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# RATIONAL FUNCTIAN AND DIFFERENTIAL POLYNOMIAL OF A MEROMORPHIC FUNCTION SHARING A SMALL FUNCTION

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**Abstract**

In this paper we have mainly dealt with the relation between a generalized differential polynomial and a rational function $\mathcal{R}(f)$ of a non-constant meromorphic function $f$ sharing a small function $a \equiv a(z)(\not\equiv 0, \infty)$. Our results will extend recent results in [4], [5] and [9] in the direction of Brück Conjecture. We have exhibited some examples which show that the result of this paper may or may not be true because non-constant entire functions and conditions obtained in the theorems cannot be removed. Other examples have also substantiated our certain claims.

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*Key words:* meromorphic function, derivative, small function, weighted sharing.

## 1 Introduction Definitions and Results

Throughout the paper, by meromorphic functions we will always mean meromorphic functions in the complex plane $\mathbb{C}$. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [10]. It will be convenient to let $E$ denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a non-constant meromorphic function $h$, we denote by $T(r,h)$ the Nevanlinna characteristic of $h$ and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$, as $r \longrightarrow \infty$ and $r \notin E$.

Let $f$ and $g$ be two non-constant meromorphic functions and let $a$ be a complex number. We say that $f$ and $g$ share $a$ CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that $f$ and $g$ share $a$ IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities.

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In addition, we say that $f$ and $g$ share $\infty$ CM, if $1/f$ and $1/g$ share 0 CM, and we say that $f$ and $g$ share $\infty$ IM, if $1/f$ and $1/g$ share 0 IM.

A meromorphic function $a$ is said to be a small function of $f$ provided that $T(r,a) = S(r,f)$, that is $T(r,a) = o(T(r,f))$ as $r \longrightarrow \infty$, $r \notin E$.

Throughout this paper we denote, $k^* = \begin{cases} \frac{k}{2}+1, & \text{if } k \text{ is even,} \\ \left[\frac{k}{2}\right]+2, & \text{if } k \text{ is odd.} \end{cases}$ and $\chi_m = \begin{cases} 0, & \text{if } m=0, \\ 1, & \text{if } m \geq 1. \end{cases}$

At the starting point of our discussion we present the following theorem of *Mues* and *Steinmetz* [16] proved in 1979. In 1979, *Mues* and *Steinmetz* [16] proved the following theorem.

**Theorem A.** *[16] Let $f$ be a non-constant entire function. If $f$ and $f'$ share two distinct values $a$, $b$ IM then $f' \equiv f$.*

The following result is due to *Brück* [6] who first dealt with the uniqueness problem of an entire function sharing one value with its derivative.

**Theorem B.** *[6] Let $f$ be a non-constant entire function. If $f$ and $f'$ share the value 1 CM and if $N(r,0;f') = S(r,f)$ then $\dfrac{f'-1}{f-1}$ is a nonzero constant.*

In the recent past, authors such as *Yang* [17], *Zhang* [20], *Yu* [19], *Liu-Gu* [14], *Zhang-Yang* [22] extended and generalized the results of *Brück*. In 2001 the notion of weighted sharing of values appeared in the uniqueness literature as follows.

**Definition 1.1.** *[11, 12] Let $k$ be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that $f, g$ share the value $a$ with weight $k$.*

The definition implies that if $f$, $g$ share a value $a$ with weight $k$ then $z_0$ is an $a$-point of $f$ with multiplicity $m$ $(\leq k)$ if and only if it is an $a$-point of $g$ with multiplicity $m$ $(\leq k)$ and $z_0$ is an $a$-point of $f$ with multiplicity $m$ $(> k)$ if and only if it is an $a$-point of $g$ with multiplicity $n$ $(> k)$, where $m$ is not necessarily equal to $n$.

We write $f$, $g$ share $(a,k)$ to mean that $f$, $g$ share the value $a$ with weight $k$. Clearly if $f$, $g$ share $(a,k)$, then $f$, $g$ share $(a,p)$ for any integer $p$, $0 \leq p < k$. Also, we note that $f$, $g$ share a value $a$ IM or CM if and only if $f$, $g$ share $(a,0)$ or $(a,\infty)$ respectively.

If $a$ is a small function we define that $f$ and $g$ share $a$ IM or $a$ CM or with weight $l$ accordingly as $f-a$ and $g-a$ share $(0,0)$ or $(0,\infty)$ or $(0,l)$ respectively.

Though we use the standard notations and definitions of the value distribution theory available in [10], we explain some definitions and notations which are used in the paper.

**Definition 1.2.** *[13]Let $p$ be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.*

*(i)* $N(r,a;f\mid\geq p)$ $(\overline{N}(r,a;f\mid\geq p))$*denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.*

*(ii)* $N(r,a;f\mid\leq p)$ $(\overline{N}(r,a;f\mid\leq p))$*denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.*

**Definition 1.3.** *[18] For* $a\in\mathbb{C}\cup\{\infty\}$ *and a positive integer p we denote by* $N_p(r,a;f)$ *the sum* $\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2)+\ldots+\overline{N}(r,a;f\mid\geq p)$. *Clearly* $N_1(r,a;f)=\overline{N}(r,a;f)$.

**Definition 1.4.** *For* $a\in\mathbb{C}\cup\{\infty\}$ *and a positive integer m, we denote by* $\overline{N}(r,a;f\mid g\neq a\mid\geq m)$ *the reduced counting function of those a-points of f with multiplicities* $\geq m$ *which are not the a-points of g.*

**Definition 1.5.** *[1] Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let* $z_0$ *be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by* $\overline{N}_L(r,1;f)$ *the counting function of those 1-points of f and g where* $p>q$, *by* $N_E^{1)}(r,1;f)$ *the counting function of those 1-points of f and g where* $p=q=1$ *and by* $\overline{N}_E^{(2}(r,1;f)$ *the counting function of those 1-points of f and g where* $p=q\geq 2$, *each point in these counting functions is counted only once. In the same way we can define* $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$, $\overline{N}_E^{(2}(r,1;g)$.

**Definition 1.6.** *[11, 12] Let f, g share a value* $(a,0)$ . *We denote by* $\overline{N}_*(r,a;f,g)$ *the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*
*Clearly* $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$ *and* $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

The notion of weighted sharing played an important role in connection with the further investigation of the *Brück*'s result [see [3], [13], [21], [23]]. In order to generalise and improve the results of *Yu* [19], recently in [7] *Chen-Wang-Zhang* initiate the problem of uniqueness of $f$ and $(f^n)^{(k)}$, when they share a small function.

Recently, in this direction *Banerjee-Majumder* [5] obtained the following two results which improve the results of *Chen-Wang-Zhang* [7].

