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HYPERSURFACE OF A FINSLER SPACE WITH RANDERS CHANGE OF SPECIAL (α, β)-METRIC

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Abstract

The aim of the present paper is to discuss different kinds of hypersurfaces of the Finsler space with Randers change of special (α, β)-metric of type $L^* = \alpha + A_1\beta + A_2\frac{\beta^{(n+1)}}{\alpha^n} + \beta$ (where A_1 and A_2 are constants) which is a generalization of the metric $L^* = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha} + \beta$ considered in [12]. We have obtained conditions for the hypersurface to be a hyperplane of first, second or third kinds.

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1 Introduction

M. Matsumoto [6] introduced the concept of (α, β)-metric on a differentiable manifold with local coordinates x^i, where $\alpha^2 = a_{ij}(x)y^iy^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M^n. M. Hashiguchi and Y. Ichijyo [2] studied some special (α, β)-metrics and obtained interesting results. B. N. Prasad [8] obtained the Cartan connection for the Finsler space whose metric is transformed by an h-vector. M. Matsumoto [7] discussed the properties of special hypersurface of a Randers space with $b_i(x)$ as gradient of a scalar function $b(x)$. M. Kitayama [5] obtained certain results related to Finslerian hypersurface given by β-change. M. K. Gupta and P. N. Pandey [1] discussed hypersurface of a Finsler space with a special metric and derived certain properties of a Finslerian hypersurface given by an h-vector. In 2011, H. Wosoughi [14] discussed hypersurface of special Finsler space with an exponential (α, β)-metric and obtained certain results. H. S. Shukla, Manmohan Pandey and B. N. Prasad [10] studied hypersurface of a Finsler space with metric $\sum_{r=0}^{m} \frac{\beta^r}{\alpha^{r-1}}$ and obtained several results.

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The hypersurface of a Finsler space with some given special metrics has been studied by authors [13, 15]. A change of a Finsler metric $L(x,y) \to L^* = L(x,y) + b_i(x)y^i$ is called Randers change of metric. The notion of a Randers change was introduced by Matsumoto, named by Hashiguchi and Ichijyo [3] and studied in detail by Shibata [11]. A hypersurface M^{n-1} of the (M^n, L) may be represented parametrically by the equation $x^i = x^i(u^\alpha)$, $\alpha = 1, 2, ..., n-1$, where u^α are Gaussian coordinate on M^{n-1}. Since the function $L^* \equiv L(x(u), y(u,v))$ gives rise to a Finsler metric of M^{n-1}, we get an $(n-1)$-dimensional Finsler space $F^{n-1} = (M^{n-1}, L^*(u,v))$. The aim of the present paper is to discuss different kinds of hypersurfaces of the Finsler space with Randers change of special (α,β)-metric of type $L^* = \alpha + A_1\beta + A_2\frac{\beta^{(n+1)}}{\alpha^n} + \beta$ (where A_1 and A_2 are constants) which is a generalization of the metric $L^* = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha} + \beta$ considered in [12]. We have obtained conditions for the hypersurface to be a hyperplane of first, second or third kinds.

2 Preliminaries

Let M^n be a real smooth manifold of dimension n and let $F^{*n} = (M^n, L^*)$ be a Finsler space on the differentiable manifold M^n endowed with a fundamental function $L^*(x,y)$, where

$$L^* = \alpha + A_1\beta + A_2\frac{\beta^{(n+1)}}{\alpha^n} + \beta. \tag{1}$$

Differentiating (1) partially with respect to α and β, we get

$$\begin{cases} a) \;\; L^*_\alpha = \frac{\alpha^{n+1} - A_2 n\beta^{n+1}}{\alpha^{n+1}}, \\ b) \;\; L^*_\beta = \frac{(A_1+1)\alpha^n + A_2(n+1)\beta^n}{\alpha^n}, \\ c) \;\; L^*_{\alpha\alpha} = \frac{A_2 n(n+1)\beta^{n+1}}{\alpha^{n+2}}, \\ d) \;\; L^*_{\beta\beta} = \frac{A_2 n(n+1)\beta^{n-1}}{\alpha^n}, \\ e) \;\; L^*_{\alpha\beta} = -\frac{A_2 n(n+1)\beta^n}{\alpha^{n+1}}, \end{cases} \tag{2}$$

where $L^*_\alpha = \partial L^*/\partial\alpha$, $L^*_\beta = \partial L^*/\partial\beta$, $L^*_{\alpha\alpha} = \partial L^*_\alpha/\partial\alpha$, $L^*_{\beta\beta} = \partial L^*_\beta/\partial\beta$ and $L^*_{\alpha\beta} = \partial L^*_\alpha/\partial\beta$.

