Bulletin of the *Transilvania* University of Braşov • Vol 11(60), No. 2 - 2018
Series III: Mathematics, Informatics, Physics, 187-194

NONLINEAR CONNECTIONS ON 2-JET BUNDLE $J^2(\mathfrak{T}, M)$

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Communicated to:
International Conference on Mathematics and Computer Science ,
June 14-16, 2018, Braşov, Romania, 3rd Edition - MACOS 2018

Abstract

The aim of this paper is to introduce the main geometrical objects on the 2-jet bundle $J^2(\mathfrak{T}, M)$, which is the natural extension of the 1-jet bundle $J^1(\mathfrak{T}, M)$. In this direction, we firstly introduce the geometrical concept of a nonlinear connection N on the 2-jet space $J^2(\mathfrak{T}, M)$, in order to construct the adapted bases of vector and covector fields. Then, we describe the Lie brackets of the d-vector fields of the adapted basis of vector fields.

2000 *Mathematics Subject Classification:* 70S05, 53C07, 53C80.
Key words: nonlinear connection, linear connection.

1 Short introduction

It is well known that the 1-jet spaces are used in the study of classical and quantum field theory. For this reason, the differential geometry of 1-jet bundles was intensively studied by authors like Saunders [12], Asanov [1], Neagu, Udrişte [10], [11] Atanasiu and Oana [3]. In the book [11] Neagu constructs a natural multi-parameter extension on 1-jet spaces of the classical Lagrange geometrical theory on the tangent bundle elaborated by Miron and Anastasiei [7]. The book [3] is a distinguished Riemannian geometrization for Hamiltonians depending on polymomenta which naturally extends the already classical Hamiltonian geometry to cotangent bundles synthesized in the Miron et al.'s book [9]. The geometry of higher order Lagrange or Hamilton spaces was elaborated by Miron in [5] and [6] and represents the starting point for the development of the 2-jet multi-time Riemann-Lagrange geometry.

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2 Components of N-linear connections on 2-jet bundle $J^2(\mathcal{T}, M)$

Let $\mathcal{T}$ and M be a *temporal* (resp. *spatial*) real, smooth manifold of dimension m (resp. n), whose coordinates are $(t^a)_{a=\overline{1,m}}$, respectively $(x^i)_{i=\overline{1,n}}$. Note that, throughout this paper, the indices a, b, c, ..., a_1,b_1, c_1,... run from 1 to m, while the indices i, j, k, .., $\overline{i}, \overline{j}, \overline{k}$, ... run from 1 to n. The Einstein convention of summation is also adopted all over this work.

Let $E = J^2(\mathcal{T}, M)$ be the 2-jet fibre bundle, whose coordinates $(t^a, x^i, y^i_a, z^i_{ab})$ are induced from $\mathcal{T}$ and M. The coordinate transformations from the product manifold $\mathcal{T}\times M$ produce on $J^2(\mathcal{T}, M)$ the following coordinate transformations:

$$\begin{gathered} \widetilde{t}^a = \widetilde{t}^a\left(t^b\right), \quad \widetilde{x}^i = \widetilde{x}^i\left(x^j\right), \quad \widetilde{y}^i_a = \frac{\partial \widetilde{x}^i}{\partial x^j}\frac{\partial t^b}{\partial \widetilde{t}^a} y^j_b, \\ 2\widetilde{z}^i_{ab} = \left(\frac{\partial \widetilde{y}^i_a}{\partial x^j}\frac{\partial t^c}{\partial \widetilde{t}^b} + \frac{\partial \widetilde{x}^i}{\partial x^j}\frac{\partial^2 t^c}{\partial \widetilde{t}^a \partial \widetilde{t}^b}\right) y^j_c + 2\frac{\partial \widetilde{y}^i_a}{\partial y^j_c}\frac{\partial t^d}{\partial \widetilde{t}^b} z^j_{cd} \end{gathered} \tag{1}$$

where $\det\left(\partial\widetilde{t}^a/\partial t^b\right) \neq 0$ and $\det\left(\partial\widetilde{x}^i/\partial x^j\right) \neq 0$.

