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ON QUANTITATIVE ESTIMATE FOR THE LIMITING SEMIGROUP OF POSITIVE OPERATORS

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Abstract

In this paper we give a general quantitative estimate of approximation of the iterates of positive linear operators which preserve constants by the limiting semigroup defined by these iterates. Applications are given for Durrmeyer operators.

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Key words: positive linear operators, Trotter theorem, limiting semigroup of operators, Durrmeyer operators.

1 Introduction

The semigroup of operators generated by iterates of Bernstein operators was given by Silva [10]. More generally, as application of the Trotter's theorem [11] the semigroups generated by linear positive operators have been considered in the last decades. For a general reference to semigroups of operators we cite [1] and [2].
The quantitative version of the Trotter's theorem was obtained first in the paper by Gonska and Raşa [6]. In Minea's paper [9] this method was improved and applied to general operators which preserve linear functions. The aim of this paper is to give more general estimates for the Trotter's theorem in the case of positive linear operators which preserve only the constants. As application we consider the Durrmeyer operators.

2 Main results

Consider a sequence of positive linear operators $(L_n)_n$, $L_n : C[0,1] \to C[0,1]$, such that $L_n(e_0) = e_0$. (We denote by $e_j(t) = t^j$, $t \in [0,1]$. Denote

$$M_n^k(x) := L_n((t-x)^k, x), \quad k,n \in \mathbb{N},\ x \in [0,1].$$
$$\tilde{M}_n^k(x) := L_n(|t-x|^k, x), \quad k,n \in \mathbb{N},\ x \in [0,1].$$

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We suppose that there are functions $\varphi_1, \varphi_2, \psi_n^1, \psi_n^2 \in C[0,1]$, $n \in \mathbb{N}$, such that

$$M_n^1(x) = \frac{1}{n}\varphi_1(x) + \psi_n^1(x),\ x \in [0,1],\ n \in \mathbb{N}, \tag{1}$$

$$M_n^2(x) = \frac{1}{n}\varphi_2(x) + \psi_n^2(x),\ x \in [0,1],\ n \in \mathbb{N}, \tag{2}$$

where

$$\|\psi_n^j\| = o\Big(\frac{1}{n}\Big),\ j = 1,2,\ (n \to \infty). \tag{3}$$

We suppose that operators L_n are convex of order i, $i \geq 0$, i.e. if $f \in C^i[0,1]$, $f^{(i)} \geq 0$, then $(L_n(f))^{(i)} \geq 0$.. We suppose also, that

$$M_n^4(x) = o(M_n^2(x)),\ x \in [0,1],\ (n \to \infty). \tag{4}$$

Then, Voronovskaya's theorem assures that

$$\lim_{n\to\infty}(L_n(f,x) - f(x)) = f'(x)\varphi_1(x) + \frac{1}{2}\varphi_2(x)f''(x). \tag{5}$$

uniformly with regard to $x \in [0,1]$, for $f \in C^2[0,1]$.

Let the differential operator $A : C^2[0,1] \to C[0,1]$, given by $A(f)(x) = \varphi_1(x)f'(x) + \frac{1}{2}\varphi_2(x)f''(x)$, $f \in C^2[0,1]$, $x \in [0,1]$. Then the domain $D(A) = C^2[0,1]$ of operator A is dense in $C[0,1]$.

The Trotter's theorem assures that there exits a C_0-semigroup $T(t)$, such that

$$\lim_{n\to\infty} L_n^{m_n} f = T(t)f, f \in C[0,1] \tag{6}$$

if $\frac{m_n}{n} \to t,\ t \geq 0$.

Lemma 1. *For every $g \in C^4[0,1]$, we have*

$$\left\|L_n g - g - \frac{1}{n}Ag\right\| \leq \frac{1}{6}||\tilde{M}_n^3||\ ||g^{(3)}|| + \|\psi_n^1\| \cdot \|g'\| + \frac{1}{2}\|\psi_n^2\| \cdot \|g''\|.$$

Proof. Let $x \in [0,1]$ be fixed. For $t \in [0,1]$, with Taylor's formula, we write:

$$\left|g(t) - g(x) - (t-x)g'(x) - \frac{1}{2}(t-x)^2 g''(x)\right| \leq \frac{1}{6}|t-x|^3||g^{(3)}||$$

Since L_n reproduces the constants we obtain:

$$\left\|L_n g - g - M_n^1 g' + \frac{1}{2}M_n^2 g''\right\| \leq \frac{1}{6}||\tilde{M}_n^3||\ \cdot ||g^{(3)}||$$

