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RICCI SOLITON, CONFORMAL RICCI SOLITON AND TORQUED VECTOR FIELDS

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Abstract

In this paper we discussed the torqued vector field and did some applications of torqued vector field on conformal Ricci soliton. We proved that if the potential field of a conformal Ricci soliton is a torqued vector field τ, then (M, g) is an almost quasi-Einstein manifold. We also dealt with multiply product. We showed that a conformal Ricci soliton with torqued potential field τ is Einstein if and only if τ is concircular vector field. We also studied Ricci soliton on multiply warped product space admitting canonical torqued potential field.

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1 Introduction

Definition 1. *A vector field v on a Riemannian or pseudo Riemannian manifold M is called torse-forming if for any vectors $X \in \chi(M)$ it satisfies [14] [20]*

$$\nabla_X v = \phi X + \psi(X)v, \tag{1}$$

where ϕ is a function, ψ is a 1-form, ∇ is Levi-Civita connection on M.

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The vector field v is called concircular [5], [20] if the 1-form ψ vanishes identically in the previous equation. The vector field v is called concurrent [18], [21] if the 1-form ψ vanishes identically and the function $\phi = 1$. The vector field v is called recurrent if the function $\phi = 0$. Finally if $\phi = \psi = 0$, then the vector field v is called a parallel vector field.

Definition 2. *On a Riemannian or pseudo-Riemannian manifold a nowhere zero vector field τ is called a torqued vector field [5] if it satisfies*

$$\nabla_X \tau = \varphi X + \alpha(X)\tau, \qquad \alpha(\tau) = 0. \tag{2}$$

The function φ is called the torqued function and the 1-form α is called the torqued form of τ.

In 1982 Hamilton [11] introduced the concept of Ricci flow and proved its existence. This concept was developed to answer Thurston's geometric conjecture which says that each closed three manifold admits a geometric decomposition. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S \tag{3}$$

on a compact Riemannian manifold M with Riemannian metric g.

A self-similar solution to the Ricci flow [11], [19] is called a Ricci soliton [12] if it moves only by a one parameter family of diffeomorphism and scaling. The Ricci soliton equation is given by

$$\pounds_X g + 2S = 2\lambda g, \tag{4}$$

where $\pounds_X$ is the Lie derivative, S is Ricci tensor, g is Riemannian metric, X is a vector field and λ is a scalar. The Ricci soliton is said to be shrinking, steady, and expanding depending to λ being positive, zero and negetive respectively.

A. E. Fischer developed the concept of conformal Ricci flow [10] during 2003-04 which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow on M where M is considered as a smooth closed connected oriented n-manifold is defined by the equation [10]

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg \tag{5}$$

and $r(g) = -1$,
where p is a scalar non-dynamical field(time dependent scalar field), $r(g)$ is the scalar curvature of the manifold and n is the dimension of manifold.

N. Basu and A. Bhattacharyya [2] in 2015 introduced the notion of conformal Ricci soliton equation as

$$\pounds_X g + 2S = [2\lambda - (p + \frac{2}{n})]g, \tag{6}$$

where λ is constant.
The equation is the generalization of the Ricci soliton equation and we have proved that it also satisfies the conformal Ricci flow equation. Many authors like C. S. Bagewadi, G. Ingalahalli[1], S. Pahan, T. Dutta, A. Bhattacharyya [17] etc studied ricci soliton on various types of contact manifolds. T. Dutta, N. Basu and A. Bhattacharyya [8] established some results on conformal Ricci soliton also.

