{n\geq 0} \frac{(-c/4)^n}{(3/2)_n (k)_n} z^n, \tag{1.5}$$

where $k = p + (b+2)/2 \neq 0, -1, -2, \dots$. The function $u_{p,b,c}$ is analytic on $\mathbb{C}$ and satisfies the second order non-homogeneous linear differential equation

$$4z^2 u''(z) + 2(2p+b+3)zu'(z) + (cz + 2p + b)u(z) = 2p + b, \tag{1.6}$$

where $p, b, c \in \mathbb{C}$.

The following results will be required in our investigation. We begin with some theorems that relate the modulus of the coefficients with the order of starlikeness or convexity with respect to symmetric points.

Lemma 1.1. *(see [2]) If $f \in \mathcal{A}$ satisfies*

$$\sum_{n\geq 2} \{2(n-1)|a_{2n-2}| + (2n-1-2\alpha)|a_{2n-1}|\} \leq 1 - 2\alpha \tag{1.7}$$

for some α, $0 \leq \alpha < 1/2$, then $f \in S_s^*(\alpha)$.

Lemma 1.2. *(see [2]) If $f \in \mathcal{A}$ satisfies*

$$\sum_{n\geq 2}\{4(n-1)^2|a_{2n-2}| + (2n-1)(2n-1-2\alpha)|a_{2n-1}|\} \leq 1-2\alpha \tag{1.8}$$

for some α, $0 \leq \alpha < 1/2$, then $f \in K_s(\alpha)$.

2 Main results

Theorem 2.1. *If $0 \leq \alpha < 1/2$, $c < 0$ and $k > 0$, then a sufficient condition for $g(z) = zu_{p,b,c}(z)$ to be in the class $S_s^*(\alpha)$ is*

$$u'_{p,b,c}(1) + (1-\alpha)u_{p,b,c}(1) - \alpha u_{p,b,c}(-1) \leq 2(1-2\alpha). \tag{2.1}$$

Proof. Since

$$g_{p,b,c}(z) = zu_{p,b,c}(z) = z + \sum_{n\geq 2}\frac{(-c/4)^{n-1}}{(3/2)_{n-1}(k)_{n-1}}z^n,$$

according to Lemma 1.1, we need to show that

$$\sum_{n\geq 2}\left\{2(n-1)\frac{(-c/4)^{2n-3}}{(3/2)_{2n-3}(k)_{2n-3}} + (2n-1-2\alpha)\frac{(-c/4)^{2n-2}}{(3/2)_{2n-2}(k)_{2n-2}}\right\} \leq 1-2\alpha. \tag{2.2}$$

We notice that

$$\sum_{n\geq 2}\frac{(-c/4)^{2n-2}}{(3/2)_{2n-2}(k)_{2n-2}} = 1/2[u_{p,b,c}(1) + u_{p,b,c}(-1) - 2] \tag{2.3}$$

and

$$\sum_{n\geq 2}\frac{(-c/4)^{2n-3}}{(3/2)_{2n-3}(k)_{2n-3}} = 1/2[u_{p,b,c}(1) - u_{p,b,c}(-1)]. \tag{2.4}$$

Next, differentiating $zu_{p,b,c}(z)$ with respect to z and setting $z = 1$ and $z = -1$ respectively, we find that

$$\sum_{n\geq 2}(2n-2)\frac{(-c/4)^{2n-2}}{(3/2)_{2n-2}(k)_{2n-2}} = 1/2[u'_{p,b,c}(1) - u'_{p,b,c}(-1)] \tag{2.5}$$

and

$$\sum_{n\geq 2}(2n-3)\frac{(-c/4)^{2n-3}}{(3/2)_{2n-3}(k)_{2n-3}} = 1/2[u'_{p,b,c}(1) + u'_{p,b,c}(-1)]. \tag{2.6}$$