Then we can write

$$\begin{aligned}\left\|L_n g - g - \frac{1}{n}Ag\right\| &= \left\|L_n g - g - \frac{1}{n}\Big(\varphi_1 g' + \frac{1}{2}\varphi_2 g''\Big)\right\| \\ &= \left\|L_n g - g - M_n^1 g' - \frac{1}{2}M_2 g''\right\| + \left\|\psi_n^1 g' + \frac{1}{2}\psi_n^2 g''\right\| \\ &\leq \frac{1}{6}||\tilde{M}_n^3||\ ||g^{(3)}|| + \|\psi_n^1\| \cdot \|g'\| + \frac{1}{2}\|\psi_n^2\| \cdot \|g''\|.\end{aligned}$$

□

Lemma 2. *For every $g \in C^4[0,1]$, we have:*

$$\begin{aligned}\left|\left|T\left(\frac{1}{n}\right)g - g - \frac{1}{n}Ag\right|\right| \;\; &\leq \;\; \frac{1}{4n^2}\|2\varphi_1\varphi_1' + \varphi_2\varphi_1''\| \cdot \|g'\| \\ &+\frac{1}{8n^2}\|4\varphi_1^2 + 2\varphi_1\varphi_2' + 4\varphi_2\varphi_1' + \varphi_2\varphi_2''\| \cdot \|g''\| \\ &+\frac{1}{4n^2}\|2\varphi_1\varphi_2 + \varphi_2\varphi_2'\| \cdot \|g^{(3)}\| + \frac{1}{8n^2}\|\varphi_2^2\| \cdot \|g^{(4)}\|.\end{aligned}$$

Proof. It is obtained from the well-known inequality

$$||T(t)g - g - tAg|| \leq \frac{t^2}{2}||A^2 g||, \; t \geq 0.$$

if we take $t = \frac{1}{n}$ and we take into account

$$A^2 g = A(Ag) = \varphi_1\Big(\varphi_1 g' + \frac{1}{2}\varphi_2 g''\Big)' + \frac{1}{2}\varphi_2\Big(\varphi_1 g' + \frac{1}{2}\varphi_2 g''\Big)''.$$

□

In [9] the folloowin lemma is proved:

Lemma 3. *If $(L_n)_n$ is a sequence of positive linear operators which are convex of any order and if $f \in C^k[0,1], \; k \geq 0$ then:*

$$||(L_n^j)^{(k)}|| \leq (\sigma_k)^j ||f^{(k)}||, \; j \geq 0.$$

where $\sigma_k := \frac{1}{k!}(L_n e_k)^{(k)}$.

Now, the main result is the following theorem

Theorem 1. *Let a sequence of positive linear operators $(L_n)_n$, $L_n : C[0,1] \to C[0,1]$, which are convex of any order. Let $f \in C^4[0,1]$, The following estimate holds:*

$$\begin{aligned}||L_n^m f - T(t)f|| \;\; &\leq \;\; \frac{1-\sigma_1^m}{1-\sigma_1}\|f'\|\left(\|\psi_n^1\| + \frac{1}{4n^2}\|2\varphi_1\varphi_1' + \varphi_2\varphi_1''\|\right) \\ &+\frac{1-\sigma_2^m}{1-\sigma_2}\|f''\|\left(\frac{1}{2}\|\psi_n^2\| + \frac{1}{8n^2}\|4\varphi_1^2 + 2\varphi_1\varphi_2' + 4\varphi_2\varphi_1' + \varphi_2\varphi_2''\|\right) \\ &+\frac{1-\sigma_3^m}{1-\sigma_3}\|f^{(3)}\|\left(\frac{1}{6}||\tilde{M}_n^3\| + \frac{1}{4n^2}\|2\varphi_1\varphi_2 + \varphi_2\varphi_2'\|\right) \\ &+\frac{1-\sigma_4^m}{1-\sigma_4}\frac{1}{8n^2}\|f^{(4)}\|\|\varphi_2^2\| \\ &+\left|\frac{m}{n} - t\right|\left(\|\varphi_1\| \cdot ||f'|| + \frac{1}{2}\|\varphi_2\| \cdot ||f''||\right), \qquad (7)\end{aligned}$$

where $\sigma_k = \frac{1}{k!}(L_n e_k)^{(k)}$.