Definition 3. *A pseudo Riemannian manifold* (M, g) *is called a quasi-Einstein manifold if*

$$S = ag + b\alpha \otimes \alpha, \tag{7}$$

where a, b *are functions and* α *is a* 1*-form.*

Definition 4. *A pseudo Riemannian manifold* (M, g) *is called an almost quasi-Einstein manifold if*

$$S = ag + b(\beta \otimes \gamma + \gamma \otimes \beta), \tag{8}$$

where a, b *are functions and* β, γ *are* 1*-forms.*

The (singly) warped product $B \times_b F$ of two pseudo Riemannian manifolds (B, g_B) and (F, g_F) with a smooth function $b : B \to (0, \infty)$ is a product manifold of form $B \times F$ with the metric tensor $g = g_B \oplus b^2 g_F$. (B, g_B) is called the base manifold and (F, g_F) is called the fiber manifold and b is called the warping function. This concept was first introduced by Bishop and O'Neill [15] to construct examples of Riemannian manifold with negative curvature.

Multiply warped product is the generalization of (singly) warped product. A multiply warped product (M, g) is the product manifold $M = B \times_{b_1} F_1 \times_{b_2} F_2 ..... \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} ..... \oplus b_m^2 g_{F_m}$ where $i \in$

$\{1, 2, ...m\}$, $b_i : B \to (0, \infty)$ are smooth and each (F_i, g_{F_i}) is a pseudo-Riemannian manifold. When $B = (c, d)$, the metric $g_B = -dt^2$ is negative and (F_i, g_{F_i}) is a Riemannian manifold. We call M as the multiply generalized Robertson-Walker space-time.

Now we shall state two propositions from [7] which would be used to derive our main results.

Proposition 1. *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 ..... \times_{b_m} F_m$ be a pseudo-Riemannian multiply warped product with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} ..... \oplus b_m^2 g_{F_m}$ [7]. Also let $X, Y \in \chi(B)$ and $V \in \chi(F_i), W \in \chi(F_j)$, then*

$$
\begin{aligned}
1. \qquad \nabla_X Y &= \nabla^B_X Y \\
2. \qquad \nabla_X V &= \nabla_V X = \frac{X(b_i)}{b_i} V \\
3. \qquad \nabla_V W &= 0, if \qquad i \neq j \\
&= \nabla^{F_i}_V W - \frac{g(V, W)}{b_i} grad_B b_i, if \qquad i = j.
\end{aligned}
$$

Proposition 2. *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 ..... \times_{b_m} F_m$ be a pseudo-Riemannian multiply warped product with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} ..... \oplus b_m^2 g_{F_m}$ [7]. Also let $X, Y, Z \in \chi(B)$ and $V \in \chi(F_i), W \in \chi(F_j)$, then*

$$
\begin{aligned}
1. \qquad S(X, Y) &= S_B(X, Y) - \Sigma_{i=1}^m \frac{s_i}{b_i} H_B^{b_i}(X, Y) \\
2. \qquad S(X, V) &= 0 \\
3. \qquad S(V, W) &= 0, if \qquad i \neq j \\
&= \nabla^{F_i}_V W - \frac{g(V, W)}{b_i} grad_B b_i, if \qquad i = j. \\
4. \qquad S(V, W) &= Ric_{F_i}(V, W) - (\frac{\Delta_B b_i}{b_i} + (s_i - 1)\frac{||grad_B b_i||_B^2}{b_i^2} \\
&+ \sum_{k=1, k \neq i}^m s_k \frac{g_B(grad_B b_i, grad_B b_k)}{b_i b_k}) g(V, W), if \qquad i = j.
\end{aligned}
$$

2 Application of torqued vector field to conformal Ricci soliton

In this section we shall discuss about the application of torqued vector field to conformal Ricci soliton and prove some results also.