**Theorem C.** *Let f be a non-constant meromorphic function, let* $k\geq 1$, $q\geq 1$, $p\geq 0$ *be integers and* $q\geq\dfrac{k}{2}+1$, *and let* $a\not\equiv 0,\infty$ *be a non-constant meromorphic small function of f. Suppose that* $f-a$ *and* $(f^q)^{(k)}-a$ *share* $(0,p)$. *If* $p=\infty$ *and*

$$\begin{aligned}&2\overline{N}(r,\infty;f)+N_2\left(r,0;(f^q)^{(k)}\right)+\overline{N}\left(r,0;(f/a)^{'}\mid f\neq 0\right)\\ <\ &(\lambda+o(1))\,T\left(r,(f^q)^{(k)}\right)\end{aligned}$$

*or if* $2 \leq p < \infty$ *and*

$$\begin{aligned} & 2\overline{N}(r,\infty;f)+N_2\left(r,0;(f^q)^{(k)}\right)+\overline{N}\left(r,0;(f/a)^{'}\mid f\neq 0\right) \\ + \quad & \overline{N}\left(r,0;(f/a)^{'}\mid f\neq 0\mid\geq l\right)<(\lambda+o(1))\,T\left(r,(f^q)^{(k)}\right) \end{aligned}$$

*or* $p = 1$ *and*

$$\begin{aligned} & 2\overline{N}(r,\infty;f)+N_2\left(r,0;(f^q)^{(k)}\right)+2\overline{N}\left(r,0;(f/a)^{'}\mid f\neq 0\right) \\ < \quad & (\lambda+o(1))\,T\left(r,(f^q)^{(k)}\right) \end{aligned}$$

*or* $p = 0$ *and*

$$\begin{aligned} & 4\overline{N}(r,\infty;f)+2N_2\left(r,0;(f^q)^{(k)}\right)+N\left(r,0;(f^q)^{(k)}\mid=1\right)+ \\ + \quad & 2\overline{N}\left(r,0;(f/a)^{'}\mid f\neq 0\right)<(\lambda+o(1))\,T\left(r,(f^q)^{(k)}\right) \end{aligned}$$

*for* $r \in I$, *where* $0 < \lambda < 1$ *then* $\dfrac{(f^q)^{(k)}-a}{f-a} = c$ *for some constant* $c \in \mathbb{C}/\{0\}$.

**Theorem D.** *Let $f$ be a non-constant meromorphic function, let $k \geq 1$, $q \geq 1$, $p \geq 0$ be integers and $q < \dfrac{k}{2}+1$, and let $a \not\equiv 0,\infty$ be a non-constant meromorphic small function of f. Suppose that $f-a$ and $(f^q)^{(k)}-a$ share $(0,p)$. If $2 \leq p < \infty$ and*

$$\begin{aligned} & 2\overline{N}(r,\infty;f)+N_2\left(r,0;(f^q)^{(k)}\right)+\overline{N}\left(r,0;(f/a)^{'}\right)+\overline{N}\left(r,0;(f/a)^{'}\mid\geq l\right) \\ & <(\lambda+o(1))\,T\left(r,(f^q)^{(k)}\right) \end{aligned}$$

*or* $p = 1$ *and*

$$\begin{aligned} & 2\overline{N}(r,\infty;f)+N_2\left(r,0;(f^q)^{(k)}\right)+2\overline{N}\left(r,0;(f/a)^{'}\right) \\ < \quad & (\lambda+o(1))\,T\left(r,(f^q)^{(k)}\right) \end{aligned}$$

*or* $p = 0$ *and*

$$\begin{aligned} & 4\overline{N}(r,\infty;f)+2N_2\left(r,0;(f^q)^{(k)}\right)+N\left(r,0;(f^q)^{(k)}\mid=1\right)+ \\ + \quad & 2\overline{N}\left(r,0;(f/a)^{'}\right)<(\lambda+o(1))\,T\left(r,(f^q)^{(k)}\right) \end{aligned}$$

*for* $r \in I$, *where* $0 < \lambda < 1$ *then,* $\dfrac{(f^q)^{(k)}-a}{f-a} = c$ *for some constant* $c \in \mathbb{C}/\{0\}$.

In this direction, very recently *Harina-Husna* [9], obtained a result as follows.

**Theorem E.** *Let $f$ be a non-constant meromorphic function and $k \geq 1$, $n \geq 1$, $m \geq 2$ and $p \geq 0$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a small meromorphic function. Suppose $f^n - a$ and $(f^{(k)})^m - a$ share $(0, p)$.*
*If $p \geq 2$ and*

$$\frac{2}{m}\overline{N}(r,\infty;f) + \frac{2}{m}\overline{N}(r,0,f^{(k)}) + N_2(r,0,(f/a)^{'}) < (\lambda + o(1))T(r,f^{(k)})$$

*or $p = 1$ and*

$$\frac{2}{m}\overline{N}(r,\infty;f) + \frac{2}{m}\overline{N}(r,0,f^{(k)}) + 2N(r,0,(f/a)^{'}) < (\lambda + o(1))T(r,f^{(k)})$$

*or $p = 0$ and*

$$\frac{4}{m}\overline{N}(r,\infty;f) + \frac{6}{m}\overline{N}(r,0,f^{(k)}) + 2\overline{N}(r,0,(f/a)^{'}) < (\lambda + o(1))T(r,f^{(k)})$$

*for $r \in I$, where $0 < \lambda < 1$ then, $\dfrac{\left(f^{(k)}\right)^m - a}{f^n - a} = c$ for some constant $c \in \mathbb{C}/\{0\}$.*

**Note 1.1.** *In the above Theorem E, the authors made a trivial mistake in the proof. Actually in the Theorem 1.1 [9], the last term on the left hand side of each of the inequalities (7), (8) and (9) a factor $\dfrac{1}{m}$ should be multiplied.*

For further extension and improvement of all the above mentioned theorems to a large extent, we recall the following well known definition.

