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UNIVALENCE CRITERIA FOR A GENERAL INTEGRAL OPERATOR

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Abstract

In this work we consider a general integral operator and we derive conditions for the univalence of this integral operator.

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1 Introduction

Let $\mathcal{A}$ be the class of the functions f which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathcal{U}$.

Let $\mathcal{P}$ denote the class of functions p which are analytic in $\mathcal{U}$, $p(0) = 1$ and $Re\ p(z) > 0$, for all $z \in \mathcal{U}$.

In [7] Pescar introduced a general integral operator

$$I_n(z) = \left[\delta \int_0^z u^{\delta-1} \left(\frac{f_1(u)}{g_1(u)}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{g_n(u)}\right)^{\alpha_n} \left(\frac{f_1'(u)}{g_1'(u)}\right)^{\beta_1} \cdots \left(\frac{f_n'(u)}{g_n'(u)}\right)^{\beta_n} du\right]^{\frac{1}{\delta}}, \tag{1}$$

for $f_j,\ g_j \in \mathcal{A}$ and complex numbers $\delta,\ \alpha_j,\ \beta_j\ (\delta \neq 0)$, $j = \overline{1, n}$, $n \in \mathbb{N} - \{0\}$.

We have the next particular cases:

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1. From (1), for $\delta = \beta$, $\alpha_1 = \gamma$, $\alpha_i = 0$, $i = \overline{2,n}$, $\beta_i = 0$, $i = \overline{1,n}$, and $g_1(z) = z$, $h(z) \equiv f_1(z)$, $z \in \mathcal{U}$, we obtain the integral operator

$$J_{\beta,\gamma}(z) = \left[\beta \int_0^z u^{\beta-1} \left(\frac{h(u)}{u}\right)^\gamma du\right]^{\frac{1}{\beta}}, \qquad (2)$$

which was defined by Pascu and Pescar [6], in the year 1990.

2. For $\beta_1 = \beta_2 = \cdots = \beta_n = 0$, from (1) we have the integral operator

$$G_n(z) = \left[\delta \int_0^z u^{\delta-1} \left(\frac{f_1(u)}{g_1(u)}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{g_n(u)}\right)^{\alpha_n} du\right]^{\frac{1}{\delta}}, \qquad (3)$$

for δ a complex number, $\delta \neq 0$ and $f_j,\ g_j \in \mathcal{A}$, $j = \overline{1,n}$, introduced by Moldoveanu, Ovesea and Pascu [4] in the year 1991.

3. For $\beta_i = 0$, $i = \overline{1,n}$ and $g_i(z) = z$, $i = \overline{1,n}$, from (1) we obtain the integral operator

$$D_n(z) = \left[\delta \int_0^z u^{\delta-1} \left(\frac{f_1(u)}{u}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{u}\right)^{\alpha_n} du\right]^{\frac{1}{\delta}}, \qquad (4)$$

α_i, δ complex numbers, $i = \overline{1,n}$, $\delta \neq 0$, defined by Breaz, D. and Breaz, N. [1], in the year 2002.

This integral operator is the particular case of the integral operator G_n, for $g_j(z) = z$, $j = \overline{1,n}$.

4. From (1), for $g_i(z) = z$, $i = \overline{1,n}$ we obtain the general integral operator

$$F_n(z) = \left[\delta \int_0^z f_1'(u) \left(\frac{f_1(u)}{u}\right)^{\alpha_1} \cdots f_n'(u) \left(\frac{f_n(u)}{u}\right)^{\alpha_n} du\right]^{\frac{1}{\delta}}, \qquad (5)$$

defined by Frasin [2], in the year 2011.

Properties of certain integral operators were studied by different authors in the following papers [8, 9, 10, 11, 12, 13].

In this paper we obtain the univalence criteria for the integral operator I_n.

2 Preliminary results

We need the following lemmas.