Proposition 3. *If the potential field of a conformal Ricci soliton is a torqued vector field τ, then (M, g) is an almost quasi-Einstein manifold.*

Proof. Let (M, g, ξ, λ) be a conformal Ricci soliton where the potential field ξ is a torqued vector field τ. From the definition of Lie-derivaive we have

$$(\pounds_\tau g)(X, Y) = g(\pounds_X \tau, Y) + g(X, \nabla_Y \tau). \tag{9}$$

Using (2) in (9) we get

$$\begin{aligned} (\pounds_\tau g)(X, Y) &= g(\varphi X + \alpha(X)\tau, Y) + g(X, \varphi Y + \alpha(Y)\tau) \\ &= g(\varphi X, Y) + \alpha(X) g(\tau, Y) + g(\varphi Y, X) + \alpha(Y) g(X, \tau) \\ &= 2\varphi g(X, Y) + \alpha(X) g(\tau, Y) + \alpha(Y) g(\tau, X), \end{aligned} \tag{10}$$

for any vector fields X, Y tangent to M.
Combining (6) and (10) we obtain

$$2S(X, Y) + 2\varphi g(X, Y) + \alpha(X) g(\tau, Y) + \alpha(Y) g(\tau, X) = [2\lambda - (p + \frac{2}{n})] g(X, Y),$$

which gives

$$\begin{aligned} S(X, Y) = \frac{1}{2}[2\lambda - (p + \frac{2}{n}) - 2\varphi] g(X, Y) - \frac{1}{2}\alpha(X) g(\tau, Y) \\ - \frac{1}{2}\alpha(Y) g(\tau, X). \end{aligned} \tag{11}$$

If we denote the dual 1-form of τ by γ, then from (11) we get

$$S = \frac{1}{2}[2\lambda - (p + \frac{2}{n}) - 2\varphi] g - \frac{1}{2}[\alpha \otimes \gamma + \gamma \otimes \alpha]. \tag{12}$$

Hence (M, g) is an almost quasi-Einstein manifold. □

Theorem 1. *If a Riemannian manifold M is locally a multiply warped product $I \times_{b_1} F_1 \times_{b_2} F_2 ..... \times_{b_m} F_m$ where I is an open interval, F_i's are Riemannian manifolds and b_i's are warping functions then M admits torqued vector field.*

Proof. Suppose that M is a multiply warped product with the metric g such that $M = I \times_{b_1} F_1 \times_{b_2} F_2 ..... \times_{b_m} F_m$ with the metric $g = g_I \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} ..... \oplus b_m^2 g_{F_m}$. Then using proposition 1 we have

$$\begin{aligned} \nabla_{\frac{\partial}{\partial s}} V &= \nabla_V (\frac{\partial}{\partial s}) \\ &= \frac{X(b_i)}{b_i} V \\ &= [\frac{\partial}{\partial s}(ln(b_i))] V, \end{aligned} \tag{13}$$

where s is the arc length parameter of I.
Let us put

$$v_i = b_i \frac{\partial}{\partial s}. \tag{14}$$

So from (13) and (14) we get

$$\begin{aligned} \nabla_{\frac{\partial}{\partial s}} v_i &= \nabla_{\frac{\partial}{\partial s}} (b_i \frac{\partial}{\partial s}) \\ &= (\frac{\partial b_i}{\partial s})(\frac{\partial}{\partial s}) \end{aligned} \tag{15}$$

and

$$\begin{aligned} \nabla_V v_i &= \nabla_V (b_i \frac{\partial}{\partial s}) \\ &= (V b_i) \frac{\partial}{\partial s} + b_i \nabla_V \frac{\partial}{\partial s} \\ &= (V b_i) \frac{\partial}{\partial s} + b_i (\frac{\partial}{\partial s}(ln(b_i)))V \\ &= (V b_i) \frac{\partial}{\partial s} + (\frac{\partial b_i}{\partial s})V, \qquad V \perp \frac{\partial}{\partial s}. \end{aligned} \tag{16}$$

Let us define a scalar function φ_i on M by

$$\varphi_i = \frac{\partial \rho_i}{\partial s}$$

and define 1 form α_i on M by

$$\left.\begin{aligned} \alpha_i(\tfrac{\partial}{\partial s}) &= 0, \\ \alpha_i(V) = V(ln\rho_i), & \qquad V \perp \tfrac{\partial}{\partial s}. \end{aligned}\right\} \tag{17}$$