The coordinates transformations (1) determines the transformation of the natural basis $(\frac{\partial}{\partial t^a}\big|_u, \frac{\partial}{\partial x^i}\big|_u, \frac{\partial}{\partial y^i_a}\big|_u, \frac{\partial}{\partial z^i_{ab}}\big|_u)$, $(a = 1, ..., m; i = 1, ..., n)$, of the tangent space TE at the point $u \in E$ as follows:

$$\begin{aligned} \frac{\partial}{\partial t^a}\Big|_u &= \frac{\partial \widetilde{t}^b}{\partial t^a}\frac{\partial}{\partial \widetilde{t}^b}\Big|_u + \frac{\partial \widetilde{y}^j_b}{\partial t^a}\frac{\partial}{\partial \widetilde{y}^j_b}\Big|_u + \frac{\partial \widetilde{z}^j_{b_1b_2}}{\partial t^a}\frac{\partial}{\partial \widetilde{z}^j_{b_1b_2}}\Big|_u \\ \frac{\partial}{\partial x^i}\Big|_u &= \frac{\partial \widetilde{x}^j}{\partial x^i}\frac{\partial}{\partial \widetilde{x}^j}\Big|_u + \frac{\partial \widetilde{y}^j_b}{\partial x^i}\frac{\partial}{\partial \widetilde{y}^j_b}\Big|_u + \frac{\partial \widetilde{z}^j_{b_1b_2}}{\partial x^i}\frac{\partial}{\partial \widetilde{z}^j_{b_1b_2}}\Big|_u \\ \frac{\partial}{\partial y^i_a}\Big|_u &= \frac{\partial \widetilde{x}^j}{\partial x^i}\frac{\partial t^a}{\partial \widetilde{t}^b}\frac{\partial}{\partial \widetilde{y}^j_b}\Big|_u + \frac{\partial \widetilde{z}^j_{b_1b_2}}{\partial y^i_a}\frac{\partial}{\partial \widetilde{z}^j_{b_1b_2}}\Big|_u \\ \frac{\partial}{\partial z^i_{ab}}\Big|_u &= \frac{\partial \widetilde{x}^j}{\partial x^i}\frac{\partial t^a}{\partial \widetilde{t}^c}\frac{\partial t^b}{\partial \widetilde{t}^d}\frac{\partial}{\partial \widetilde{z}^j_{cd}}\Big|_u \end{aligned} \tag{2}$$

Proposition 1. *Under a change of coordinates (1), the elements of the dual basis*

$(dt^a, dx^i, dy^i_a, dz^i_{a_1a_2}) \subset \mathcal{X}^*(E)$ *of 1-forms on* E *satisfy the transformation laws*

$$\begin{aligned}
dt^a &= \frac{\partial t^a}{\partial \widetilde{t}^b} d\widetilde{t}^b \\
dx^i &= \frac{\partial x^i}{\partial \widetilde{x}^j} d\widetilde{x}^j \\
dy^i_a &= \frac{\partial y^i_a}{\partial \widetilde{t}^b} d\widetilde{t}^b + \frac{\partial y^i_a}{\partial \widetilde{x}^j} d\widetilde{x}^j + \frac{\partial x^i}{\partial \widetilde{x}^j}\frac{\partial \widetilde{t}^b}{\partial t^a} d\widetilde{y}^j_b \\
dz^i_{a_1a_2} &= \frac{\partial z^i_{a_1a_2}}{\partial \widetilde{t}^b} d\widetilde{t}^b + \frac{\partial z^i_{a_1a_2}}{\partial \widetilde{x}^j} d\widetilde{x}^j + \frac{\partial z^i_{a_1a_2}}{\partial \widetilde{y}^j_b} d\widetilde{y}^j_b + \frac{\partial \widetilde{x}^j}{\partial x^i}\frac{\partial t^{a_1}}{\partial \widetilde{t}^{b_1}}\frac{\partial t^{a_2}}{\partial \widetilde{t}^{b_2}} d\widetilde{z}^j_{b_1b_2}.
\end{aligned} \tag{3}$$

A set of local functions on $E = J^2(\mathcal{T}, M)$, denoted by

$$N = \left(\underset{01}{N}{}^{(i)}_{(a)b}, \underset{02}{N}{}^{(i)}_{(a)c}, \underset{11}{N}{}^{(i)}_{(a)j}, \underset{12}{N}{}^{(i)}_{(a)j}, \underset{21}{N}{}^{(i)(b)}_{(a)(j)} \right),$$