Lemma 1. *([5]). Let γ, δ be complex numbers, $Re\ \gamma > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1 - |z|^{2Re\ \gamma}}{Re\ \gamma} \left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \qquad (1)$$

for all $z \in \mathcal{U}$*, then for any complex number* δ*,* $Re\ \delta \geq Re\ \gamma$*, the function* F_δ *defined by*

$$F_\delta(z) = \left[\delta \int_0^z u^{\delta-1} f'(u)du\right]^{\frac{1}{\delta}} \tag{2}$$

is regular and univalent in $\mathcal{U}$*.*

Lemma 2. *(Schwarz [3]). Let* f *be the function regular in the disk* $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ *with* $|f(z)| < M$*,* M *fixed. If* $f(z)$ *has in* $z = 0$ *one zero with multiply* $\geq m$*, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m,\ (z \in \mathcal{U}_R)\,, \tag{3}$$

the equality (in the inequality (3) *for* $z \neq 0$*) can hold only if*

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ *is constant.*

3 Main results

Theorem 1. *Let* γ*,* δ*,* α_j*,* β_j *be complex numbers,* $c = Re\ \gamma > 0$*,* $j = \overline{1,n}$*,* M_{ij}*,* L_{ij} *real positive numbers,* $i = \overline{1,2}$*,* $j = \overline{1,n}$ *and* f_j*,* $g_j \in \mathcal{A}$*,* $f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots$*,* $g_j(z) = z + b_{2j}z^2 + b_{3j}z^3 + \ldots$*,* $j = \overline{1,n}$*.*

If

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \leq M_{1j}, \tag{1}$$

$$\left|\frac{zg_j'(z)}{g_j(z)} - 1\right| \leq M_{2j}, \tag{2}$$

$$\left|\frac{zf_j''(z)}{f_j'(z)}\right| \leq L_{1j}, \tag{3}$$

$$\left|\frac{zg_j''(z)}{g_j'(z)}\right| \leq L_{2j}, \tag{4}$$

for all $z \in \mathcal{U}$*,* $j = \overline{1,n}$ *and*

$$\sum_{j=1}^{n}\left[(M_{1j} + M_{2j})|\alpha_j| + (L_{1j} + L_{2j})|\beta_j|\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2}, \tag{5}$$

then for all δ *complex numbers,* $Re\ \delta \geq Re\ \gamma$*, the integral operator* I_n*, given by* (1) *is in the class* $\mathcal{S}$*.*

Proof. Let us consider the function

$$h_n(z) = \int_0^z \left(\frac{f_1(u)}{g_1(u)}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{g_n(u)}\right)^{\alpha_n} \left(\frac{f_1'(u)}{g_1'(u)}\right)^{\beta_1} \cdots \left(\frac{f_n'(u)}{g_n'(u)}\right)^{\beta_n} du, \tag{6}$$

for $f_j,\ g_j \in \mathcal{A},\ j = \overline{1,n}$.

The function h_n is regular in $\mathcal{U}$ and $h(0) = h'(0) - 1 = 0$.

We have:

$$\frac{zh_n''(z)}{h_n'(z)} = \sum_{j=1}^{n} \left[\alpha_j \left(\frac{zf_j'(z)}{f_j(z)} - \frac{zg_j'(z)}{g_j(z)}\right) + \beta_j \left(\frac{zf_j''(z)}{f_j'(z)} - \frac{zg_j''(z)}{g_j'(z)}\right)\right], \tag{7}$$

for all $z \in \mathcal{U}$, and hence, we get

$$\frac{zh_n''(z)}{h_n'(z)} = \sum_{j=1}^{n} \left\{\alpha_j \left[\left(\frac{zf_j'(z)}{f_j(z)} - 1\right) - \left(\frac{zg_j'(z)}{g_j(z)} - 1\right)\right] + \right.$$
$$\left. +\beta_j \left(\frac{zf_j''(z)}{f_j'(z)} - \frac{zg_j''(z)}{g_j'(z)}\right)\right\} \tag{8}$$

for all $z \in \mathcal{U}$.

From (8) we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le \frac{1-|z|^{2c}}{c}\left\{\sum_{j=1}^{n}\left[|\alpha_j|\left(\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| + \left|\frac{zg_j'(z)}{g_j(z)} - 1\right|\right) + \right.\right.$$
$$\left.\left. + \ |\beta_j|\left(\left|\frac{zf_j''(z)}{f_j'(z)}\right| + \left|\frac{zg_j''(z)}{g_j'(z)}\right|\right)\right]\right\} \tag{9}$$

for all $z \in \mathcal{U}$.

Applying Lemma 2 we get

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \le M_{1j}|z|, \tag{10}$$

$$\left|\frac{zg_j'(z)}{g_j(z)} - 1\right| \le M_{2j}|z|, \tag{11}$$

$$\left|\frac{zf_j''(z)}{f_j'(z)}\right| \le L_{1j}|z|, \tag{12}$$

$$\left|\frac{zg_j''(z)}{g_j'(z)}\right| \le L_{2j}|z|, \tag{13}$$

for all $z \in \mathcal{U}$, $j = \overline{1, n}$.