**Definition 1.7.** *[4] Let $n_{0j}, n_{1j}, \ldots, n_{kj}$ be non-negative integers. Also let $g = f^q$.*
- *The expression $\mathcal{M}_j[g] = (g)^{n_{0j}}(g')^{n_{1j}} \ldots (g^{(k)})^{n_{kj}}$ is called a differential monomial generated by $g$ of degree $d(\mathcal{M}_j) = \sum_{i=0}^{k} n_{ij}$ and weight $\Gamma_{\mathcal{M}_j} = \sum_{i=0}^{k}(1+i)n_{ij}$.*
- *The sum $\mathcal{P}[g] = \sum_{j=1}^{t} b_j\mathcal{M}_j[g]$ is called a differential polynomial generated by $g$ of degree $\overline{d}(\mathcal{P}) = \max\{d(\mathcal{M}_j) : 1 \leq j \leq t\}$ and weight $\Gamma_{\mathcal{P}} = \max\{\Gamma_{\mathcal{M}_j} : 1 \leq j \leq t\}$, where $T(r, b_j) = S(r, g)$ for $j = 1, 2, \ldots, t$.*
- *The numbers $\underline{d}(\mathcal{P}) = \min\{d(\mathcal{M}_j) : 1 \leq j \leq t\}$ and $k$ the highest order of the derivative of $g$ in $\mathcal{P}[g]$ are called respectively the lower degree and order of $\mathcal{P}[g]$.*
- *$\mathcal{P}[g]$ is called* homogeneous *if $\overline{d}(\mathcal{P}) = \underline{d}(\mathcal{P})$.*
- *$\mathcal{P}[g]$ is called a* linear differential polynomial *generated by $g$ if $\overline{d}(\mathcal{P}) = 1$. Otherwise $\mathcal{P}[g]$ is called* non-linear differential polynomial. *We denote by $Q = \max\{\Gamma_{\mathcal{M}_j} - d(\mathcal{M}_j) : 1 \leq j \leq t\}$.*

In the meantime the present authors [4], extended the above theorems to differential polynomial and elaborately studied the sharing condition under the light of weighted sharing. Below we demonstrate the theorem in [4].

**Theorem F.** *[4] Let $f$ be a non-constant meromorphic function and $n(\geq 1)$, and $p(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose further that $\mathcal{P}[f]$ is a differential polynomial generated by $f$ such that $\mathcal{P}[f]$ contains at least one derivative. Suppose that $f^n - a$ and $\mathcal{P}[f] - a$ share $(0, p)$. If $p = \infty$ and*

$$2\overline{N}(r,\infty;f) + N_2(r,0;\mathcal{P}[f]) + \overline{N}(r,0;(f^n/a)') < (\lambda + o(1))T(r, f^{(k)}),$$

*or $p \geq 2$ and*

$$2\overline{N}(r,\infty;f) + N_2(r,0;\mathcal{P}[f]) + N_2(r,0;(f^n/a)') < (\lambda + o(1))T(r, f^{(k)},$$

*or $p = 1$ and*

$$\begin{aligned} & 2\overline{N}(r,\infty;f) + N_2(r,0;\mathcal{P}[f]) + \overline{N}(r,0;(f^n/a)') + \overline{N}(r,0;(f^n/a)'|(f^n/a) \neq 0) \\ < \;& (\lambda + o(1))T(r, f^{(k)}), \end{aligned}$$

*or $p = 0$ and*

$$\begin{aligned} & 4\overline{N}(r,\infty;f) + N_2(r,0;\mathcal{P}[f]) + 2\overline{N}(r,0;\mathcal{P}[f]) + \overline{N}(,0;(f^n/a)') + \\ + \;& \overline{N}(r,0;(f^n/a)'|(f^n/a) \neq 0) < (\lambda + o(1))T(r, f^{(k)}) \end{aligned}$$

*for $r \in I$, where $0 < \lambda < 1$, then $\dfrac{\mathcal{P}[f] - a}{f^n - a} = c$, for some non-zero constant $c$.*

Now since $f$ in [5] and $f^n$ in [4, 9] are both polynomials and $(f^n)^{(k)}$ in [5] and $(f^{(k)})^m$ in [9] are both special forms of a linear differential polynomial, from the above observation it will be a natural inquisition to investigate the possible answer of the following question:

**Question 1.1.** *Is it possible to replace, $f$ or $f^n$ more generally, by a non-zero rational function $\mathcal{R}(f)$ and $(f^q)^{(k)}$, $(f^{(k)})^m$ or $\mathcal{P}[f]$ by the differential polynomial $\mathcal{P}[f^q]$ in the Theorems C, D, E and F in order to get the similar conclusions?*

Henceforth we defined $\mathcal{R}(f)$ as in Lemma 2.3, $d_i$ $(1 \leq i \leq u)$ and $c_j$ $(1 \leq j \leq l)$ are the roots of the the polynomial $P_n(z) = \displaystyle\sum_{i=0}^{n} a_i z^i$ and $1 \leq u \leq n$ and $P_m(z) = \displaystyle\sum_{j=0}^{m} b_j z^j$ and $1 \leq l \leq m$ respectively, where $u$ and $l$ are two positive integers. Let $c_0 \neq c_j (j = 1, .., l)$ be a non-zero constant.

Let us define $u^* = \begin{cases} u, & \text{if none of } d_i \text{ is zero,} \\ u-1, & \text{if if one of the of } d_i \text{ is zero.} \end{cases}$ and $l^* = \begin{cases} \chi_m, & \text{if m=0,} \\ l\chi_m, & \text{if } m \geq 1. \end{cases}$

Finding out the possible answer to the Question 1.1 is the motivation of the paper. In this paper, we have obtained a combined result which improves and extends all the Theorems $A$ - $E$ by giving an affirmative answer of the above question. Actually we will place the improved version of all the above theorems under a single umbrella. The following are the main results of this paper.

**Theorem 1.1.** *Let $f$ be a non-constant meromorphic function, let $k \geq 1$, $n \geq 1$, $p \geq 0$ and $q \geq 1$ be integers such that $q \geq k^*$ and $a \not\equiv 0, \infty$ be a meromorphic small function of $f$. Let $\mathcal{P}[f^q]$ be a differential polynomial containing at least one derivative. Suppose $\mathcal{R}(f) - a$ and $\mathcal{P}[f^q] - a$ share $(0, p)$ with $\overline{N}(r, 0; (\mathcal{R}(f)/a)') \neq S(r, f)$. If $p = \infty$ and*

$$\begin{aligned} & 2\overline{N}(r,\infty;f) + \sum_{j=0}^{l^*} \chi_j \overline{N}(r, c_j; f \mid\geq 2) + \sum_{i=1}^{u^*} \overline{N}(r, d_i; f \mid\geq 2) \\ + \quad & N_2\left(r, 0; \mathcal{P}[f^q]\right) + \overline{N}\left(r, 0; \left(\mathcal{R}(f)/a\right)' \mid \mathcal{R}(f) \neq 0\right) < \left(\lambda + o(1)\right) T\left(r, \mathcal{P}[f^q]\right) \end{aligned} \tag{1}$$