The normalized supporting element, the metric tensor, the angular metric tensor and Cartan tensor are defined by [9]

$$\begin{cases} a) \;\; l_i = \dot{\partial}_i L, \\ b) \;\; g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \\ c) \;\; h_{ij} = L\dot{\partial}_i\dot{\partial}_j L, \\ d) \;\; C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}, \end{cases} \tag{3}$$

respectively, where $\dot{\partial}_i \equiv \frac{\partial}{\partial y^i}$.

In the Finsler space $F^{*n} = (M^n, L^*)$ the normalized element of support $(3a)$ and the angular metric tensor $(3c)$ are given by [14]

$$l_i^* = \alpha^{-1} L_\alpha^* y_i + L_\beta^* b_i, \tag{4}$$

$$h_{ij}^* = p a_{ij} + q_0 b_i b_j + q_1 (b_i Y_j + b_j Y_i) + q_2 Y_i Y_j, \tag{5}$$

where

$$Y_i = a_{ij} y^j, \tag{6}$$

$$\begin{aligned} p &= L^* L_\alpha^* \alpha^{-1} \\ &= \frac{1}{\alpha^{2n+2}} \Big[\alpha^{2n+2} - A_2(n-1)\alpha^{n+1}\beta^{n+1} + (A_1+1)\beta\alpha^{2n+1} \\ &\qquad - A_2 n (A_1\alpha^n + A_2\beta^n + \alpha^n)\beta^{n+2} \Big], \end{aligned} \tag{7}$$

$$\begin{aligned} q_0 &= L^* L_{\beta\beta}^* \\ &= \frac{A_2 n(n+1)}{\alpha^{2n}} \Big[\alpha^{n+1}\beta^{n-1} + (A_1+1)\alpha^n\beta^n + A_2\beta^{2n} \Big], \end{aligned} \tag{8}$$

$$\begin{aligned} q_1 &= L^* L_{\alpha\beta}^* \alpha^{-1} \\ &= -\frac{A_2 n(n+1)\beta^n}{\alpha^{2n+2}} \Big[\alpha^{n+1} + A_1\beta\alpha^n + A_2\beta^{n+1} + \beta\alpha^n \Big], \end{aligned} \tag{9}$$

$$\begin{aligned} q_2 &= L^* \alpha^{-2} (L_{\alpha\alpha}^* - L_\alpha^* \alpha^{-1}) \\ &= \frac{1}{\alpha^{2n+4}} \Big[A_2(n^2+2n-1)\alpha^{n+1}\beta^{n+1} + A_2 n(n+2)\{(A_1+1)\alpha^n \\ &\qquad + A_2\beta^n\}\beta^{n+2} - (A_1+1)\beta\alpha^{2n+1} - \alpha^{2n+2} \Big]. \end{aligned} \tag{10}$$

In the Finsler space $F^{*n} = (M^n, L^*)$ the fundamental metric tensor $(3b)$ is given by [14]

$$g_{ij}^* = p a_{ij} + p_0 b_i b_j + p_1 (b_i Y_j + b_j Y_i) + p_2 Y_i Y_j, \tag{11}$$

where

$$\begin{aligned} p_0 &= q_0 + L_\beta^{*2} \\ &= \frac{1}{\alpha^{2n}} \Big[A_2(A_1+1)(n^2+3n+2)\alpha^n\beta^n + (A_1+1)^2\alpha^{2n} \\ &\qquad + A_2 n(n+1)\alpha^{n+1}\beta^{n-1} + A_2^2(2n^2+3n+1)\beta^{2n} \Big], \end{aligned} \tag{12}$$

$$\begin{aligned} p_1 &= q_1 + L^{*-1} p L_\beta^* \\ &= \frac{1}{\alpha^{2n+2}} \Big[A_2\alpha^n\beta^n\{(1-n^2)\alpha - n\beta(A_1+1)(n+2)\} \\ &\qquad - 2A_2^2 n(n+1)\beta^{2n+1} + (A_1+1)\alpha^{2n+1} \Big], \end{aligned} \tag{13}$$

$$\begin{aligned} p_2 &= q_2 + p^2 L^{*-2} \\ &= \frac{1}{\alpha^{2n+4}}\Big[A_2(n^2-1)\alpha^{n+1}\beta^{n+1} + 2A_2^2 n(n+1)\beta^{2n+2} \\ &\qquad + (A_1+1)\left\{A_2 n(n+2)\alpha^n\beta^{n+2} - \beta\alpha^{2n+1}\right\}\Big]. \end{aligned} \tag{14}$$