Using these inequalities from (9) we have

$$\begin{aligned} &\frac{1-|z|^{2c}}{c}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \leq \\ &\leq \frac{1-|z|^{2c}}{c}|z|\left\{\sum_{j=1}^{n}\left[(M_{1j}+M_{2j})|\alpha_j|+(L_{1j}+L_{2j})|\beta_j|\right]\right\} \end{aligned} \tag{14}$$

for all $z \in \mathcal{U}$.

Since

$$\max_{|z|\leq 1}\frac{(1-|z|^{2c})|z|}{c} = \frac{2}{(2c+1)^{\frac{2c+1}{2c}}},$$

from (14) we obtain

$$\begin{aligned} &\frac{1-|z|^{2c}}{c}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \leq \\ &\leq \frac{2}{(2c+1)^{\frac{2c+1}{2c}}}\left\{\sum_{j=1}^{n}\left[(M_{1j}+M_{2j})|\alpha_j|+(L_{1j}+L_{2j})|\beta_j|\right]\right\} \end{aligned}$$

and hence, by (5) we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \leq 1, \quad (z \in \mathcal{U}). \tag{15}$$

From (6) we obtain

$$h_n'(z) = \left(\frac{f_1(z)}{g_1(z)}\right)^{\alpha_1}\cdots\left(\frac{f_n(z)}{g_n(z)}\right)^{\alpha_n}\left(\frac{f_1'(z)}{g_1'(z)}\right)^{\beta_1}\cdots\left(\frac{f_n'(z)}{g_n'(z)}\right)^{\beta_n}$$

and using (15), by Lemma 1, it results that the integral operator I_n given by (1) is in the class $\mathcal{S}$. □

Corollary 1. *Let γ, α_j, β_j be complex numbers, $0 < Re\ \gamma \leq 1$, $c = Re\ \gamma$, M_{ij}, L_{ij} positive real numbers, $i = \overline{1, 2}$, $j = \overline{1, n}$ and f_j, $g_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots$, $g_j(z) = z + b_{2j}z^2 + b_{3j}z^3 + \ldots$, $j = \overline{1, n}$.*

If

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \leq M_{1j}, \tag{16}$$

$$\left|\frac{zg_j'(z)}{g_j(z)} - 1\right| \leq M_{2j}, \tag{17}$$

$$\left|\frac{zf_j''(z)}{f_j'(z)}\right| \le L_{1j}, \tag{18}$$

$$\left|\frac{zg_j''(z)}{g_j'(z)}\right| \le L_{2j}, \tag{19}$$

for all $z \in \mathcal{U}$, $j = \overline{1,n}$ *and*

$$\sum_{j=1}^{n}\left[(M_{1j}+M_{2j})|\alpha_j| + (L_{1j}+L_{2j})|\beta_j|\right] \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2}, \tag{20}$$

then the integral operator T_n *defined by*

$$T_n(z) = \int_0^z \left(\frac{f_1(u)}{g_1(u)}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{g_n(u)}\right)^{\alpha_n} \left(\frac{f_1'(u)}{g_1'(u)}\right)^{\beta_1} \cdots \left(\frac{f_n'(u)}{g_n'(u)}\right)^{\beta_n} du, \tag{21}$$

is in the class $\mathcal{S}$.

Proof. For $\delta = 1$, from Theorem 1, we obtain Corollary 1. □

Corollary 2. *Let* γ, α_j *be complex numbers,* $0 < Re\ \gamma \le 1$, $c = Re\ \gamma$, M_{ij} *positive real numbers,* $i = \overline{1,2}$, $j = \overline{1,n}$ *and* $f_j,\ g_j \in \mathcal{A}$,
$f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots$, $g_j(z) = z + b_{2j}z^2 + b_{3j}z^3 + \ldots$, $j = \overline{1,n}$.

If

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \le M_{1j}, \tag{22}$$

$$\left|\frac{zg_j'(z)}{g_j(z)} - 1\right| \le M_{2j}, \tag{23}$$

for all $z \in \mathcal{U}$, $j = \overline{1,n}$ *and*

$$\sum_{j=1}^{n}(M_{1j}+M_{2j})|\alpha_j| \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2}, \tag{24}$$

then the integral operator H_n *defined by*

$$H_n(z) = \int_0^z \left(\frac{f_1(u)}{g_1(u)}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{g_n(u)}\right)^{\alpha_n} du, \tag{25}$$

is in the class $\mathcal{S}$.