*or, if $2 \leq p < \infty$ and*

$$\begin{aligned} & 2\overline{N}(r,\infty;f) + \sum_{j=0}^{l^*} \chi_j \overline{N}(r, c_j; f \mid\geq 2) + \sum_{i=1}^{u^*} \overline{N}(r, d_i; f \mid\geq 2) + N_2\left(r, 0; \mathcal{P}[f^q]\right) \\ + \quad & \overline{N}\left(r, 0; \left(\mathcal{R}(f)/a\right)' \mid \mathcal{R}(f) \neq 0\right) + \overline{N}\left(r, 0; \left(\mathcal{R}(f)/a\right)' \mid \mathcal{R}(f) \neq 0 \mid\geq p\right) \\ < \quad & \left(\lambda + o(1)\right) T\left(r, \mathcal{P}[f^q]\right) \end{aligned} \tag{2}$$

*or, if $p = 1$ and*

$$\begin{aligned} & 2\overline{N}(r,\infty;f) + \sum_{j=0}^{l^*} \chi_j \overline{N}(r, c_j; f \mid\geq 2) + \sum_{i=1}^{u^*} \overline{N}(r, d_i; f \mid\geq 2) \\ + \quad & N_2\left(r, 0; \mathcal{P}[f^q]\right) + 2\overline{N}\left(r, 0; \left(\mathcal{R}(f)/a\right)' \mid \mathcal{R}(f) \neq 0\right) \\ < \quad & \left(\lambda + o(1)\right) T\left(r, \mathcal{P}[f^q]\right) \end{aligned} \tag{3}$$

*or, if $p = 0$ and*

$$\begin{aligned} & 4\overline{N}(r,\infty;f) + \sum_{j=0}^{l^*} \chi_j \overline{N}(r, c_j; f \mid\geq 2) + \sum_{i=1}^{u^*} \overline{N}(r, d_i; f \mid\geq 2) \\ + \quad & 2N_2\left(r, 0; \mathcal{P}[f^q]\right) + N\left(r, 0; \mathcal{P}[f^q] \mid= 1\right) + 2\overline{N}\left(r, 0; \left(\mathcal{R}(f)/a\right)' \mid \mathcal{R}(f) \neq 0\right) \\ < \quad & \left(\lambda + o(1)\right) T\left(r, \mathcal{P}[f^q]\right) \end{aligned} \tag{4}$$

*for $r \in I$, where $0 < \lambda < 1$, then $\dfrac{\mathcal{P}[f^q] - a}{\mathcal{R}(f) - a} = c$ for some constant $c \in \mathbb{C}/\{0\}$.*

**Theorem 1.2.** *Let $f$ be a non-constant meromorphic function, let $k \geq 1$, $n \geq 1$, $p \geq 0$ and $q \geq 1$ be integers such that $q < k^*$ and $a \not\equiv 0, \infty$ be a meromorphic small function of $f$. Let $\mathcal{P}[f^q]$ be a differential polynomial containing at least one derivative. Suppose $\mathcal{R}(f) - a$ and $\mathcal{P}[f^q] - a$ share $(0, p)$ with $\overline{N}(r, 0; (\mathcal{R}(f)/a)') \neq S(r, f)$. If $2 \leq p < \infty$ and*

$$\begin{aligned} & 2\overline{N}(r,\infty;f) + \sum_{j=0}^{l^*} \chi_j \overline{N}(r, c_j; f \mid\geq 2) + \sum_{i=1}^{u} \overline{N}(r, d_i; f \mid\geq 2) \\ + \quad & N_2\left(r, 0; \mathcal{P}[f^q]\right) + \overline{N}\left(r, 0; \left(\mathcal{R}(f)/a\right)'\right) + \overline{N}\left(r, 0; \left(\mathcal{R}(f)/a\right)' \mid\geq p\right) \\ < \quad & \left(\lambda + o(1)\right) T\left(r, \mathcal{P}[f^q]\right) \end{aligned} \tag{5}$$

*or, if* $p=1$ *and*

$$\begin{aligned} & 2\overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f\mid\geq 2)+\sum_{i=1}^{u}\overline{N}(r,d_i;f\mid\geq 2) \\ + & N_2\left(r,0;\mathcal{P}[f^q]\right)+2\overline{N}\left(r,0;(\mathcal{R}(f)/a)'\right) \\ < & (\lambda+o(1))\,T\left(r,\mathcal{P}[f^q]\right) \end{aligned} \tag{6}$$

*or,if* $p=0$ *and*

$$\begin{aligned} & 4\overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f\mid\geq 2)+\sum_{i=1}^{u}\overline{N}(r,d_i;f\mid\geq 2) \\ + & 2N_2\left(r,0;\mathcal{P}[f^q]\right)+N\left(r,0;\mathcal{P}[f^q]\mid=1\right)+2\overline{N}\left(r,0;(\mathcal{R}(f)/a)'\right) \\ < & (\lambda+o(1))\,T\left(r,\mathcal{P}[f^q]\right) \end{aligned} \tag{7}$$

*for* $r\in I$*, where* $0<\lambda<1$*, then* $\dfrac{\mathcal{P}[f^q]-a}{\mathcal{R}(f)-a}=c$ *for some constant* $c\in\mathbb{C}/\{0\}$*.*

The following examples show that $a\neq 0$ is necessary in Theorem 1.1 and Theorem 1.2.

**Example 1.1.** *For* $n,m\in\mathbb{N}$*, let* $\mathcal{R}(f)=\dfrac{f^n}{f^m-1}$ *and* $\mathcal{P}[f^4]=\dfrac{1}{4}\left(f^4\right)'\left(f^4\right)^2$*, where* $f=e^z$*. Here we see that* $f$ *is a non-constant non-entire meromorphic function and* $q\geq k^*$ *as* $q=4$*,* $k=1$*. Clearly* $\mathcal{R}(f)=\dfrac{e^{nz}}{e^{mz}-1}$ *and* $\mathcal{P}[f^4]=e^{12z}$ *share* $(0,\infty)$*. All the conditions* (1) - (4) *in Theorem 1.1 are satisfied, but* $\dfrac{\mathcal{P}[f^4]}{\mathcal{R}(f)}=\dfrac{e^{(n-12)z}}{e^{mz}-1}\neq c$*, where* $c$ *is a non-zero constant.*

**Example 1.2.** *Let* $\mathcal{R}(f)=\dfrac{f^n}{P_m(f)}$*, where* $P_m(z)=\sum_{j=0}^{m}b_mz^m$*,* $b_mb_0\neq 0$ *and for* $n,k\in\mathbb{N}$ $\mathcal{P}[f]=\dfrac{1}{i^{5k}}\left(f^{(iv)}\right)^3\left(f^{(k)}\right)^5(f)^{n-8}$*, where* $f=e^{iz}$*. Here we see that* $f$ *is a non-constant meromorphic function and* $q<k^*$ *as* $q=1=k$*. Clearly* $\mathcal{R}(f)=\dfrac{e^{inz}}{P_m(e^{iz})}$ *and* $\mathcal{P}[f]=e^{inz}$ *share* $(0,\infty)$*. All the conditions* (5) - (7) *in Theorem 1.2 are satisfied, but* $\dfrac{\mathcal{P}[f]}{\mathcal{R}(f)}=P_m(e^{iz})\neq c$*, where* $c$ *is a non-zero constant.*

The following examples show that the conditions (1) - (7) in Theorem 1.1 and Theorem 1.2 are sufficient but not necessary.