The reciprocal tensor g^{*ij} of g^*_{ij} is given by

$$g^{*ij} = p^{-1}a^{ij} - S_0 b^i b^j - S_1(b^i y^j + b^j y^i) - S_2 y^i y^j, \tag{15}$$

$$\begin{cases} a)\ \ b^i = a^{ij}b_j,\ \ b^2 = a_{ij}b^i b^j \\ b)\ \ S_0 = \frac{\{pp_0 + (p_0p_2 - p_1^2)\alpha^2\}}{\tau p}, \\ c)\ \ S_1 = \frac{\{pp_1 - (p_0p_2 - p_1^2)\beta}{\tau p}, \\ d)\ \ S_2 = \frac{\{pp_2 + (p_0p_2 - p_1^2)b^2}{\tau p}, \\ e)\ \ \tau = p(p + p_0 b^2 + p_1\beta) + (p_0p_2 - p_1^2)(\alpha^2 b^2 - \beta^2). \end{cases} \tag{16}$$

For the Finsler space F^{*n} the hv-torsion tensor is given by

$$C^*_{ijk} = p_1(h^*_{ij}m_k + h^*_{jk}m_i + h^*_{ki}m_j) + \gamma_1 m_i m_j m_k, \tag{17}$$

where

$$\text{(a)}\ \ \gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_1 q_0,\ \ \text{(b)}\ \ m_i = b_i - \alpha^{-2}\beta Y_i. \tag{18}$$

Here m_i is a non-vanishing covariant vector orthogonal to the element of support y^i.

Let $\left\{ {i \atop jk} \right\}$ be the components of Christoffel symbols of the associated Riemannian space R^n and ∇_k denote the covariant differential operator with respect to x^k relative to the Chriestoffel symbols. We will use the following tensors:

$$\text{(a)}\ \ 2E^*_{ij} = b_{ij} + b_{ji},\ \ \text{(b)}\ \ 2F^*_{ij} = b_{ij} - b_{ji}, \tag{19}$$

where $b_{ij} = \nabla_j b_i$.

Let $C\Gamma^* = (F^{*i}_{jk}, G^{*i}_j, C^{*i}_{jk})$ be the Cartan connection of F^{*n}. The difference tensor $D^{*i}_{jk} = F^{*i}_{jk} - \left\{ {i \atop jk} \right\}$ of the Finsler space F^{*n} is given by

$$\begin{aligned} D^{*i}_{jk} &= B^{*i}E^*_{jk} + F^{*i}_k B^*_j + F^{*i}_j B^*_k + B^{*i}_j b_{0k} + B^{*i}_k b_{0j} \\ &\quad - b_{0m}g^{*im}B^*_{jk} - C^{*i}_{jm}A^{*m}_k - C^{*i}_{km}A^{*m}_j + C^*_{jkm}A^{*m}_s g^{*is} \\ &\quad + \lambda^{*s}(C^{*i}_{jm}C^{*m}_{sk} + C^{*i}_{km}C^{*m}_{sj} - C^{*m}_{jk}C^{*i}_{ms}), \end{aligned} \tag{20}$$

where

$$\begin{cases} a)\ \ B^*_k = p_0 b_k + p_1 Y_k,\ \ B^{*i} = g^{*ij}B^*_j, \\ b)\ \ B^*_{ij} = \frac{p_1(a_{ij} - \alpha^{-2}Y_iY_j) + (\partial p_0/\partial\beta)m_i m_j}{2}, \\ c)\ \ A^{*m}_k = B^{*m}_k E^*_{00} + B^{*m}E^*_{k0} + B^*_k F^{*m}_0 + B^*_0 F^{*m}_k, \\ d)\ \ \lambda^{*m} = B^*E^*_{00} + 2B^*_0 F^{*m}_0,\ \ F^{*k} = g^{*kj}F^*_{ji} \\ e)\ \ B^{*k}_i = g^{*kj}B^*_{ji},\ \ B^*_0 = B^*_i y^i, \end{cases} \tag{21}$$

where 0 denote the contraction with y^i except for the quantities p_0, q_0 and S_0.

