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SECOND HANKEL DETERMINANT FOR A CERTAIN CLASS OF ANALYTIC FUNCTIONS

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Abstract

In this paper, we investigate the upper bounds for second Hankel determinant $H_2(2)$ for a class of analytic functions defined by a q-derivative operator.

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1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

defined in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

An analytic function f is said to be subordinated to an analytic function g, written $f \prec g$, if there exists an analytic function w in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$, $z \in \mathcal{U}$.

Denote by $\mathcal{P}$ the class of analytic functions in $\mathcal{U}$ satisfying the conditions $p(0) = 1$ and $\Re p(z) > 0$.

In [15], Ma and Minda considered the following two classes of functions:

$$S^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z),\ z \in \mathcal{U} \right\}$$

$$C(\phi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z),\ z \in \mathcal{U} \right\}$$

where $\phi \in \mathcal{P}$ with $\phi'(0) > 0$ and such that $\phi(\mathcal{U})$ is a starlike region with respect to 1 and symmetric with respect to the real axis. The classes $S^*(\phi)$ and $C(\phi)$ unify

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various subclasses of starlike (S^*) or convex (C) functions in $\mathcal{U}$. Similar classes defined by subordination were investigated in [3], [18], [21] etc.

For a function $f \in \mathcal{A}$ given by (1) and $q \in \mathbb{N} = \{1, 2, \ldots\}$, the qth Hankel determinant (see [17]) is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1).$$

Note that the Hankel determinant $H_2(1) = a_3 - a_2^2$ is related to the well-known Fekete-Szegö functional [8]. The second Hankel determinant $H_2(2)$ is given by $H_2(2) = a_2a_4 - a_3^2$. In [4], [6], [9], [20], [22], [23] estimates for the second Hankel determinant for various subclasses of univalent and bi-univalent functions were investigated.

For $q \in (0, 1)$ and for $n \in \mathbb{N}$, the *q-integer number* n, is defined by (see e.g. [1], [10])

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \ldots + q^{n-1}. \tag{2}$$

Note that $\lim_{q \to 1^-} [n]_q = n$.

Let $q \in (0, 1)$ and $f \in \mathcal{A}$. The *q-derivative* or *q-difference operator* of f is defined by (see e.g. [1], [10])

$$\mathcal{D}_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z} & , z \neq 0 \\ f'(0) & , z = 0. \end{cases} \tag{3}$$

Note that $\lim_{q \to 1^-} \mathcal{D}_q f(z) = f'(z)$.

If $f(z) = z^n$ then

$$\mathcal{D}_q f(z) = \mathcal{D}_q(z^n) = \frac{1 - q^n}{1 - q} z^{n-1} = [n]_q z^{n-1} \tag{4}$$

and $\lim_{q \to 1^-} \mathcal{D}_q(z^n) = \lim_{q \to 1^-} [n]_q z^{n-1} = nz^{n-1}$.

Let $f \in \mathcal{A}$ be given by (1). From (3) we have

$$\mathcal{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \tag{5}$$

Recently, several classes of analytic functions defined by q-derivative operators have been investigated (see e.g. [11], [19]).

Making use of the q-derivative $\mathcal{D}_q f(z)$, we consider the following class of functions defined by means of subordination.

Definition 1. *Let $\phi : \mathcal{U} \to \mathbb{C}$ be analytic and let $q \in (0,1)$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}_q(\phi)$ if*

$$\mathcal{D}_q f(z) \prec \phi(z),\ z \in \mathcal{U}. \tag{6}$$

Note that for $\phi(z) = (1+z)/(1-z)$ and $q \to 1^-$ the class $\mathcal{R}_q(\phi)$ reduces to the class $\mathcal{R}$ of analytic functions whose derivative has positive real part in $\mathcal{U}$, studied in [16].

In this paper, motivated by the results in [12], we find the upper bound for the second Hankel determinant $H_2(2)$ for functions in the class $\mathcal{R}_q(\phi)$.

2 Main results

Unless otherwise mentioned, we assume throughout this section that

$$\phi(z) = 1 + A_1 z + A_2 z^2 + A_3 z^3 + \dots,\quad A_1 > 0. \tag{7}$$

The following lemmas will be used to prove our results.

Lemma 1. *([7]) Let the function $p \in \mathcal{P}$ be given by*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots,\quad z \in \mathcal{U}. \tag{8}$$

Then the sharp estimate

$$|c_n| \le 2,\quad n \in \mathbb{N} \tag{9}$$

holds.

Lemma 2. *([13], [14]) If the function $p \in \mathcal{P}$ is given by (8), then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{10}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \tag{11}$$

for some x, z with $|x| \le 1$ and $|z| \le 1$.

In the next theorem, we obtain the upper bound of the Hankel determinant $H_2(2)$ for the class $\mathcal{R}_q(\phi)$.