**Example 1.3.** *Let* $\mathcal{R}(f)=f^q$ *and* $\mathcal{P}[f^q]=\dfrac{1}{qN}(f^q)'$*, where* $f=e^{Nz},N\in\mathbb{Z}-\{0\}$ *and* $q\geq 2$*. Here* $q\geq k^*$ *as* $k=1$*. Let* $a\equiv a(z)$ *be any small function for* $f$*. Then clearly* $\mathcal{R}(f)-a=e^{Nqz}-a$ *and* $\mathcal{P}[f^q]-a=e^{Nqz}-a$ *share* $(0,\infty)$ *and* $f$ *satisfies all the conditions* (1) - (4) *in Theorem 1.1. Also* $\dfrac{\mathcal{P}[f^q]-a}{\mathcal{R}(f)-a}=1$*.*

**Example 1.4.** *Let* $\mathcal{R}(f) = \dfrac{2f^2-1}{f^2}$ *and* $\mathcal{P}[f^2] = \dfrac{1}{2}(f^2)'$, *where* $f = e^z$ *and* $a(\neq 0,\infty)$ *and* $q \geq 2$. *Here* $k = 1$ *and hence* $q \geq k^*$ *and it is clear that* $\mathcal{R}(f)-1 = \dfrac{e^{2z}-1}{e^{2z}}$ *and* $\mathcal{P}[f^2]-1 = e^{2z}-1$ *share* $(0,\infty)$. *We see that all the conditions* (1) - (4) *in Theorem 1.1 are satisfied. But* $\dfrac{\mathcal{P}[f^2]-1}{\mathcal{R}(f)-1} = e^{2z} \neq c$, *where* $c$ *is a non-zero constant.*

**Example 1.5.** *Let* $\mathcal{R}(f) = (f^2-1)^2$ *and* $\mathcal{P}[f] = 4(f')^2$, *where* $f = \dfrac{e^z-1}{e^z+1}$. *Here we see that* $f$ *is a non-constant non-entire meromorphic function. Here* $q < k^*$ *as* $q = 1 = k$. *Let* $a \equiv a(z)$ *be a small function for* $f$. *Clearly* $\mathcal{R}(f)-a$ *and* $\mathcal{P}[f^q]-a$ *share* $(0,\infty)$. *But none of the conditions* (5) - (7) *in Theorem 1.2 is satisfied, although* $\dfrac{\mathcal{P}[f^q]-a}{\mathcal{R}(f)-a} = 1$.

The following examples show that Theorem 1.1 and Theorem 1.2 may or may not be valid for the condition $\overline{N}(r,0;(\mathcal{R}(f)/a)') = S(r,f)$.

**Example 1.6.** *Let* $\mathcal{R}(f) = \dfrac{2f}{f+1}$ *and* $\mathcal{P}[f^q] = \dfrac{1}{2}f' + \dfrac{1}{2}f''$, *where* $f = e^z$. *Here* $q = 1$, $k = 2$ *and hence* $q < k^*$ *and note that* $\mathcal{R}(f)-1 = \dfrac{e^z-1}{e^z+1}$ *and* $\mathcal{P}[f^q]-1 = e^z-1$ *share* $(0,\infty)$. *We see that all the conditions* (5) - (7) *in Theorem 1.2 are satisfied. But* $\dfrac{\mathcal{P}[f^q]-1}{\mathcal{R}(f)-1} = e^z+1 \neq c$, *where* $c$ *is a non-zero constant.*

**Example 1.7.** *Let* $\mathcal{R}(f) = f$ *and* $\mathcal{P}[f^q] = \dfrac{1}{2N}f' + \dfrac{1}{2N^4}f^{(4)}$, *where* $f = e^{Nz}, N \in \mathbb{Z}-\{0\}$. *Here* $q < k^*$ *as* $q = 1$ *and* $k = 4$. *Let* $a \equiv a(z)$ *be any small function for* $f$. *Then clearly* $\mathcal{R}(f)-a$ *and* $\mathcal{P}[f^q]-a$ *share* $(0,\infty)$. *We see that* $f$ *satisfies all the conditions* (5) - (7) *in Theorem 1.2. Also* $\dfrac{\mathcal{P}[f^q]-a}{\mathcal{R}(f)-a} = 1$.

**Example 1.8.** *Let* $\mathcal{R}(f) = \dfrac{f+1}{f-1}$ *and* $\mathcal{P}[f^q] = f'$, *where* $f(z) = e^z+1$. *Here* $q < k^*$ *as* $q = 1$, $k = 1$. *Also* $\mathcal{R}(f)-b = \dfrac{(1-b)e^z+2}{e^z}$ *and* $\mathcal{P}[f^q]-b = e^z-b$, *where* $b$ *is a complex number such that* $b^2-b-2 = 0$. *Then* $\mathcal{R}(f)-b$ *and* $\mathcal{P}[f^q])-b$ *share* $(0,\infty)$. *All the conditions* (5) - (7) *in Theorem 1.2 are satisfied but* $\dfrac{\mathcal{P}[f^q]-2}{\mathcal{R}(f)-2} = -e^z \neq C$, *where* $C$ *is a non-zero constant.*

Next we shall show by the following examples that all the conditions (1) - (7) in Theorem 1.1 and Theorem 1.2 cannot be removed.

**Example 1.9.** *Let* $\mathcal{R}(f) = f$ *and* $\mathcal{P}[f^q] = f'$, *where* $f = \dfrac{z}{e^{-z}+1}$. *Here* $q < k^*$ *as* $q = 1$, $k = 1$. *Then* $\mathcal{R}(f)-1 = \dfrac{z-e^{-z}-1}{e^{-z}+1}$ *and* $\mathcal{P}[f^q]-1 = \dfrac{e^{-z}(z-e^{-z}-1)}{(e^{-z}+1)^2}$.