Then from (14) to (17) we obtain

$$\begin{aligned} \nabla_X v_i &= \nabla_X (b_i \frac{\partial}{\partial s}) \\ &= \varphi_i X + \alpha_i(X) v_i \qquad \forall X \in \chi(M) \end{aligned} \tag{18}$$

and

$$\alpha_i(v_i) = 0.$$

So the multiply warped product $I \times_{b_1} F_1 \times_{b_2} F_2 ..... \times_{b_m} F_m$ admits a torqued vector field given by $v_i = b_i \frac{\partial}{\partial s}$. □

Now we shall prove the following theorem:

Theorem 2. *A conformal Ricci soliton with torqued potential field τ is Einstein if and only if τ is a concircular vector field.*

Proof. Let (M, g, τ, λ) be a conformal Ricci soliton on a manifold M of dimension n with a torqued potential field τ. Let V be any vertical vector field. Then from (11) we have

$$S(\tau, V) = -\frac{1}{2}\alpha(\tau)g(\tau, V) - \frac{1}{2}\alpha(V)g(\tau, \tau). \tag{19}$$

Now from the definition of torqued vector field we have $\alpha(\tau) = 0$. So

$$S(\tau, V) = -\frac{1}{2}\alpha(V)g(\tau, \tau). \tag{20}$$

Now if M is an Einstein manifold then

$$S(\tau, V) = \lambda g(\tau, V) = 0. \tag{21}$$

Using (21) in (20) we have

$$\alpha(V)g(\tau, \tau) = 0. \tag{22}$$

Since τ is torqued potential field so τ is nowhere zero and so $g(\tau, \tau) \neq 0$ which gives

$$\alpha(V) = 0, \tag{23}$$

for any vector V orthogonal to τ.

Combining (2) and (23) we get

$$\alpha = 0. \tag{24}$$

Hence the potential field τ is a concircular vector field.

Conversely, let (M, g, τ, λ) be a conformal Ricci soliton with concircular vector field V. Then we have

$$\nabla_X V = \varphi X, \tag{25}$$

for some function φ. From the definition of Lie derivative we have

$$\begin{aligned}(\pounds_V g)(X,Y) &= g(\nabla_X V, Y) + g(\nabla_Y V, X) \\ &= g(\varphi X, Y) + g(\varphi Y, X) \\ &= 2g\varphi(X,Y), \end{aligned} \tag{26}$$

for some $X, Y \in \chi(M)$.

Now using (26) in (6) we get

$$2g\varphi(X,Y) + 2S(X,Y) = [2\lambda - (p + \frac{2}{n})]g(X,Y),$$

which gives

$$S(X,Y) = \frac{1}{2}[2\lambda - (p + \frac{2}{n}) - 2\varphi]g(X,Y). \tag{27}$$

So (M, g) is an Einstein manifold. □

3 Ricci soliton on multiply warped product space admitting canonical torqued potential field

Canonical vector field: For a multiply warped product $M = I \times_{b_1} F_1 \times_{b_2} F_2 ..... \times_{b_m} F_m$, the torqued vector field $b_i \frac{\partial}{\partial s}$ is called the canonical torqued vector field of M, where s is an arc length parameter on I. It is denoted by $\tau_{ca}^{b_i}$.

Let τ^{b_i} be arbitrary torqued vector fields associated with the multiply warped product $I \times_{b_1} F_1 \times_{b_2} F_2 ..... \times_{b_m} F_m$, then by definition of torqued vector field we get

$$\nabla_X \tau^{b_i} = \varphi_i X + \alpha(X)\tau^{b_i} \quad \text{and} \quad \alpha(\tau^{b_i}) = 0. \tag{28}$$

Since τ^{b_i} is tangent to I, we can write

$$\tau^{b_i} = \rho_i \frac{\partial}{\partial s}, \qquad \rho_i = |\tau^{b_i}|. \tag{29}$$