Thus, we have

$$\begin{aligned}
&\sum_{n\geq 2}\left\{2(n-1)\frac{(-c/4)^{2n-3}}{(3/2)_{2n-3}(k)_{2n-3}}+(2n-1-2\alpha)\frac{(-c/4)^{2n-2}}{(3/2)_{2n-2}(k)_{2n-2}}\right\}\\
&=\sum_{n\geq 2}(2n-3)\frac{(-c/4)^{2n-3}}{(3/2)_{2n-3}(k)_{2n-3}}+\sum_{n\geq 2}\frac{(-c/4)^{2n-3}}{(3/2)_{2n-3}(k)_{2n-3}}\\
&+\sum_{n\geq 2}(2n-2)\frac{(-c/4)^{2n-2}}{(3/2)_{2n-2}(k)_{2n-2}}+\sum_{n\geq 2}(1-2\alpha)\frac{(-c/4)^{2n-2}}{(3/2)_{2n-2}(k)_{2n-2}}\\
&=u'_{p,b,c}(1)+(1-\alpha)u_{p,b,c}(1)-\alpha u_{p,b,c}(-1)+2\alpha-1. \qquad (2.7)
\end{aligned}$$

Because the last expression is bounded above by $1-2\alpha$ if and only if (2.1) holds, we obtain that $zu_{p,b,c}\in S_s^*(\alpha)$. □

If we set $\alpha=0$, we obtain the following result:

Corollary 2.1. *If $c<0$ and $k>0$, then a sufficient condition for $g(z)=zu_{p,b,c}(z)$ to be in the class S_s^* is*

$$u'_{p,b,c}(1)+u_{p,b,c}(1)\leq 2. \qquad (2.8)$$

Theorem 2.2. *If $0\leq\alpha<1/2$, $c<0$ and $k>0$, then a sufficient condition for $g(z)=zu_{p,b,c}(z)$ to be in the class $K_s(\alpha)$ is*

$$\begin{aligned}
&u''_{p,b,c}(1)+(3-\alpha)u'_{p,b,c}(1)+\alpha u'_{p,b,c}(-1)\\
&+(1-\alpha)u_{p,b,c}(1)-\alpha u_{p,b,c}(-1)\leq 2(1-2\alpha). \qquad (2.9)
\end{aligned}$$

Proof. By virtue of Lemma 1.2, it is sufficient to show that

$$\begin{aligned}
&\sum_{n\geq 2}\{4(n-1)^2\frac{(-c/4)^{2n-3}}{(3/2)_{2n-3}(k)_{2n-3}}\\
&+(2n-1)(2n-1-2\alpha)\frac{(-c/4)^{2n-2}}{(3/2)_{2n-2}(k)_{2n-2}}\}\leq 1-2\alpha. \qquad (2.10)
\end{aligned}$$

Differentiating $zu'_{p,b,c}(z)$ with respect to z and taking $z=1$ and $z=-1$ respectively, we get

$$\sum_{n\geq 2}(2n-3)(2n-2)\frac{(-c/4)^{2n-2}}{(3/2)_{2n-2}(k)_{2n-2}}=1/2[u''_{p,b,c}(1)+u''_{p,b,c}(-1)] \qquad (2.11)$$

and

$$\sum_{n\geq 2}(2n-3)(2n-4)\frac{(-c/4)^{2n-3}}{(3/2)_{2n-3}(k)_{2n-3}}=1/2[u''_{p,b,c}(1)-u''_{p,b,c}(-1)]. \qquad (2.12)$$

Now, if we write $4(n-1)^2 = (2n-3)(2n-4)+6n-8$ and $(2n-1)(2n-1-2\alpha) = (2n-2)(2n-3)+(3-2\alpha)(2n-2)+(1-2\alpha)$, after some computation, we obtain

$$\sum_{n\geq 2}\{4(n-1)^2\frac{(-c/4)^{2n-3}}{(3/2)_{2n-3}(k)_{2n-3}} + (2n-1)(2n-1-2\alpha)$$
$$\cdot\frac{(-c/4)^{2n-2}}{(3/2)_{2n-2}(k)_{2n-2}}\} = u''_{p,b,c}(1) + (3-\alpha)u'_{p,b,c}(1) + \alpha u'_{p,b,c}(-1)$$
$$+ (1-\alpha)u_{p,b,c}(1) - \alpha u_{p,b,c}(-1) + 2\alpha - 1. \tag{2.13}$$

Since the last sum is bounded above by $1-2\alpha$ if and only if (2.9) holds, we obtain that $zu_{p,b,c} \in K_s(\alpha)$. □

If we set $\alpha = 0$, we obtain the following result:

Corollary 2.2. *If $c < 0$ and $k > 0$, then a sufficient condition for $zu_{p,b,c}$ to be in the class K_s is*

$$u''_{p,b,c}(1) + 3u'_{p,b,c}(1) + u_{p,b,c}(1) \leq 2. \tag{2.14}$$

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