*Therefore* $\mathcal{R}(f)-1$ *and* $\mathcal{P}[f^q]-1$ *share* $(0,\infty)$ *and none of the conditions* (5) - (7) *in Theorem 1.2 is satisfied and hence* $\dfrac{\mathcal{P}[f^q]-1}{\mathcal{R}(f)-1}=\dfrac{e^{-z}}{(e^{-z}+1)}\neq C$*, where* $C$ *is a non-zero constant.*

**Example 1.10.** *Let* $\mathcal{R}(f)=f$ *and* $\mathcal{P}[f^q]=f'$*, where* $f=\dfrac{4}{1-5e^{-2z}}$*. Here* $q<k^*$ *as* $q=1$*,* $k=1$*. Then* $\mathcal{R}(f)-2=\dfrac{2(1+5e^{-2z})}{1-5e^{-2z}}$ *and* $\mathcal{P}[f^q]-2=-\dfrac{2(1+5e^{-2z})^2}{(1-5e^{-2z})^2}$. *Therefore* $\mathcal{R}(f)-2$ *and* $\mathcal{P}[f^q]-2$ *share* $(0,0)$ *. Since the condition* (7) *in Theorem 1.2 is not satisfied and hence* $\dfrac{\mathcal{P}[f^q]-2}{\mathcal{R}(f)-2}=-\dfrac{(1+5e^{-2z})}{(1-5e^{-2z})}\neq C$*, where* $C$ *is a non-zero constant.*

## 2 Lemmas

In this section we present some lemmas which will be needed in the sequel. Let $F$, $G$ be two non-constant meromorphic functions. Henceforth we shall denote by $H$ the following function.

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \tag{8}$$

**Lemma 2.1.** *[23] Let* $f$ *be a non-constant meromorphic function and let* $p$ *and* $k$ *be two positive integers. Then*

$$N_s\left(r,0;f^{(k)}\right)\leq T\left(r,f^{(k)}\right)-T(r,f)+N_{s+k}(r,0;f)+S(r,f),$$

$$N_s\left(r,0;f^{(k)}\right)\leq N_{s+k}(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f).$$

**Lemma 2.2.** *[2] Let* $f$*,* $g$ *share* $(1,0)$*. Then*

$$\overline{N}_L(r,1;f)\leq\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r),$$

*where* $S(r)=o\{T(r)\}$ *and* $T(r)=\max\{T(r,f),T(r,g)\}$

**Lemma 2.3.** *[15] Let* $f$ *be a non-constant meromorphic function and let*

$$\mathcal{R}(f)=\frac{\sum\limits_{i=0}^{n}a_if^i}{\sum\limits_{j=0}^{m}b_jf^j}$$

*be an irreducible rational function in* $f$ *with constant coefficients* $\{a_i\}$ *and* $\{b_j\}$ *with* $a_n\neq0$ *and* $b_m\neq0$*. Then*

$$T(r,\mathcal{R}(f))=\max\{n,m\}T(r,f)+O(1).$$

**Lemma 2.4.** *Let $f$ be a meromorphic function and $P[f]$ be a differential polynomial. Then*

$$m\left(r, \frac{P[f^q]}{(f^q)^{\bar{d}(P)}}\right) \leq \left(\bar{d}(P) - \underline{d}(P)\right) m\left(r, \frac{1}{f^q}\right) + S(r, f).$$

*Proof.* The Lemma can be proven the same way as in [8]. □

**Lemma 2.5.** *Let $f$ be a meromorphic function and $\mathcal{P}[f^q]$ be a differential polynomial. Then we have*

$$\begin{aligned} & N\left(r, \infty; \frac{\mathcal{P}[f^q]}{(f^q)^{\bar{d}(\mathcal{P})}}\right) \\ \leq \; & \left(\Gamma_{\mathcal{P}} - \bar{d}(\mathcal{P})\right) \overline{N}(r, \infty; f) + \left(\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P})\right) N(r, 0; f^q| \geq k+1) \\ & + Q\overline{N}(r, 0; f^q| \geq k+1) + \bar{d}(\mathcal{P}) N(r, 0; f^q| \leq k) + S(r, f). \end{aligned}$$

*Proof.* The Lemma can be proven the same way as in the proof of [4, Lemma 2.5]. □

**Lemma 2.6.** *Let $\mathcal{P}[f^q]$ be a differential polynomial. Then*

$$T(r, \mathcal{P}[f^q]) \leq \Gamma_P T(r, f^q) + S(r, f).$$

*Proof.* The Lemma can be proven in line of the proof [4, Lemma 2.6]. □

**Lemma 2.7.** *Let $f$ be a non-constant meromorphic function and $\mathcal{P}[f^n]$ be a differential polynomial. Then $S(r, \mathcal{P}[f^q])$ can be replaced by $S(r, f)$.*

*Proof.* From Lemma 2.7 it is clear that $T(r, \mathcal{P}[f^q] = O(T(r, f))$ and so the Lemma follows. □

# 3 Proofs of the theorems

*Proof of Theorem 1.1.* Let $F = \dfrac{\mathcal{R}(f)}{a}$ and $G = \dfrac{\mathcal{P}[f^q]}{a}$. Then $F - 1 = \dfrac{\mathcal{R}(f) - a}{a}$ and $G - 1 = \dfrac{\mathcal{P}[f^q] - a}{a}$. Since $\mathcal{R}(f) - a$ and $\mathcal{P}[f^q] - a$ share $(0, p)$ it follows that $F$, $G$ share $(1, p)$ except the zeros and poles of $a$. Now we consider the following cases.

**Case 1** Let $H \not\equiv 0$.

**Subcase 1.1** Let $l \geq 1$

From (8) it can be easily calculated that the possible poles of $H$ occur at (i) multiple zeros of $F$ and $G$, (ii) those 1 points of $F$ and $G$ whose multiplicities are different related to $F$ and $G$, (iii) those common poles of $F$ and $G$ whose multiplicities are different, (iv) zeros of $F^{'}$ ($G^{'}$) which are not the zeros of $F(F-1)$ ($G(G-1)$).