From (28) and (29) we get

$$\begin{aligned}\rho_i \varphi_i \frac{\partial}{\partial s} &= \nabla_{\tau^{b_i}} \tau^{b_i} \\ &= (\tau^{b_i} \rho_i)\frac{\partial}{\partial s} + \rho_i^2 \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s}. \end{aligned} \tag{30}$$

Now the metric tensor on M is given by

$$g = g_I \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} ..... \oplus b_m^2 g_{F_m}. \tag{31}$$

So from (31) and the definition of Lie derivative it follows that

$$\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} = 0 \tag{32}$$

and

$$\begin{aligned} \nabla_{\frac{\partial}{\partial s}} V &= \nabla_V \frac{\partial}{\partial s} \\ &= \frac{\partial(ln b_i)}{\partial s} V, \end{aligned} \tag{33}$$

for any vector field $V \in \chi(F_i)$. So from (30) and (32) we have

$$\varphi_i = \rho_i \frac{\partial(ln \rho_i)}{\partial s}. \tag{34}$$

Also from (28), (29) and (33) we get

$$\begin{aligned} \varphi_i V + \alpha_i(V) \tau^{b_i} &= \nabla_V \tau^{b_i} \\ &= (V \rho_i) \frac{\partial}{\partial s} + \rho_i \frac{\partial(ln b_i)}{\partial s} V, \end{aligned} \tag{35}$$

for any vector field $V \in \chi(F_i)$. So we obtain

$$\varphi_i = \rho_i \frac{\partial(ln b_i)}{\partial s}, \quad \alpha_i(V) = V(ln \rho_i). \tag{36}$$

Theorem 3. *If $(I \times_{b_1} F_1 \times_{b_2} F_2 ..... \times_{b_m} F_m, \tau_{ca}^{b_i}, \lambda)$ is a multiply warped product Ricci soliton, then the following relations hold:(where $dim(F_i) = s_i$)*

$$1. \quad (\lambda - \rho_i b_i') b_i^2 = -\sum_{i=1}^{m} \frac{s_i}{b_i} b_i''$$

$$2. \quad F_i \textit{are Einstein manifolds}, i = 1, 2, ..., m$$

$$3. \quad (\lambda - \rho_i b_i') b_i^2 - \alpha - [b_i'' b_i + (s_i - 1) b_i^2 + b_i b_i' \sum_{k=1, k \neq i}^{m} s_k \frac{b_k'}{b_k}] = 0.$$

Proof. Let $(I \times_{b_1} F_1 \times_{b_2} F_2 ..... \times_{b_m} F_m, g, \tau_{ca}^{b_i}, \lambda)$ be a multiply warped product Ricci soliton with the canonical torqued vector field $\tau_{ca}^{b_i}$ as its potential field. So from the definition of Lie derivative we get

$$(\pounds_{\tau_{ca}^{b_i}} g)(X,Y) = 2\varphi g(X,Y) + \alpha(X) g(\tau_{ca}^{b_i}, Y) + \alpha(Y) g(\tau_{ca}^{b_i}, X), \tag{37}$$

for any $X, Y \in \chi(M)$.

Using (4) in (37) we get

$$\begin{aligned} S(X,Y) &= (\lambda - \varphi_i) g(X,Y) - \frac{1}{2}\alpha_i(X) g(\tau_{ca}^{b_i}, Y) \\ &= -\frac{1}{2}\alpha_i(Y) g(\tau_{ca}^{b_i}, X). \end{aligned} \tag{38}$$

Since (2) holds for $\tau = \tau_{ca}^{b_i}$ so $\alpha_i(\tau_{ca}^{b_i}) = 0$. Also from (28) and (35) we get

$$\varphi_i = \rho_i b_i'$$

and

$$\alpha_i(V) = V(ln \rho_i).$$

So from (38) we have

$$S(\tau_{ca}^{b_i}, \tau_{ca}^{b_i}) = (\lambda - \rho_i b_i') b_i^2. \tag{39}$$