Proof. First, we have

$$\begin{aligned}||L_n^m f - T(t)f|| &\le \left|\left|L_n^m f - T\left(\frac{m}{n}\right) f\right|\right| + \left|\left|T\left(\frac{m}{n}\right) f - T(t)f\right|\right| \\ &\le \left|\left|L_n^m f - T\left(\frac{m}{n}\right) f\right|\right| + \left|\left|\int_t^{\frac{m}{n}} T(u)Afdu\right|\right| \\ &\le \left|\left|L_n^m f - T\left(\frac{m}{n}\right) f\right|\right| + \left|\frac{m}{n} - t\right| ||Af|| \\ &\le \left|\left|L_n^m f - T\left(\frac{m}{n}\right) f\right|\right| + \left|\frac{m}{n} - t\right| \left(\|\varphi_1\| \cdot ||f'|| + \frac{1}{2}\|\varphi_2\| \cdot ||f''||\right)\end{aligned}$$

Here we used $\|T(t)\| = 1$, for $t \ge 0$. Using a telescopic sum, we write:

$$\begin{aligned}\left|\left|L_n^m f - T\left(\frac{m}{n}\right) f\right|\right| &= \left|\left|\sum_{j=0}^{m-1} T\left(\frac{m-1-j}{n}\right)\left(L_n - T\left(\frac{1}{n}\right)\right) L_n^j f\right|\right| \\ &\le \sum_{j=0}^{m-1} \left|\left|\left(L_n - T\left(\frac{1}{n}\right)\right) L_n^j f\right|\right|.\end{aligned}$$

Let j be fixed for the moment. Denote $g := L_n^j f \in C^4[0,1]$. We write then

$$\left|\left|\left(L_n - T\left(\frac{1}{n}\right)\right) g\right|\right| \le \left|\left|L_n g - g - \frac{1}{n}Ag\right|\right| + \left|\left|T\left(\frac{1}{n}\right) g - g - \frac{1}{n}Ag\right|\right|. \qquad (8)$$

Lemma 1 and Lemma 2 get

$$\begin{aligned}\left|\left|\left(L_n - T\left(\frac{1}{n}\right)\right) g\right|\right| \le\; & \|g'\| \left(\|\psi_n^1\| + \frac{1}{4n^2}\|2\varphi_1\varphi_1' + \varphi_2\varphi_1''\|\right) \\ & + \|g''\| \left(\frac{1}{2}\|\psi_n^2\| + \frac{1}{8n^2}\|4\varphi_1^2 + 2\varphi_1\varphi_2' + 4\varphi_2\varphi_1' + \varphi_2\varphi_2''\|\right) \\ & + \|g^{(3)}\| \left(\frac{1}{6}||\tilde{M}_n^3\| + \frac{1}{4n^2}\|2\varphi_1\varphi_2 + \varphi_2\varphi_2'\|\right) \\ & + \|g^{(4)}\| \frac{1}{8n^2}\|\varphi_2^2\|.\end{aligned}$$

From Lemma 3 we have for $1 \le j, k \le 4$, that $\|g^{(k)}\| \le (\sigma_k)^j \|f^{(k)}\|$. Then the above relation becomes

$$\begin{aligned}\left|\left|\left(L_n - T\left(\frac{1}{n}\right)\right) g\right|\right| \le\; & (\sigma_1)^j \|f'\| \left(\|\psi_n^1\| + \frac{1}{4n^2}\|2\varphi_1\varphi_1' + \varphi_2\varphi_1''\|\right) \\ & + (\sigma_2)^j \|f''\| \Big(\frac{1}{2}\|\psi_n^2\| + \frac{1}{8n^2}\|4\varphi_1^2 + 2\varphi_1\varphi_2' + 4\varphi_2\varphi_1' \\ & \qquad + \varphi_2\varphi_2''\|\Big) \\ & + (\sigma_3)^j \|f^{(3)}\| \left(\frac{1}{6}||\tilde{M}_n^3\| + \frac{1}{4n^2}\|2\varphi_1\varphi_2 + \varphi_2\varphi_2'\|\right) \\ & + (\sigma_4)^j \frac{1}{8n^2}\|f^{(4)}\|\|\varphi_2^2\|.\end{aligned}$$

If we take the sum of these inequalities for $0 \le j \le m-1$ and we take into account the relations above we get relation (7). $\square$

3 Application to Durrmeyer operators

The Durrmeyer operators $D_n : L_1[0,1] \to C[0,1]$ are defined as

$$(D_n f)(x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt,$$

where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k},\ k = 0,1,\cdots,\ x \in [0.1]$. They are convex of any orders $i \ge 0$. i.e. if $f \in C^i[0,1]$ and $f^{(i)} \ge 0$ on $[0,1]$, then $(D_n(f))^{(i)} \ge 0$ on $[0,1]$.