Let $z_0$, a zero of $f$ with multiplicity $r \geq 2$ such that $a(z_0) \neq 0, \infty$. Then since $G$ contains at least one derivative then $z_0$ would be a zero of $G$ with multiplicity at least $2q-k$. Since $q \geq k^*$, it follows that $z_0$ will be a multiple zero of $G$ too. Since $H$ has only simple poles we get

$$\begin{aligned}
&\overline{N}(r,\infty;H) \qquad (9)\\
\leq\;& \overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f\mid\geq 2)+\overline{N}_*(r,1;F,G)+\sum_{i=1}^{u^*}\overline{N}(r,d_i;f\mid\geq 2)\\
&+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+\overline{N}(r,0;a)+\overline{N}(r,\infty;a),
\end{aligned}$$

where $\overline{N}_0(r,0;F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G^{'})$ is similarly defined. Let $z_1$ be a simple zero of $F-1$ but $a(z_1)\neq 0,\infty$. Then $z_1$ is a simple zero of $G-1$ and a zero of $H$. So

$$N(r,1;F\mid=1)\leq\overline{N}(r,0;H)+N(r,\infty;a)+N(r,0;a)\leq\overline{N}(r,\infty;H)+S(r,f).\qquad(10)$$

Hence

$$\begin{aligned}
&\overline{N}(r,1;G) \qquad (11)\\
\leq\;& N(r,1;F\mid=1)+\overline{N}(r,1;F\mid\geq 2)\\
\leq\;& \overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f\mid\geq 2)+\sum_{i=1}^{u^*}\overline{N}(r,d_i;f\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)\\
&+\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f).
\end{aligned}$$

Note that $\overline{N}(r,\infty;G)=\overline{N}(r,\infty;f)+S(r,f)$.

By the Second Fundamental Theorem and (11), we get

$$\begin{aligned}
&T(r,G) \qquad (12)\\
\leq\;& \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,1;G)-N_0(r,0;G^{'})+S(r,G)\\
\leq\;& 2\overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f\mid\geq 2)+N_2(r,0;G)+\sum_{i=1}^{u^*}\overline{N}(r,d_i;f\mid\geq 2)\\
&+\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0(r,0;F^{'})+S(r,f).
\end{aligned}$$

**Subcase 1.1.1.** While $p=\infty$, we have $\overline{N}_*(r,1;F,G)=S(r,f)$.
So we have

$$\begin{aligned}
&\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0\left(r,0;F^{'}\right) \qquad (13)\\
\leq\;& \overline{N}\left(r,0;F'\mid F\neq 0\right)+S(r,f).
\end{aligned}$$

Hence from (12) we have

$$\begin{aligned}
&T\left(r,\mathcal{P}[f^q]\right)\\
\leq\;& 2\,\overline{N}(r,\infty;f)+\sum_{j=0}^{l^*}\chi_j\overline{N}(r,c_j;f\mid\geq 2)+\sum_{i=1}^{u^*}\overline{N}(r,d_i;f\mid\geq 2)+N_2\left(r,0;\mathcal{P}[f^q]\right)\\
&+\overline{N}\left(r,0;(\mathcal{R}(f)/a)^{'}\mid\mathcal{R}(f)\neq 0\right)+S(r,f).
\end{aligned}$$

which contradicts (1).
**Subcase 1.1.2.** While $2 \leq p < \infty$, (13) changes to

$$\begin{aligned}
& \overline{N}(r,1;F \mid\geq p+1) + \overline{N}(r,1;F \mid\geq 2) + \overline{N}_0\left(r,0;F^{'}\right) \\
\leq\ & \overline{N}\left(r,0;F^{'} \mid F \neq 0 \mid\geq p\right) + \overline{N}\left(r,0;F^{'} \mid F \neq 0\right) + S(r,f).
\end{aligned}$$

So from (12) we have

$$\begin{aligned}
& T(r,\mathcal{P}[f^q]) \\
\leq\ & 2\,\overline{N}(r,\infty;f) + \sum_{j=0}^{l^*} \chi_j \overline{N}(r,c_j;f \mid\geq 2) + \sum_{i=1}^{u^*} \overline{N}(r,d_i;f \mid\geq 2) + N_2\left(r,0;\mathcal{P}[f^q]\right) \\
& + \overline{N}\left(r,0;(\mathcal{R}(f)/a)^{'} \mid \mathcal{R}(f) \neq 0\right) + \overline{N}\left(r,0;(\mathcal{R}(f)/a)^{'} \mid \mathcal{R}(f) \neq 0 \mid\geq p\right) \\
& + S(r,f),
\end{aligned}$$

which contradicts (2).
**Subcase 1.1.3.** While $p = 1$, (13) changes to

$$\begin{aligned}
& \overline{N}_*(r,1;F,G) + \overline{N}(r,1;F \mid\geq 2) + \overline{N}_0(r,0;F^{'}) \\
\leq\ & 2\,\overline{N}(r,0;F^{'} \mid F \neq 0) + S(r,f)
\end{aligned}$$

Similarly as above we have

$$\begin{aligned}
& T(r,\mathcal{P}[f^q]) \\
\leq\ & 2\,\overline{N}(r,\infty;f) + \sum_{j=0}^{l^*} \chi_j \overline{N}(r,c_j;f \mid\geq 2) + \sum_{i=1}^{u^*} \overline{N}(r,d_i;f \mid\geq 2) + N_2(r,0;\mathcal{P}[f^q]) \\
& + 2\,\overline{N}(r,0;\mathcal{R}(f)^{'} \mid \mathcal{R}(f) \neq 0) + S(r,f),
\end{aligned}$$

which contradicts (3)
**Subcase 1.2** Let $p = 0$.
Here proceeding in the same way as in [4, Subcase 1.2, Proof of Theorem 1.1], we obtain

$$\begin{aligned}
& T(r,G) \\
\leq\ & 4\overline{N}(r,\infty;f) + \sum_{j=0}^{l^*} \chi_j \overline{N}(r,c_j;f \mid\geq 2) + \sum_{i=1}^{u^*} \overline{N}(r,d_i;f \mid\geq 2) + 2N_2(r,0;G) \\
& + N\left(r,0;G \mid= 1\right) + 2\,\overline{N}\left(r,0;F^{'} \mid F \neq 0\right) + S(r,f).
\end{aligned}$$

i.e.,

$$\begin{aligned}
& T(r,\mathcal{P}[f^q]) \\
\leq\ & 4\,\overline{N}(r,\infty;f) + \sum_{j=0}^{l^*} \chi_j \overline{N}(r,c_j;f \mid\geq 2) + \sum_{i=1}^{u^*} \overline{N}(r,d_i;f \mid\geq 2) + 2\,N_2(r,0;\mathcal{P}[f^q]) \\
& + N\left(r,0;\mathcal{P}[f^q] \mid= 1\right) + 2\,\overline{N}(r,0;(\mathcal{R}/a)^{'} \mid \mathcal{R}(f) \neq 0) + S(r,f).
\end{aligned}$$