3 Induced Cartan connection

Let F^{*n-1} be a hypersurface of F^{*n} given by the equation $x^i = x^i(u)$. The element of support y^i of F^{*n} is to be taken tangential to F^{*n-1}, i.e.

$$y^i = B^{*i}_\alpha(u)v^\alpha. \tag{22}$$

The metric tensor $g^*_{\alpha\beta}$ and v-torsion tensor $C^*_{\alpha\beta\gamma}$ are given by

$$\text{(a) } g^*_{\alpha\beta} = g^*_{ij}B^{*i}_\alpha B^{*j}_\beta, \ \text{(b) } C^*_{\alpha\beta\gamma} = C^*_{ijk}B^{*i}_\alpha B^{*j}_\beta B^{*k}_\gamma. \tag{23}$$

At each point u^α of F^{*n-1}, a unit normal vector $N^{*i}(u,v)$ is defined by

$$\text{(a) } g^*_{ij}B^{*i}_\alpha N^{*j} = 0, \ \text{(b) } g^*_{ij}N^{*i}N^{*j} = 1. \tag{24}$$

The angular metric tensor h^*_{ij} of F^{*n}, satisfies the following relations

$$\text{(a) } h^*_{\alpha\beta} = h^*_{ij}B^{*i}_\alpha B^{*j}_\beta, \ \text{(b) } h^*_{ij}B^{*i}_\alpha N^{*j} = 0, \ \text{(c) } h^*_{ij}N^{*i}N^{*j} = 1. \tag{25}$$

The inverse projection factor $B^{*\alpha}_i(u,v)$ of B^{*i}_α is given by

$$B^{*\alpha}_i = g^{*\alpha\beta}g^*_{ij}B^{*j}_\beta, \tag{26}$$

where $g^{*\alpha\beta}$ is the inverse of the metric tensor $g^*_{\alpha\beta}$ of F^{*n-1}.
From (24) and (26), we get

$$B^{*i}_\alpha B^{*\beta}_i = \delta^\beta_\alpha, \ B^{*i}_\alpha N^*_i = 0, \ N^{*i}B^{*\alpha}_i = 0, \ N^{*i}N^*_i = 1, \tag{27}$$

and further

$$B^{*i}_\alpha B^{*\alpha}_j + N^{*i}N^*_j = \delta^i_j. \tag{28}$$

For induced cartan connection $IC\Gamma = (F^{*\alpha}_{\beta\gamma}, G^{*\alpha}_\beta, C^{*\alpha}_{\beta\gamma})$ on F^{*n-1}, the second fundamental h-tensor $H^*_{\alpha\beta}$ and the normal curvature vector H^*_α are given by

$$H^*_{\alpha\beta} = N^*_i(B^*_{\alpha\beta} + F^{*i}_{jk}B^{*j}_\alpha B^{*k}_\beta) + M^*_\alpha H^*_\beta, \ H^*_\alpha = N^*_i(B^{*i}_{0\alpha} + G^{*i}_j B^{*j}_\alpha), \tag{29}$$

where $M^*_\alpha = C^*_{ijk}B^{*i}_\alpha N^{*j}N^{*k}$, $B^{*i}_{\alpha\beta} = \partial^2 x^i/\partial u^\alpha \partial U^\beta$ and $B^{*i}_{0\alpha} = B^{*i}_{\beta\alpha}v^\beta$.

It is clear that $H^*_{\alpha\beta}$ is not symmetric and

$$H^*_{\alpha\beta} - H^*_{\beta\alpha} = M^*_\alpha H^*_\beta - M^*_\beta H^*_\alpha. \tag{30}$$

Equation (29) yields

$$H^*_{\alpha 0} = H^*_{\alpha\beta}v^\beta = H^*_\alpha + M^*_\alpha + M^*_\alpha H^*_0. \tag{31}$$