We have (see [**?**]):

$$\begin{aligned}
M_n^1(x) &= \frac{1-2x}{n+2}\\
M_n^2(x) &= \frac{(2n-6)x(1-x)+2}{(n+2)(n+3)}\\
M_n^4(x) &= O\left(\frac{1}{n^2}\right), \text{ uniformly with respect to } x \in [0,1],
\end{aligned}$$

Then

$$\varphi_1(x) = 1-2x, \quad \psi_n^1(x) = \frac{2(2x-1)}{n(n+2)}$$
$$\varphi_2(x) = 2x(1-x), \quad \psi_n^2(x) = -\frac{x(1-x)}{n(n+2)(n+3)} + \frac{2}{(n+2)(n+3)}.$$

Hence

$$\|\psi_n^1\| \le \frac{2}{n(n+2)} \le \frac{2}{n^2}, \quad \|\psi_n^2\| \le \frac{2}{(n+2)(n+3)} \le \frac{2}{n^2}.$$

From [**?**] it follows

$$\begin{aligned}
\sigma_1 &= \frac{n}{n+2}\\
\sigma_2 &= \frac{n(n-1)}{(n+2)(n+3)}\\
\sigma_3 &= \frac{n(n-1)(n-2)}{(n+2)(n+3)(n+4)}\\
\sigma_4 &= \frac{n(n-1)(n-2)(n-3)}{(n+2)(n+3)(n+4)(n+5)}
\end{aligned}$$

We obtain

$$\begin{aligned}
\frac{1}{1-\sigma_1} &= \frac{n+2}{(n+2)-n} \leq \frac{3}{2}n \\
\frac{1}{1-\sigma_2} &= \frac{(n+2)(n+3)}{(n+2)(n+3)-n(n-1)} \leq n \\
\frac{1}{1-\sigma_3} &= \frac{(n+2)(n+3)(n+4)}{(n+2)(n+3)(n+4)-n(n-1)(n-2)} \leq n \\
\frac{1}{1-\sigma_4} &= \frac{(n+2)(n+3)(n+4)(n+5)}{(n+2)(n+3)(n+4)(n+5)-n(n-1)(n-2)(n-3)} \leq n.
\end{aligned}$$

Since $2\varphi_1(x)\varphi_1'(x) + \varphi_2(x)\varphi_1''(x) = -4(1-2x)$ we deduce

$$\|\psi_n^1\| + \frac{1}{4n^2}\|2\varphi_1\varphi_1' + \varphi_2\varphi_1''\| \leq \frac{2}{n^2} + \frac{1}{n^2} = \frac{3}{n^2}.$$

Since $4\varphi_1^2(x) + 2\varphi_1(x)\varphi_2'(x) + 4\varphi_2(x)\varphi_1'(x) + \varphi_2(x)\varphi_2''(x) = 8 - 56x(1-x)$ we deduce

$$\frac{1}{2}\|\psi_n^2\| + \frac{1}{8n^2}\|4\varphi_1^2 + 2\varphi_1\varphi_2' + 4\varphi_2\varphi_1' + \varphi_2\varphi_2''\| \leq \frac{2}{n^2}.$$

We have $2\varphi_1(x)\varphi_2(x) + \varphi_2(x)\varphi_2'(x) = 8x(1-x)(1-2x)$. Also $\tilde{M}_n^3(x) \leq \sqrt{M_n^2(x)M_n^4(x)}$ and $M_n^2(x) \leq \frac{n+1}{2(n+2}n+3)$ and $M_n^4(x) \leq \frac{C}{n^2}$. We deduce

$$\frac{1}{6}||\tilde{M}_n^3\| + \frac{1}{4n^2}\|2\varphi_1\varphi_2 + \varphi_2\varphi_2'\| \leq \frac{\sqrt{C}}{6n\sqrt{n}} + \frac{1}{2n^2}.$$

Since $\varphi_2^2(x) = 4x^2(1-x)^2$ we deduce

$$\frac{1}{8n^2}\|f^{(4)}\|\|\varphi_2^2\| \leq \frac{1}{32n^2}.$$

Finally $\|\varphi_1\| \leq 1$ and $\frac{1}{2}\|\varphi_2\| \leq \frac{1}{4}$.

Finally the following estimate is obtained.

Theorem 2. *For any $f \in C^4[0,1]$ and any $0 \leq m \leq n$ we have*

$$\begin{aligned}
||D_n^m f - T(t)f|| \leq\ & \frac{9}{2n}\|f'\| + \frac{2}{n}\|f''\| \\
& + \left(\frac{\sqrt{C}}{6\sqrt{n}} + \frac{1}{2n}\right)\|f^{(3)}\| + \frac{1}{32n}\|f^{(4)}\| \\
& + \left|\frac{m}{n} - t\right|\left(\|f'\| + \frac{1}{4}\|f''\|\right).
\end{aligned}$$

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