This contradicts (4).
**Case 2** Let $H \equiv 0$.
On integration we get from (8)

$$\frac{1}{F-1} \equiv \frac{C}{G-1} + D, \tag{14}$$

where $C$, $D$ are constants and $C \neq 0$. We will prove that $D = 0$.
**Subcase 1.2.a.** Let $D \neq 0$.
**Subcase 1.2.a.1.** Suppose $n > m$. If $z_0$ is a pole of $f$ with multiplicity $r$ such that $a(z_0) \neq 0, \infty$, then it is a pole of $F$ and $G$ of multiplicities $nr - mr$ and $nr + k$ respectively. This contradicts (14).
**Subcase 1.2.a.2.** Suppose $n = m$. If $z_0$ is a pole of $f$ with multiplicity $r$ such that $a(z_0) \neq 0, \infty$, then it is not pole of $F$ but of $G$ of multiplicity $nr + k$. This contradicts (14) again.
**Subcase 1.2.a.3.** Suppose $n < m$. If $z_0$ is a pole of $f$ with multiplicity $r$ such that $a(z_0) \neq 0, \infty$, then it is a zero of $F$ but a pole $G$ of multiplicities $nr + k$. This contradicts (14) again.
**Subcase 1.2.a.4.** if there exist some $c_j$, $j = 1, 2, \ldots, m$ points of $f$, then that would be a pole of $F$ but not of $G$ this again contradicts (14).

Then it follows that

$$N(r, \infty; f) \leq \overline{N}(r, 0; a) + \overline{N}(r, \infty; a) = S(r, f).$$

So from (14) we get

$$\frac{1}{F-1} = \frac{D\left(G - 1 + \dfrac{C}{D}\right)}{G-1}. \tag{15}$$

Clearly

$$\overline{N}\left(r, 1 - \frac{C}{D}; G\right) = \overline{N}(r, \infty; F) + S(r, f). \tag{16}$$

**Subcase 1.2.a.5.** When $n > m$, then

$$\overline{N}(r, \infty; F) \leq \overline{N}(r, \infty; f) + \sum_{j=0}^{l^*} \chi_j \overline{N}(r, c_j; f) + S(r, f). \tag{17}$$

**Subcase 1.2.a.6.** When $n = m$ or $n < m$, then

$$\overline{N}(r, \infty; F) \leq \sum_{j=0}^{l^*} \chi_j \overline{N}(r, c_j; f) + S(r, f). \tag{18}$$

**Subcase 1.2.a.7.** If $\dfrac{C}{D} \neq 1$, by the Second Fundamental Theorem and (16) and (17) or (18), we have

$$\begin{aligned} T(r, G) &\leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}\left(r, 1 - \frac{C}{D}; G\right) + S(r, G) \\ &\leq N_2(r, 0; G) + \sum_{j=0}^{l^*} \chi_j \overline{N}(r, c_j; f) + S(r, f). \end{aligned}$$

i.e.,

$$T(\mathcal{P}[f^q]) \leq N_2(r,0;\mathcal{P}[f^q]) + \sum_{j=0}^{l^*} \chi_j \overline{N}(r,c_j;f) + S(r,f),$$

which contradicts (1) - (4).

**Subcase 1.2.a.7.** If $\dfrac{C}{D} = 1$, we get

$$\left(F - 1 - \frac{1}{C}\right) G \equiv -\frac{1}{C}. \tag{19}$$

From (19) it follows that

$$N(r,0;f^q| \geq k+1) \leq N(r,0;\mathcal{P}[f^q]) \leq N(r,0;G) = S(r,f). \tag{20}$$

Again from (19) we see that

$$\frac{1}{(f^q)^{\bar{d}(\mathcal{P})}\left(\mathcal{R}(f) - (1+\frac{1}{C})a\right)} \equiv -\frac{C}{a^2}\frac{\mathcal{P}[f]}{(f^q)^{\bar{d}(\mathcal{P})}}.$$

Hence by the *First Fundamental Theorem*, (20), Lemmas 2.3, 2.4 and 2.5 we get

$$\begin{aligned}
& \left(\max\{m,n\} + \bar{d}(\mathcal{P})\right) T(r,f^q) \\
= \;& T\left(r, (f^q)^{\bar{d}(\mathcal{P})}\left(\mathcal{R}(f) - (1+\frac{1}{C}a)\right)\right) + S(r,f) \\
= \;& T\left(r, \frac{1}{(f^q)^{\bar{d}(\mathcal{P})}\left(\mathcal{R}(f) - (1+\frac{1}{C}a)\right)}\right) \\
= \;& T\left(r, \frac{\mathcal{P}[f]}{(f^q)^{\bar{d}(\mathcal{P})}}\right) + S(r,f) \\
\leq \;& m\left(r, \frac{\mathcal{P}[f]}{(f^q)^{\bar{d}(\mathcal{P})}}\right) + N\left(r, \frac{\mathcal{P}[f]}{(f^q)^{\bar{d}(\mathcal{P})}}\right) + S(r,f) \\
\leq \;& \left(\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P})\right)\left[T(r,f^q) - \{N(r,0;f^q| \leq k) + N(r,0;f^q| \geq k+1)\}\right] \\
& + \left(\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P})\right) N(r,0;f^q| \geq k+1) + Q\overline{N}(r,0;f^q| \geq k+1) \\
& + \bar{d}(\mathcal{P}) N(r,0;f^q| \leq k) + S(r,f) \\
\leq \;& \left(\bar{d}(\mathcal{P}) - \underline{d}(\mathcal{P})\right) T(r,f^q) + \underline{d}(\mathcal{P}) N(r,0;f^q| \leq k) + S(r,f)
\end{aligned}$$

i.e.,

$$q\left(\max\{m,n\}\right) T(r,f) \leq S(r,f),$$

which is not possible.

Hence $D = 0$ and so $\dfrac{G-1}{F-1} = C$ i.e, $\dfrac{\mathcal{P}[f] - a}{\mathcal{R}(f) - a} = C$, where $C$ is a non-zero constant. □

*Proof of Theorem 1.2.* Let $F$ and $G$ be given as in the proof of *Theorem 1.1*. When $H \not\equiv 0$ we observe that (9) can be changed to

$$\begin{aligned} & N(r,\infty;H) \qquad (21) \\ \leq\ & \overline{N}(r,\infty;f) + \sum_{j=0}^{l^*} \chi_j \overline{N}(r,c_j;f \mid\geq 2) + \overline{N}_*(r,1;F,G) + \overline{N}(r,0;F \mid\geq 2) \\ & + \overline{N}(r,0;G \mid\geq 2) + \overline{N}_0(r,0;F^{'}) + \overline{N}_0(r,0;G^{'}) + \overline{N}(r,0;a) + \overline{N}(r,\infty;a). \end{aligned}$$

We omit the rest of the proof as that is simalar to the proof of *Theorem 1.1*.

□

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