whose local components obey the transformation rules

$$\begin{cases}
\underset{01}{\widetilde{N}}{}^{(j)}_{(b)c} \dfrac{\partial \widetilde{t}^c}{\partial t^a} = \underset{01}{N}{}^{(k)}_{(c)a} \dfrac{\partial t^c}{\partial \widetilde{t}^b}\dfrac{\partial \widetilde{x}^j}{\partial x^k} - \dfrac{\partial \widetilde{y}^j_b}{\partial t^a}, \\[2ex]
\underset{02}{\widetilde{N}}{}^{(j)}_{(b_1b_2)c} \dfrac{\partial \widetilde{t}^c}{\partial t^a} = \underset{02}{N}{}^{(k)}_{(c_1c_2)a} \dfrac{\partial t^{c_1}}{\partial \widetilde{t}^{b_1}}\dfrac{\partial t^{c_2}}{\partial \widetilde{t}^{b_2}}\dfrac{\partial \widetilde{x}^j}{\partial x^k} - \dfrac{\partial \widetilde{z}^j_{b_1b_2}}{\partial t^a} + \underset{01}{N}{}^{(k)}_{(c)a} \dfrac{\partial \widetilde{z}^j_{b_1b_2}}{\partial y^k_c},
\end{cases}$$

$$\begin{cases}
\underset{11}{\widetilde{N}}{}^{(j)}_{(b)k} \dfrac{\partial \widetilde{x}^k}{\partial x^i} = \underset{11}{N}{}^{(k)}_{(c)i} \dfrac{\partial t^c}{\partial \widetilde{t}^b}\dfrac{\partial \widetilde{x}^j}{\partial x^k} - \dfrac{\partial \widetilde{y}^j_b}{\partial x^i}, \\[2ex]
\underset{12}{\widetilde{N}}{}^{(j)}_{(b_1b_2)k} \dfrac{\partial \widetilde{x}^k}{\partial x^i} = \underset{12}{N}{}^{(k)}_{(c_1c_2)i} \dfrac{\partial t^{c_1}}{\partial \widetilde{t}^{b_1}}\dfrac{\partial t^{c_2}}{\partial \widetilde{t}^{b_2}}\dfrac{\partial \widetilde{x}^j}{\partial x^k} - \dfrac{\partial \widetilde{z}^j_{b_1b_2}}{\partial x^i} + \underset{11}{N}{}^{(k)}_{(c)i} \dfrac{\partial \widetilde{z}^j_{b_1b_2}}{\partial y^k_c},
\end{cases}$$

$$\underset{21}{\widetilde{N}}{}^{(i)(b)}_{(a_1a_2)(j)} \frac{\partial \widetilde{x}^j}{\partial x^l}\frac{\partial \widetilde{t}^d}{\partial t^b} = \underset{21}{N}{}^{(k)(d)}_{(c_1c_2)(l)} \frac{\partial t^{c_1}}{\partial \widetilde{t}^{a_1}}\frac{\partial t^{c_2}}{\partial \widetilde{t}^{a_2}}\frac{\partial \widetilde{x}^i}{\partial x^k} - \frac{\partial \widetilde{z}^k_{c_1c_2}}{\partial y^l_d}$$

is called a *nonlinear connection* on E. The components $\underset{01}{N}{}^{(i)}_{(a)b}, \underset{02}{N}{}^{(i)}_{(a_1a_2)c}$ (resp. $\underset{11}{N}{}^{(i)}_{(a)j}, \underset{12}{N}{}^{(i)}_{(a_1a_2)j}$) are called the *temporal* (resp. *spatial*) *components* of N.
In what follows, we fix a nonlinear connection on E, and we consider the *adapted bases* of the nonlinear connection N, defined by

$$\left\{ \frac{\delta}{\delta t^a}, \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^i_a}\frac{\partial}{\partial z^i_{ab}} \right\} \subset \mathcal{X}(E), \quad \left\{ dt^a, dx^i, \delta y^i_a, \delta z^i_{ab} \right\} \subset \mathcal{X}^*(E), \tag{4}$$

where

$$\frac{\delta}{\delta t^a} = \frac{\partial}{\partial t^a} - \underset{01}{N}{}^{(j)}_{(b)a}\frac{\partial}{\partial y^j_b} - \underset{02}{N}{}^{(j)}_{(b_1b_2)a}\frac{\partial}{\partial z^j_{b_1b_2}},$$

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{11}{N}{}^{(j)}_{(b)i}\frac{\partial}{\partial y^j_b} - \underset{12}{N}{}^{(j)}_{(b_1b_2)i}\frac{\partial}{\partial z^j_{b_1b_2}},$$

$$\frac{\delta}{\delta y^i_a} = \frac{\partial}{\partial y^i_a} - \underset{21}{N}{}^{(j)(a)}_{(b_1b_2)(i)}\frac{\partial}{\partial z^j_{b_1b_2}},$$

$$\delta y^i_a = dy^i_a + \underset{21}{M}{}^{(i)}_{(a)j}dx^j + \underset{22}{M}{}^{(i)}_{(a)b}dt^b$$