The second fundamental v-tensor $M^*_{\alpha\beta}$ is given by

$$M^*_{\alpha\beta} = C^*_{ijk} B^{*i}_\alpha B^{*j}_\beta N^{*k}. \tag{32}$$

The relative h and v-covariant derivatives of B^{*i}_α and N^{*i} are given by

$$\begin{cases} a)\ B^{*i}_{\alpha|\beta} = H^*_{\alpha\beta} N^{*i}, \\ b)\ B^{*i}_\alpha|\beta = M^*_{\alpha\beta} N^{*i}, \\ c)\ N^{*i}_{|\beta} = -H^*_{\alpha\beta} B^{*\alpha}_j g^{*ij}, \\ d)\ N^{*i}|\beta = -M^*_{\alpha\beta} B^{*\alpha}_j g^{*ij}. \end{cases} \tag{33}$$

$$M^*_\beta = N^*_i C^{*i}_{jk} B^{*j}_\beta N^{*k}. \tag{34}$$

Let $X_i(x,y)$ be a vector field of F^{*n}, the relative h and v-covariant derivatives of X_i are given by

$$X_{i|\beta} = X_{i|j} B^{*j}_\beta + X_i|_j N^{*j} H^*_\beta,\ \ X_{i|\beta} = X_i|_j B^{*j}_\beta. \tag{35}$$

M. Matsumoto [4] defined different kinds of hypersurfaces and obtained their characteristic conditions, which are given in the following lemmas.

Lemma 1. *A Finslerian hypersurface F^{n-1} is a hyperplane of the first kind if and only if $H_\alpha = 0$.*

Lemma 2. *A Finslerian hypersurface F^{n-1} is a hyperplane of the second kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.*

Lemma 3. *A Finslerian hypersurface F^{n-1} is a hyperplane of the third kind if and only if $H_{\alpha\beta} = 0 = M_{\alpha\beta}$ and $H_\alpha = 0$.*

4 Hypersurface $F^{*n-1}(c)$ of the Finsler space F^{*n}

Let us consider a special Finsler metric $L^* = \alpha + A_1\beta + A_2\frac{\beta^{(n+1)}}{\alpha^n} + \beta$ with gradient $b_i(x) = \partial_i b$. From parametric equation $x^i = x^i(u^\alpha)$ of $F^{*n-1}(c)$, we get $\partial_\alpha b(x(u)) = 0 = b_i B^{*i}_\alpha$ so that $b_i(x)$ are regarded as covariant components of a normal vector field of $F^{*n-1}(c)$. Therefore along the $F^{*n-1}(c)$, we have

$$\text{(a)}\ \ b_i B^{*i}_\alpha = 0, \quad \text{(b)}\ \ b_i y^i = 0. \tag{36}$$

Hence the induced metric $\underline{L}^*(u,v)$ of $F^{*n-1}(c)$ is given by

$$\underline{L}^*(u,v) = \sqrt{a_{\alpha\beta} v^\alpha v^\beta}, \quad a_{\alpha\beta} = a_{ij} B^{*i}_\alpha B^{*j}_\beta, \tag{37}$$

which is a Riemannian metric.
At a point of $F^{*n-1}(c)$, from (7), (8), (9), (10), (12), (13), (14) and (16), we get

$$\begin{aligned} &p = 1,\ \ q_0 = 0,\ \ q_1 = 0,\ \ q_2 = -\frac{1}{\alpha^2},\ \ \tau = 1, \\ &p_0 = (A_1+1)^2,\ \ p_1 = \frac{A_1+1}{\alpha},\ \ p_2 = 0, \\ &S_0 = 0,\ \ S_1 = \frac{A_1+1}{\alpha},\ \ S_2 = -\frac{(A_1+1)^2}{\alpha^2} b^2. \end{aligned} \tag{38}$$

Using (38) in (15), we find

$$g^{*ij} = a^{ij} - \frac{A_1+1}{\alpha}(b^i y^j + b^j y^i) + \frac{(A_1+1)^2}{\alpha^2} b^2 y^i y^j. \tag{39}$$

Transvecting (39) with $b_i b_j$ and using ($36b$), we have

$$g^{*ij} b_i b_j = b^2, \tag{40}$$

which gives

$$b_i(x) = bN_i^*, \tag{41}$$

where b is the length of vector b^i.
Transvecting (39) with b_j and using ($36b$) and (41), we get

$$b^i = a^{ij} b_j = bN^{*i} + (A_1+1)b^2\alpha^{-1}y^i. \tag{42}$$

This leads to

Theorem 1. *In the Finslerian hypersurface* $F^{*n-1}(c)$ *of a Finsler space with Randers change of special* (α,β)*-metric, the induced metric is a Riemannian metric given by* (37) *and the scalar function* $b(x)$ *is given by* (40) *and* (41)*.*