Proof. For $\delta = 1$ and $\beta_1 = \beta_2 = \cdots = \beta_n = 0$, from Theorem 1, we have the Corollary 2. □

Corollary 3. *Let γ, β_j be complex numbers, $0 < Re\ \gamma \leq 1$, $c = Re\ \gamma$, L_{ij} positive real numbers, $i = \overline{1,2}$, $j = \overline{1,n}$ and $f_j,\ g_j \in \mathcal{A}$,*
$f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots$, $g_j(z) = z + b_{2j}z^2 + b_{3j}z^3 + \ldots$, $j = \overline{1,n}$.

If

$$\left|\frac{zf_j''(z)}{f_j'(z)}\right| \leq L_{1j}, \tag{26}$$

$$\left|\frac{zg_j''(z)}{g_j'(z)}\right| \leq L_{2j}, \tag{27}$$

for all $z \in \mathcal{U}$, $j = \overline{1,n}$ and

$$\sum_{j=1}^{n}(L_{1j} + L_{2j})|\beta_j| \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2}, \tag{28}$$

then the integral operator K_n defined by

$$K_n(z) = \int_0^z \left(\frac{f_1'(u)}{g_1'(u)}\right)^{\beta_1} \cdots \left(\frac{f_n'(u)}{g_n'(u)}\right)^{\beta_n} du, \tag{29}$$

is in the class $\mathcal{S}$.

Proof. We take $\delta = 1$, $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, in Theorem 1. □

Corollary 4. *Let γ, δ, α_j, be complex numbers, $c = Re\ \gamma > 0$, $Re\ \delta \geq Re\ \gamma$, $j = \overline{1,n}$, M_{ij} real positive numbers, $i = \overline{1,2}$, $j = \overline{1,n}$ and $f_j,\ g_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots$, $g_j(z) = z + b_{2j}z^2 + b_{3j}z^3 + \ldots$, $j = \overline{1,n}$.*

If

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \leq M_{1j}, \tag{30}$$

$$\left|\frac{zg_j'(z)}{g_j(z)} - 1\right| \leq M_{2j}, \tag{31}$$

for all $z \in \mathcal{U}$, $j = \overline{1,n}$ and

$$\sum_{j=1}^{n}\left[(M_{1j} + M_{2j})|\alpha_j|\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2}, \tag{32}$$

then the integral operator G_n defined by

$$G_n(z) = \left[\delta \int_0^z u^{\delta-1} \left(\frac{f_1(u)}{g_1(u)}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{g_n(u)}\right)^{\alpha_n} du\right]^{\frac{1}{\delta}}, \tag{33}$$

is in the class $\mathcal{S}$.

Proof. For $\beta_1 = \beta_2 = \cdots = \beta_n = 0$, from Theorem 1, we obtain Corollary 4. $\square$

Corollary 5. *Let γ, δ, β_j be complex numbers, $c = Re\ \gamma > 0$, $Re\ \delta \geq Re\ \gamma$, $j = \overline{1,n}$, L_{ij} real positive numbers, $i = \overline{1,2}$, $j = \overline{1,n}$ and $f_j,\ g_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots$, $g_j(z) = z + b_{2j}z^2 + b_{3j}z^3 + \ldots$, $j = \overline{1,n}$.*

If

$$\left| \frac{z f_j''(z)}{f_j'(z)} \right| \leq L_{1j}, \tag{34}$$

$$\left| \frac{z g_j''(z)}{g_j'(z)} \right| \leq L_{2j}, \tag{35}$$

for all $z \in \mathcal{U}$, $j = \overline{1,n}$ and

$$\sum_{j=1}^{n} (L_{1j} + L_{2j})|\beta_j| \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2}, \tag{36}$$

then the integral operator Q_n defined by

$$Q_n(z) = \left[\delta \int_0^z u^{\delta-1} \left(\frac{f_1'(u)}{g_1'(u)} \right)^{\beta_1} \cdots \left(\frac{f_n'(u)}{g_n'(u)} \right)^{\beta_n} du \right]^{\frac{1}{\delta}}, \tag{37}$$

is in the class $\mathcal{S}$.