Theorem 1. *Let $q \in (0,1)$ and let $m = (1+1/q)^2(1+q^2)$. Assume that $f \in \mathcal{R}_q(\phi)$ is given by (1).*

1. *If A_1, A_2 and A_3 satisfy the conditions*

$$2|A_2| - (m-1)A_1 \le 0 \;\; \text{and} \;\; |(m+1)A_1A_3 - mA_2^2| - mA_1^2 \le 0$$

then

$$|a_2 a_4 - a_3^2| \le \frac{A_1^2}{q^2(m+1)}.$$

2. *If A_1, A_2 and A_3 satisfy the conditions*

$$2|A_2| - (m-1)A_1 \geq 0 \;\; \textit{and} \;\; 2|(m+1)A_1A_3 - mA_2^2| - 2A_1|A_2| - (m+1)A_1^2 \geq 0$$

or the conditions

$$2|A_2| - (m-1)A_1 \leq 0 \;\; \textit{and} \;\; |(m+1)A_1A_3 - mA_2^2| - mA_1^2 \geq 0$$

then

$$|a_2a_4 - a_3^2| \leq \frac{1}{q^2m(m+1)}|(m+1)A_1A_3 - mA_2^2|.$$

3. If A_1, A_2 and A_3 satisfy the conditions

$$2|A_2| - (m-1)A_1 > 0 \;\; \textit{and} \;\; 2|(m+1)A_1A_3 - mA_2^2| - 2A_1|A_2| - (m+1)A_1^2 \leq 0$$

then

$$|a_2a_4 - a_3^2| \leq$$
$$\frac{A_1^2}{4q^2m(m+1)} \cdot \frac{4m|(m+1)A_1A_3 - mA_2^2| - 4(m+1)A_1|A_2| - 4|A_2|^2 - (m+1)^2A_1^2}{|(m+1)A_1A_3 - mA_2^2| - 2A_1|A_2| - A_1^2}.$$

Proof. Let $f \in \mathcal{R}_q(\phi)$. Then there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathcal{U}$ such that

$$\mathcal{D}_q f(z) = \phi(w(z)), \quad z \in \mathcal{U}. \tag{12}$$

Let the function $p(z)$ be defined by

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + c_3z^3 + \ldots, \quad z \in \mathcal{U}$$

or equivalently

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2}\left[c_1z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3 + \ldots\right]. \tag{13}$$

It is easy to see that the function p is in the class $\mathcal{P}$.

Making use of (13) together with (7) we have

$$\phi\left(\frac{p(z) - 1}{p(z) + 1}\right) = 1 + \frac{1}{2}A_1c_1z + \frac{1}{2}\left[A_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_2c_1^2}{2}\right]z^2$$
$$+\frac{1}{2}\left[A_1\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) + A_2c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_3c_1^3}{4}\right]z^3 + \ldots. \tag{14}$$

Since

$$\mathcal{D}_q f(z) = 1 + [2]_q a_2 z + [3]_q a_3 z^2 + [4]_q a_4 z^3 + \ldots$$

equating the coefficients of z, z^2 and z^3, from (2), (12) and (14), we find

$$a_2 = \frac{A_1c_1}{2(1+q)} \tag{15}$$

$$a_3 = \frac{1}{4(1+q+q^2)}(2A_1c_2 - A_1c_1^2 + A_2c_1^2) \tag{16}$$

$$a_4 = \frac{1}{8(1+q)(1+q^2)}(4A_1c_3 - 4A_1c_1c_2 + 4A_2c_1c_2 + A_1c_1^3 - 2A_2c_1^3 + A_3c_1^3). \tag{17}$$

Thus we have

$$a_2a_4 - a_3^2 = \frac{A_1}{16q^2m(m+1)}\left[(A_1 - 2A_2 + (m+1)A_3 - mA_2^2/A_1)c_1^4\right.$$
$$\left. -4(A_1 - A_2)c_1^2c_2 + 4(m+1)A_1c_1c_3 - 4mA_1c_2^2\right], \tag{18}$$

where $m = (1+1/q)^2(1+q^2)$.

Let

$$\Omega = \frac{A_1}{16q^2m(m+1)},$$

$$\omega_1 = A_1 - 2A_2 + (m+1)A_3 - m\frac{A_2^2}{A_1}, \quad \omega_2 = -4(A_1 - A_2), \tag{19}$$

$$\omega_3 = 4(m+1)A_1, \quad \omega_4 = -4mA_1. \tag{20}$$

Then (18) is equivalent to

$$a_2a_4 - a_3^2 = \Omega(\omega_1c_1^4 + \omega_2c_1^2c_2 + \omega_3c_1c_3 + \omega_4c_2^2). \tag{21}$$