The angular metric tensor and metric tensor of F^{*n} are given by

$$h^*_{ij} = a_{ij} - \frac{Y_i Y_j}{\alpha^2} \tag{43}$$

and

$$g^*_{ij} = a_{ij} + (A_1+1)^2 b_i b_j + \frac{A_1+1}{\alpha}(b_i Y_j + b_j Y_i), \tag{44}$$

respectively.
Transvecting (43) with $B^{*i}_\alpha B^{*j}_\beta$ and using (25), we get

$$h^*_{\alpha\beta} = h^{*(\alpha)}_{\alpha\beta}, \tag{45}$$

where $h^{*(\alpha)}_{\alpha\beta}$ denote the angular metric tensor of induced Riemannian metric. Differentiating (12) with respect to β, we get

$$\begin{aligned} \frac{\partial p_0}{\partial \beta} = \frac{n(n+1)A_2}{\alpha^{2n}} \Big[& (n+2)(A_1+1)\alpha^n\beta^{n-1} + (n-1)\alpha^{n+1}\beta^{n-2} \\ & + 2(2n+1)A_2\beta^{2n-1} \Big]. \end{aligned} \tag{46}$$

Thus along $F^{*n-1}(c)$, $\frac{\partial p_0}{\partial \beta} = 0$. Therefore (18) gives $\gamma_1 = 0$, $m_i = b_i$.
Then from (17), we get

$$C^*_{ijk} = \frac{(A_1+1)}{\alpha}(h^*_{ij} b_k + h^*_{jk} b_i + h^*_{ki} b_j). \tag{47}$$

Transvecting (47) with $B^{*i}_\alpha B^{*j}_\beta N^{*k}$ and using (25), (32) and (41), we have

$$M^*_{\alpha\beta} = (\frac{A_1+1}{\alpha})bh^*_{\alpha\beta}. \tag{48}$$

From (25), (34), (36b) and (47), we have

$$M^*_\alpha = 0. \tag{49}$$

Using (49) in (30), we have

$$H^*_{\alpha\beta} = H^*_{\beta\alpha}. \tag{50}$$

Thus, we have the following:

Theorem 2. *The second fundamental v-tensor $M^*_{\alpha\beta}$ of F^{*n-1} is given by* (48) *and the second fundamental h-tensor $H^*_{\alpha\beta}$ is symmetric.*

Taking h-covariant derivative of (36) with respect to the induced connection, we find

$$b_{i|\beta}B^{*i}_\alpha + b_i B^{*i}_{\alpha|\beta} = 0. \tag{51}$$

Applying (35) for the vector b_i, we get

$$b_{i|\beta} = b_{i|j}B^{*j}_\beta + b_i|_j N^{*j}H^*_\beta. \tag{52}$$

In view of (33) and (52), (51) implies

$$b_{i|j}B^{*i}_\alpha B^{*j}_\beta + b_i|_j B^{*i}_\alpha N^{*j}H^*_\beta + b_i H^*_{\alpha\beta}N^{*i} = 0. \tag{53}$$

From $b_i|_j = -b_h C^{*h}_{ij}$, (34), (41) and (49) together imply

$$b_i|_j B^{*i}_\alpha N^{*j} = -bM^*_\alpha = 0. \tag{54}$$

Using (41) and (54) in (53), we find

$$bH^*_{\alpha\beta} + b_{i|j}B^{*i}_\alpha B^{*j}_\beta = 0. \tag{55}$$

Since $H^*_{\alpha\beta}$ is symmetric, $b_{i|j}$ is symmetric. Transvecting (55) with v^β, we get

$$bH^*_\alpha + b_{i|j}B^{*i}_\alpha y^i = 0. \tag{56}$$

Again transvecting with v^α, we get

$$bH^*_0 + b_{i|j}y^i y^j = 0. \tag{57}$$

In view of Lemma 1, the hypersurface $F^{*n-1}(c)$ is a hyperplane of first kind if and only if $b_{i|j}y^i y^j = 0$. Here $b_{i|j}$ being the covariant derivative with respect to the Cartan connection of F^{*n} may depend on y^i. Since b_i is gradient vector, from (19) for induced metric L^*, we have $E^*_{ij} = b_{ij}$, $F^*_{ij} = 0$. Thus, (20) reduces to

$$\begin{aligned} D^{*i}_{jk} = {} & B^{*i}b_{jk} + B^{*i}_j b_{0k} + B^{*i}_k b_{0j} - b_{0m}g^{*im}B^*_{jk} - C^{*i}_{jm}A^{*m}_k - C^{*i}_{km}A^{*m}_j \\ & + C^*_{jkm}A^{*m}_s g^{*is} + \lambda^{*s}\left(C^{*i}_{jm}C^{*m}_{sk} + C^{*i}_{km}C^{*m}_{sj} - C^{*m}_{jk}C^{*i}_{ms}\right). \end{aligned} \tag{58}$$