$$\delta z^i_{ab} = dz^i_{ab} + \underset{31}{M}{}^{(i)(c)}_{(ab)(k)}dy^k_c + \underset{32}{M}{}^{(i)}_{(ab)k}dx^k + \underset{33}{M}{}^{(i)}_{(ab)c}dt^c.$$

where

$$\begin{aligned}
\underset{21}{M}{}^{(i)}_{(a)j} &= \underset{11}{N}{}^{(i)}_{(a)j}\\
\underset{22}{M}{}^{(i)}_{(a)b} &= \underset{01}{N}{}^{(i)}_{(a)b}\\
\underset{31}{M}{}^{(i)(b)}_{(a_1a_2)(j)} &= \underset{21}{N}{}^{(i)(b)}_{(a_1a_2)(j)}\\
\underset{32}{M}{}^{(i)}_{(a_1a_2)j} &= \underset{12}{N}{}^{(i)}_{(a_1a_2)j} + \underset{21}{N}{}^{(i)(c)}_{(a_1a_2)(k)}\underset{11}{N}{}^{(k)}_{(c)j}\\
\underset{33}{M}{}^{(i)}_{(a_1a_2)b} &= \underset{02}{N}{}^{(i)}_{(a_1a_2)b} + \underset{21}{N}{}^{(i)(c)}_{(a_1a_2)(k)}\underset{01}{N}{}^{(k)}_{(c)b}.
\end{aligned} \tag{5}$$

It is important to note that the transformation rules of the elements of the adapted bases (4) are tensorial ones:

$$\begin{aligned}
&\frac{\delta}{\delta t^a} = \frac{\partial \tilde{t}^b}{\partial t^a}\frac{\delta}{\delta \tilde{t}^b}, \qquad \frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{x}^j},\\
&\frac{\delta}{\delta y^i_a} = \frac{\partial t^a}{\partial \tilde{t}^b}\frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \widetilde{y}^j_b}, \quad \frac{\partial}{\partial z^i_{a_1a_2}} = \frac{\partial t^{a_1}}{\partial \tilde{t}^{b_1}}\frac{\partial t^{a_2}}{\partial \tilde{t}^{b_2}}\frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \widetilde{z}^i_{b_1b_2}},\\
&dt^a = \frac{\partial t^a}{\partial \tilde{t}^b}d\tilde{t}^b, \qquad dx^i = \frac{\partial x^i}{\partial \tilde{x}^j}d\tilde{x}^j,\\
&\delta y^i_a = \frac{\partial \tilde{t}^b}{\partial t^a}\frac{\partial x^i}{\partial \tilde{x}^j}\delta \widetilde{y}^j_b, \quad \delta z^i_{a_1a_2} = \frac{\partial \tilde{t}^{b_1}}{\partial t^{a_1}}\frac{\partial \tilde{t}^{b_2}}{\partial t^{a_2}}\frac{\partial x^i}{\partial \tilde{x}^j}\delta \widetilde{z}^j_{b_1b_2}.
\end{aligned} \tag{6}$$

Remark 2. *The simple tensorial transformation rules (6) of the adapted bases (4) determined us to prefer them instead of those from the tangent bases whose transformation rules (2) are not tensorial ones. We describe in what follows the geometrical objects on the* 2*-jet space* $J^2(\mathcal{T}, M)$ *in adapted local components.*

Proposition 3. *A transformation of coordinates (1) on the differentiable manifold* $J^2(\mathcal{T}, M)$ *implies the following transformation of the dual coefficients*

$$\underset{21}{\widetilde{M}}{}^{(j)}_{(b)k}\frac{\partial \widetilde{x}^k}{\partial x^i} = \underset{21}{M}{}^{(k)}_{(c)i}\frac{\partial t^c}{\partial \widetilde{t}^b}\frac{\partial \widetilde{x}^j}{\partial x^k} - \frac{\partial \widetilde{y}^j_b}{\partial x^i},$$

$$\underset{22}{\widetilde{M}}{}^{(j)}_{(b)c}\frac{\partial \widetilde{t}^c}{\partial t^a} = \underset{22}{M}{}^{(k)}_{(c)a}\frac{\partial t^c}{\partial \widetilde{t}^b}\frac{\partial \widetilde{x}^j}{\partial x^k} - \frac{\partial \widetilde{y}^j_b}{\partial t^a},$$

$$\underset{31}{\widetilde{M}}{}^{(i)(b)}_{(a_1a_2)(j)}\frac{\partial \widetilde{x}^j}{\partial x^l}\frac{\partial \widetilde{t}^d}{\partial t^b} = \underset{31}{M}{}^{(k)(d)}_{(c_1c_2)(l)}\frac{\partial t^{c_1}}{\partial \widetilde{t}^{a_1}}\frac{\partial t^{c_2}}{\partial \widetilde{t}^{a_2}}\frac{\partial \widetilde{x}^i}{\partial x^k} - \frac{\partial \widetilde{z}^k_{c_1c_2}}{\partial y^l_d}$$