Since the functions p and $p(e^{i\theta}z)$ $(\theta \in \mathbb{R})$ are in the class $\mathcal{P}$, without loss of generality, we can assume that $c_1 = c \in [0,2]$. Substituting in (21) the values of c_2 and c_3 from (10) and (11), it follows that

$$|a_2a_4 - a_3^2| = \Omega\left|c^4\left(\omega_1 + \frac{\omega_2}{2} + \frac{\omega_3}{4} + \frac{\omega_4}{4}\right) + c^2x(4-c^2)\left(\frac{\omega_2}{2} + \frac{\omega_3}{2} + \frac{\omega_4}{2}\right)\right.$$
$$\left. + x^2(4-c^2)\left(-c^2\frac{\omega_3}{4} + \frac{\omega_4}{4}(4-c^2)\right) + \frac{\omega_3}{2}c(4-c^2)(1-|x|^2)z\right|.$$

Furthermore, substituting the values of $\omega_1, \omega_2, \omega_3$ and ω_4 we get

$$|a_2a_4 - a_3^2| \le \Omega[|(m+1)A_3 - mA_2^2/A_1|c^4 + 2\mu c^2(4-c^2)|A_2|$$
$$+2(m+1)A_1c(4-c^2) + \mu^2(4-c^2)A_1(c-2)(c-2m)] := F(c,\mu), \tag{22}$$

where $\mu = |x| \in [0,1]$.

A simple calculation shows that the partial derivative of $F(c,\mu)$ with respect to μ is strictly positive when $c \in [0,2]$ and $\mu \in [0,1]$. Thus

$$\max_{0\le\mu\le1} F(c,\mu) = F(c,1) := G(c), \tag{23}$$

where

$$G(c) = \Omega[(|(m+1)A_3 - mA_2^2/A_1| - A_1 - 2|A_2|)c^4 + 4(2|A_2| + (1-m)A_1)c^2 + 16mA_1].$$

Define

$A = |(m+1)A_3 - mA_2^2/A_1| - A_1 - 2|A_2|$

$B = 4[\,2|A_2| + (1-m)A_1]$

$C = 16mA_1$ and $c^2 = t$.

Then, in view of (22) and (23), we have

$$|a_2a_4 - a_3^2| \leq \max_{0\leq c\leq 2} G(c) = \Omega \max_{0\leq t\leq 4}(At^2 + Bt + C).$$

Since

$$\max_{0\leq t\leq 4}(At^2+Bt+C) = \begin{cases} C & , B \leq 0,\;\; A \leq -B/4 \\ 16A + 4B + C & , B \geq 0,\; A \geq -B/8 \;\text{ or }\; B \leq 0,\; A \geq -B/4 \\ \dfrac{4AC - B^2}{4A} & , B > 0,\;\; A \leq -B/8. \end{cases}$$

a routine calculation yields the desired result. □

If in Theorem 1 we let $q \to 1^-$, we obtain the upper bound for the second Hankel determinant for the class $\mathcal{R}(\phi) = \{f \in \mathcal{A} : f'(z) \prec \phi(z),\, z \in \mathcal{U}\}$.

Corollary 1. *Let $f \in \mathcal{A}$, given by (1), be in the class $\mathcal{R}(\phi)$.*

1. If A_1, A_2 and A_3 satisfy the conditions

$$|A_2| \leq \frac{7}{2}A_1 \;\; \textit{and} \;\; |9A_1A_3 - 8A_2^2| - 8A_1^2 \leq 0$$

then

$$|a_2a_4 - a_3^2| \leq \frac{A_1^2}{9}.$$

2. If A_1, A_2 and A_3 satisfy the conditions

$$|A_2| \geq \frac{7}{2}A_1 \;\; \textit{and} \;\; 2|9A_1A_3 - 8A_2^2| - 2A_1|A_2| - 9A_1^2 \geq 0$$

or the conditions

$$|A_2| \leq \frac{7}{2}A_1 \;\; \textit{and} \;\; |9A_1A_3 - 8A_2^2| - 8A_1^2 \geq 0$$

then

$$|a_2a_4 - a_3^2| \leq \frac{|9A_1A_3 - 8A_2^2|}{72}.$$

3. If A_1, A_2 and A_3 satisfy the conditions

$$|A_2| > \frac{7}{2}A_1 \;\; \textit{and} \;\; 2|9A_1A_3 - 8A_2^2| - 2A_1|A_2| - 9A_1^2 \leq 0$$

then

$$|a_2a_4 - a_3^2| \leq$$
$$\frac{A_1^2}{288} \cdot \frac{32|9A_1A_3 - 8A_2^2| - 36A_1|A_2| - 4|A_2|^2 - 81A_1^2}{|9A_1A_3 - 8A_2^2| - 2A_1|A_2| - A_1^2}.$$

Remark 1.

1. *For* $\phi(z) = (1+Az)/(1+Bz)\ -1 \le B < A \le 1$*, Corollary 1 reduces to ([2], Theorem 2.1) for* $\gamma = 0$ *and* $\tau = 1$*.*

2. *For* $\phi(z) = (1+z)/(1-z)$*, Corollary 1 reduces to ([5], Theorem 3.1).*

3. *Corollary 1 reduces to ([12], Theorem 3) for* $\gamma = 0$ *and* $\tau = 1$*.*

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