Using (38) and (39) in (21), we get

$$\begin{aligned} B^*_j &= (A_1+1)^2 b_j + \frac{A_1+1}{\alpha} Y_j, \ \ B^{*i} = \frac{A_1+1}{\alpha} y^i, \\ B^*_{ij} &= \frac{A_1+1}{2\alpha}(a_{ij} - \alpha^{-2} Y_i Y_j), \ \ \lambda^{*m} = B^{*m} b_{00}, \\ B^{*i}_j &= \frac{A_1+1}{2\alpha}(\delta^i_j - \alpha^{-2} y^i Y_j) - \frac{(A_1+1)^2}{2\alpha^2} b_j y^i, \\ A^{*m}_k &= B^{*m}_k b_{00} + B^{*m} b_{k0}. \end{aligned} \tag{59}$$

In view of (36), we have $B^{*i}_0 = 0$ and $B^*_{i0} = 0$, which together with (4.24) gives $A^{*m}_0 = B^{*m} b_{00}$.
Transvecting (58) with y^k, we get

$$D^{*i}_{j0} = B^{*i} b_{j0} + B^{*i}_j b_{00} - B^{*m} C^{*i}_{jm} b_{00}. \tag{60}$$

Again transvecting (60) with y^j, we find

$$D^{*i}_{00} = B^{*i} b_{00} = \frac{A_1+1}{\alpha} y^i b_{00}. \tag{61}$$

Transvecting (61) with b_i and using (36), we have

$$b_i D^{*i}_{00} = 0. \tag{62}$$

Thus, the relation $b_{i|j} = b_{ij} - b_r D^{*r}_{ij}$ and (62) gives

$$b_{i|j} y^i y^j = b_{00}. \tag{63}$$

Using (63) in (57), we get

$$b H^*_0 + b_{00} = 0. \tag{64}$$

From equation (64) and Lemmas 1 and 2, it is clear that the necessary and sufficient condition for $F^{*n-1}(c)$ to be a hyperplane of first kind is that $b_{00} = 0$. Since $b_{ij} = \nabla_j b_i$ does not depend on y^i and satisfy (36b), this condition may be written as $b_{ij} y^i y^j = (b_i y^i)(c_j y^j) = 0$ for some $c_j(x)$. Therefore

$$2b_{ij} = b_i c_j + b_j c_i. \tag{65}$$

From (36) and (65) it follows that

$$b_{00} = 0, \ \ b_{ij} B^{*i}_\alpha B^{*j}_\beta = 0, \ \ b_{ij} B^{*i}_\alpha y^j = 0. \tag{66}$$

Hence (64) gives $H^*_0 = 0$. Again from (58), (59) and (65), we have

$$b_{i0} b^i = \frac{1}{2} c_0 b^2, \ \ \lambda^{*m} = 0, \ \ A^{*i}_j B^{*j}_\beta = 0, \ \ B^*_{ij} B^{*i}_\alpha B^{*j}_\beta = \frac{A_1+1}{2\alpha} h^*_{\alpha\beta}. \tag{67}$$

Using (32), (39), (42), (48) and (67) in (58), we get

$$b_r D^{*r}_{ij} B^{*i}_\alpha B^{*j}_\beta = -\frac{(A_1+1) c_0 b^2}{2\alpha} h^*_{\alpha\beta}. \tag{68}$$

Also from the relation $b_{i|j} = b_{ij} - b_r D^{*r}_{ij}$ and (4.31), we get

$$b_{i|j} B^{*i}_{\alpha} B^{*j}_{\beta} = -b_r D^{*r}_{ij} B^{*i}_{\alpha} B^{*j}_{\beta} = \frac{(A_1 + 1)c_0 b^2}{2\alpha} h^*_{\alpha\beta}. \tag{69}$$

Therefore equation (55) reduces to

$$bH^*_{\alpha\beta} + \frac{(A_1 + 1)c_0 b^2}{2\alpha} h^*_{\alpha\beta} = 0. \tag{70}$$