Proof. We take $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ in Theorem 1. $\square$

Theorem 2. *Let γ, α_j, β_j be complex numbers, $j = \overline{1,n}$, $c = Re\ \gamma > 0$ and f_j, $g_j \in \mathcal{S}$, f_j', $g_j' \in \mathcal{P}$, $f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots$, $g_j(z) = z + b_{2j}z^2 + b_{3j}z^3 + \ldots$, $j = \overline{1,n}$.*

If

$$2\sum_{j=1}^{n} |\alpha_j| + \sum_{j=1}^{n} |\beta_j| \leq \frac{c}{4},\ for\ 0 < c < 1 \tag{38}$$

or

$$2\sum_{j=1}^{n} |\alpha_j| + \sum_{j=1}^{n} |\beta_j| \leq \frac{1}{4},\ for\ c \geq 1 \tag{39}$$

then for any complex numbers δ, $Re\ \delta \geq c$, the integral operator I_n, defined by (1) is in the class $\mathcal{S}$.

Proof. We consider the function

$$h_n(z) = \int_0^z \left(\frac{f_1(u)}{g_1(u)}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{g_n(u)}\right)^{\alpha_n} \left(\frac{f_1'(u)}{g_1'(u)}\right)^{\beta_1} \cdots \left(\frac{f_n'(u)}{g_n'(u)}\right)^{\beta_n} du, \qquad (40)$$

for $f_j,\ g_j \in \mathcal{S},\ f_j',\ g_j' \in \mathcal{P},\ j = \overline{1,n}$. The function h_n is regular in $\mathcal{U}$ and $h_n(0) = h_n'(0) - 1 = 0$. We obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le \frac{1-|z|^{2c}}{c}\sum_{j=1}^{n}\left[|\alpha_j|\left(\left|\frac{zf_j'(z)}{f_j(z)}\right| + \left|\frac{zg_j'(z)}{g_j(z)}\right|\right) + |\beta_j|\left(\left|\frac{zf_j''(z)}{f_j'(z)}\right| + \left|\frac{zg_j''(z)}{g_j'(z)}\right|\right)\right], \qquad (41)$$

for all $z \in \mathcal{U}$. Since $f_j,\ g_j \in \mathcal{S}$ we have

$$\left|\frac{zf_j'(z)}{f_j(z)}\right| \le \frac{1+|z|}{1-|z|}, \qquad (42)$$

$$\left|\frac{zg_j'(z)}{g_j(z)}\right| \le \frac{1+|z|}{1-|z|}, \qquad (43)$$

for all $z \in \mathcal{U}$, $j = \overline{1,n}$.

For $f_j',\ g_j' \in \mathcal{P}$ we have

$$\left|\frac{zf_j''(z)}{f_j'(z)}\right| \le \frac{2|z|}{1-|z|^2}, \qquad (44)$$

$$\left|\frac{zg_j''(z)}{g_j'(z)}\right| \le \frac{2|z|}{1-|z|^2}, \qquad (45)$$

for all $z \in \mathcal{U}$, $j = \overline{1,n}$.

Using these relations we get

$$\frac{1-|z|^{2c}}{c}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le \frac{1-|z|^{2c}}{c}\,\frac{2(1+|z|)}{1-|z|}\sum_{j=1}^{n}|\alpha_j| + \frac{1-|z|^{2c}}{c}\,\frac{4|z|}{1-|z|^2}\sum_{j=1}^{n}|\beta_j|, \qquad (46)$$

for all $z \in \mathcal{U}$.

For $0 < c < 1$, we have $1 - |z|^{2c} \le 1 - |z|^2$, $z \in \mathcal{U}$ and by (46) we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le \frac{8}{c}\sum_{j=1}^{n}|\alpha_j| + \frac{4}{c}\sum_{j=1}^{n}|\beta_j|, \quad (z \in \mathcal{U}). \qquad (47)$$

From (38) and (47) we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \leq 1 \tag{48}$$

for all $z \in \mathcal{U}$ and $0 < c \leq 1$.

For $c > 1$, we have $\frac{1-|z|^{2c}}{c} \leq 1-|z|^2$, $z \in \mathcal{U}$ and using (46) we get

$$\frac{1-|z|^{2c}}{c}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \leq 8\sum_{j=1}^{n}|\alpha_j| + 4\sum_{j=1}^{n}|\beta_j|, \tag{49}$$

for all $z \in \mathcal{U}$, $c \geq 1$.