Also

$$S(\tau_{ca}^{b_i}, V) = -\frac{V(b_i^2)}{4} \tag{40}$$

and

$$S(V,W) = (\lambda - \rho_i b_i') g(V,W). \tag{41}$$

Again from proposition 2 we get

$$S(\tau_{ca}^{b_i}, \tau_{ca}^{b_i}) = -\sum_{i=1}^{m} \frac{s_i}{b_i} b_i ''. \tag{42}$$

Therefore from (39) and (42) it follows that

$$(\lambda - \rho_i b_i') b_i^2 = -\sum_{i=1}^{m} \frac{s_i}{b_i} b_i''. \tag{43}$$

Hence first part of the theorem is proved.

Again from [7] we have

$$\Delta b_i = -b_i'', \tag{44}$$

$$||grad\ b_i||^2 = -(b_i')^2, \tag{45}$$

$$g_B(grad\ b_i, grad\ b_i) = -(b_i')^2. \tag{46}$$

Putting (44), (45), (46) in part (4) of proposition 2 we have

$$\begin{aligned} S(V,W) &= S_{F_i}(V,W) + [\frac{b_i''}{b_i} + \frac{(s_i-1)}{b_i^2}(b_i')^2 \\ &+ \sum_{k=1,k\neq i}^{m} s_k s_i(\frac{b_i' b_s'}{b_i b_s})]g(V,W). \end{aligned} \tag{47}$$

So

$$\begin{aligned} (\lambda - \rho_i b_i')b_i^2 g_{F_i}(V,W) &= S_{F_i}(V,W) + [\frac{b_i''}{b_i} + \frac{(s_i-1)}{b_i^2}(b_i')^2 \\ &+ \sum_{k=1,k\neq i}^{m} s_k s_i(\frac{b_i' b_s'}{b_i b_s})]b_i^2 g_{F_i}(V,W). \end{aligned} \tag{48}$$

Therefore

$$S_{F_i}(V,W) = \beta_{F_i} g_{F_i}(V,W), \tag{49}$$

where

$$\begin{aligned} \beta_{F_i} &= [(\lambda - \rho_i b_i')b_i^2 - [\frac{b_i''}{b_i} + \frac{(s_i-1)}{b_i^2}(b_i')^2 \\ &+ \sum_{k=1,k\neq i}^{m} s_k s_i(\frac{b_i' b_s'}{b_i b_s})]b_i^2]. \end{aligned} \tag{50}$$

Hence second part of the theorem is proved.

Also from (50) we get

$$\beta_{F_i} - [(\lambda - \rho_i b_i')b_i^2 + [\frac{b_i''}{b_i} + \frac{(s_i-1)}{b_i^2}(b_i')^2 + \sum_{k=1,k\neq i}^{m} s_k s_i(\frac{b_i' b_s'}{b_i b_s})]b_i^2] = 0.$$

So third part of the theorem is proved. □

Since conformal Ricci soliton is the generalization of Ricci soliton, by using equation (6) and theorem 3 we can derive the following corollary:

Corollary 1. *If* $(I \times_{b_1} F_1 \times_{b_2} F_2 ..... \times_{b_m} F_m, \tau_{ca}^{b_i}, \lambda)$ *is a multiply warped product conformal Ricci soliton then the following relations hold:(where* $dim(F_i) = s_i$*)*

$$1. \quad (\frac{1}{2}[2\lambda - (p + \frac{2}{n})] - \rho_i b_i')b_i^2 = -\sum_{i=1}^{m} \frac{s_i}{b_i} b_i''$$

$$2. \quad F_i \quad \textit{are Einstein manifolds}, i = 1, 2, ..., m$$

$$3. \quad (\frac{1}{2}[2\lambda - (p + \frac{2}{n})] - \rho_i b_i')b_i^2 - \alpha - [b_i'' b_i + (s_i - 1)b_i^2 + b_i b_i' \sum_{k=1, k\neq i}^{m} s_k \frac{b_k'}{b_k}] = 0.$$

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