Hence the hypersurface $F^{*n-1}(c)$ is umbilical. Thus, we have

Theorem 3. *The necessary and sufficient condition for a Finslerian hypersurface $F^{*n-1}(c)$ of a Finsler space with Randers change of special (α, β)-metric to be a hyperplane of the first kind is that* (70) *holds and $F^{*n-1}(c)$ is umbilical.*

From lemma 3, the hypersurface $F^{*n-1}(c)$ is a hyperplane of second kind if and only if $H^*_{\alpha} = 0$ and $H^*_{\alpha\beta} = 0$. Thus (70) gives $c_0 = c_i(x)y^i = 0$. Therefore there exists a function $\phi(x)$ such that

$$c_i(x) = \phi(x) b_i(x). \tag{71}$$

Hence (65) reduces to $b_{ij} = \phi b_i b_j$.

Theorem 4. *The necessary and sufficient condition for a Finslerian hypersurface $F^{*n-1}(c)$ of a Finsler space with Randers change of special (α, β)-metric to be a hyperplane of second kind is $b_{ij} = \phi b_i b_j$.*

In view of (48) and (49), Lemma 3 shows that $F^{*n-1}(c)$ does not become a hyperplane of the third kind. Thus, we have

Theorem 5. *The Finslerian hypersurface $F^{*n-1}(c)$ of a Finsler space with Randers change of special (α, β)-metric is not a hyperplane of the third kind.*

References

[1] Gupta M. K., and Pandey, P. N., *On hypersurface of a Finsler space with a special metric*, Acta Math. Hungar. **120** (2008), no. 1-2, 165-177.

[2] Hashiguchi M. and Ichijyo Y., *On some special (α, β)-metrics*, Rep. Fac. Sci. Kagoshima Univ. **8** (1975), 39-46.

[3] Hashiguchi M. and Ichijiyo Y., *Randers spaces with rectilinear geodesics*, Rep. Fac. Sci. Kagasima Univ. **13** (1980), no. 13, 33-40.

[4] Hashiguchi M., Hojo S. and Matsumoto M., *On Landsberg spaces of two dimensions with (α, β)-metric*, J. Korean Math. Soc. **10**, (1973), no. 1, 17-26.

[5] Kitayama M., *On Finslerian hypersurface given by β-changes*, Balkan J. Geom. Appl., **7** (2002), no. 2, 49-55.

[6] Matsumoto M.,*Theory of Finsler spaces with (α, β)-metric*, Rep. Math. Phys. **31** (1992), no. 1, 43-83.

[7] Matsumoto M., *The induced and intrinsic Finsler connections of a hypersurface and Finslerian projective geometry*, J. Math. Kyoto Univ. **25** (1985), 107-144.

[8] Prasad B.N., *On the torsion tensors R_{hjk} and P_{hjk} of Finsler spaces with a metric $ds = \left(g_{ij}(dx)dx^i dx^j\right)^{1/2} + b_i(x, y)dx^i$*, Indian J. Pure Appl. Math. **21** (1990), 27-39.

[9] Rund H., *The Differential Geometry of Finsler Spaces*, Springer-Verlag, Berlin, 1959.

[10] Shukla H. S., Pandey Manmohan, and Prasad B. N., *Hypersurface of a Finsler space with metric $\sum_{r=0}^{m} \frac{\beta^r}{\alpha^{r-1}}$*, J. Int. Acad. of Phy. Sci. **20** (2016), 57-71.

[11] Shibata C., *On invariant tensor of β-change of Finsler metric*, J. Math. Kyoto Univ. **24** (1984), 163-188.

[12] Shankar Gauree and Singh Vijeta, *On the hypersurface of a Finsler space with randers change of generalized (α, β)-metric*, Int. J. Pure Appl. Math. **105** (2015), 223-234.

[13] Singh U. P. and Kumari Bindu, *On a hypersurface of a Matsumoto space*, Indian J. Pure Appl. Math. **32** (2001), 521-531.

[14] Wosoughi H., *On a hypersurface of special Finsler space with an exponential (α, β)-metric*, Int. J. Contemp. Math. Sci. **6** (2011), 1969-1980.

[15] Yong Lee I. L., Park Ha-Yong and Lee Yong-Duk, *On a hypersurface of a special Finsler space with a metric $\alpha + \beta^2/\alpha$*, J. Korean Math. Soc. **8** (2001), 93-101.