From (39) and (49) we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \leq 1 \tag{50}$$

for all $z \in \mathcal{U}$, $c \geq 1$.

Using (40) we have

$$h_n'(z) = \left(\frac{f_1(u)}{g_1(u)}\right)^{\alpha_1} \cdots \left(\frac{f_n(u)}{g_n(u)}\right)^{\alpha_n} \left(\frac{f_1'(u)}{g_1'(u)}\right)^{\beta_1} \cdots \left(\frac{f_n'(u)}{g_n'(u)}\right)^{\beta_n},$$

and by (48), (50) and Lemma 1 it results that the integral operator I_n, given by (1), is in the class $\mathcal{S}$. □

Corollary 6. *Let γ, α_j, β_j be complex numbers, $0 < Re\ \gamma \leq 1$ and $f_j,\ g_j \in \mathcal{S}$, $f_j',\ g_j' \in \mathcal{P}$, $f_j(z) = z + a_{2j}z^2 + \ldots$, $g_j(z) = z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$2\sum_{j=1}^{n}|\alpha_j| + \sum_{j=1}^{n}|\beta_j| \leq \frac{Re\ \gamma}{4}, \tag{51}$$

then the integral operator T_n defined by (21) *belongs to the class $\mathcal{S}$.*

Proof. We take $\delta = 1$ in Theorem 2. □

Corollary 7. *Let γ, α_j, be complex numbers, $j = \overline{1,n}$, $0 < Re\ \gamma \leq 1$, $f_j,\ g_j \in \mathcal{S}$, $f_j(z) = z + a_{2j}z^2 + \ldots$, $g_j(z) = z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\sum_{j=1}^{n}|\alpha_j| \leq \frac{Re\ \gamma}{8}, \tag{52}$$

then the integral operator H_n given by (25) *is in the class $\mathcal{S}$.*

Proof. For $\delta = 1$ and $\beta_1 = \beta_2 = \cdots = \beta_n = 0$, from Theorem 2 we obtain Corollary 7. □

Corollary 8. *Let γ, β_j be complex numbers, $j = \overline{1,n}$, $0 < Re\ \gamma \leq 1$ and $f_j,\ g_j \in \mathcal{S}$, $f_j',\ g_j' \in \mathcal{P}$, $f_j(z) = z + a_{2j}z^2 + \ldots$, $g_j(z) = z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\sum_{j=1}^{n} |\beta_j| \leq \frac{Re\ \gamma}{4}, \tag{53}$$

then the integral operator K_n defined by (29) *belongs to the class $\mathcal{S}$.*

Proof. We take $\delta = 1$ and $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ in Theorem 2. □

Corollary 9. *Let γ, δ, α_j be complex numbers, $c = Re\ \gamma > 0$, $Re\ \delta \geq Re\ \gamma$, $j = \overline{1,n}$, $f_j,\ g_j \in \mathcal{S}$, $f_j(z) = z + a_{2j}z^2 + \ldots$, $g_j(z) = z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\sum_{j=1}^{n} |\alpha_j| \leq \frac{Re\ \gamma}{8},\ for\ 0 < Re\ \gamma < 1 \tag{54}$$

or

$$\sum_{j=1}^{n} |\alpha_j| \leq \frac{1}{8},\ for\ Re\ \gamma \geq 1 \tag{55}$$

then the integral operator G_n defined by (33) *is in the class $\mathcal{S}$.*

Proof. For $\beta_1 = \beta_2 = \cdots = \beta_n = 0$ in Theorem 2 we obtain the Corollary 9. □

Corollary 10. *Let γ, δ, β_j be complex numbers, $c = Re\ \gamma > 0$, $Re\ \delta \geq Re\ \gamma$, $j = \overline{1,n}$ and $f_j,\ g_j \in \mathcal{S}$, $f_j',\ g_j' \in \mathcal{P}$, $f_j(z) = z + a_{2j}z^2 + \ldots$, $g_j(z) = z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\sum_{j=1}^{n} |\beta_j| \leq \frac{Re\ \gamma}{4},\ for\ 0 < Re\ \gamma \leq 1 \tag{56}$$

or

$$\sum_{j=1}^{n} |\beta_j| \leq \frac{1}{4},\ for\ Re\ \gamma > 1 \tag{57}$$

then the integral operator Q_n defined by (37) *belongs to the class $\mathcal{S}$.*

Proof. For $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ in Theorem 2 we obtain the Corollary 10. □

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