$$\underset{32}{\widetilde{M}}{}^{(i)}_{(a_1a_2)k}\frac{\partial \widetilde{x}^k}{\partial x^j} = \underset{33}{M}{}^{(k)}_{(c_1c_2)j}\frac{\partial t^{c_1}}{\partial \widetilde{t}^{a_1}}\frac{\partial t^{c_2}}{\partial \widetilde{t}^{a_2}}\frac{\partial x^i}{\partial \widetilde{x}^k} - \frac{\partial \widetilde{z}^i_{a_1a_2}}{\partial x^j} - \underset{21}{N}{}^{(i)(d)}_{(a_1a_2)(l)}\frac{\partial \widetilde{y}^l_d}{\partial x^j},$$

$$\underset{33}{\widetilde{M}}{}^{(i)}_{(a_1a_2)b}\frac{\partial \widetilde{t}^b}{\partial t^a} = \underset{33}{M}{}^{(k)}_{(c_1c_2)b}\frac{\partial t^{c_1}}{\partial \widetilde{t}^{a_1}}\frac{\partial t^{c_2}}{\partial \widetilde{t}^{a_2}}\frac{\partial x^i}{\partial \widetilde{x}^k} - \frac{\partial \widetilde{z}^i_{a_1a_2}}{\partial t^a} - \underset{21}{N}{}^{(i)(b)}_{(a_1a_2)(j)}\frac{\partial \widetilde{y}^j_b}{\partial t^a}.$$

In order to develop the geometrical theory of N-linear connections on the 2-jet space E, we need the following result:

Proposition 4. (i) *The Lie algebra* $\mathcal{X}(E)$ *of vector fields decomposes as*

$$\mathcal{X}(E) = \mathcal{X}(\mathcal{H}_{\mathcal{T}}) \oplus \mathcal{X}(\mathcal{H}_M) \oplus \mathcal{X}(\mathcal{V}_1) \oplus \mathcal{X}(\mathcal{V}_2),$$

where

$$\mathcal{X}(\mathcal{H}_{\mathcal{T}}) = \text{Span}\left\{\frac{\delta}{\delta t^a}\right\}, \quad \mathcal{X}(\mathcal{H}_M) = \text{Span}\left\{\frac{\delta}{\delta x^i}\right\},$$

$$\mathcal{X}(\mathcal{V}_1) = \text{Span}\left\{\frac{\delta}{\delta y^i_a}\right\}, \quad \mathcal{X}(\mathcal{V}_2) = \text{Span}\left\{\frac{\partial}{\partial z^i_{a_1a_2}}\right\}.$$

(ii) *The Lie algebra* $\mathcal{X}^*(E)$ *of covector fields decomposes as*

$$\mathcal{X}^*(E) = \mathcal{X}^*(\mathcal{H}_{\mathcal{T}}) \oplus \mathcal{X}^*(\mathcal{H}_M) \oplus \mathcal{X}^*(\mathcal{V}_1) \oplus \mathcal{X}^*(\mathcal{V}_2),$$

where

$$\mathcal{X}^*(\mathcal{H}_{\mathcal{T}}) = \text{Span}\{dt^a\}, \quad \mathcal{X}^*(\mathcal{H}_M) = \text{Span}\{dx^i\},$$

$$\mathcal{X}^*(\mathcal{V}_1) = \text{Span}\{\delta y^i_a\}, \quad \mathcal{X}^*(\mathcal{V}_2) = \text{Span}\{\delta z^i_{a_1a_2}\}.$$

3 Lie brakets

In order to obtain an adapted local characterization of the torsion tensor associated to an N-linear connection ∇, we deduce, by direct computations, the following result:

Proposition 5. *The following identities of the Lie brackets are true:*

$$\left[\frac{\delta}{\delta t^b},\frac{\delta}{\delta t^c}\right]=\overset{2}{\underset{00}{R}}{}^{(i)}_{(a)bc}\frac{\delta}{\delta y^i_a}+\overset{3}{\underset{00}{R}}{}^{(i)}_{(a_1a_2)bc}\frac{\partial}{\partial z^i_{a_1a_2}}$$

$$\left[\frac{\delta}{\delta t^b},\frac{\delta}{\delta x^j}\right]=\overset{2}{\underset{01}{R}}{}^{(i)}_{(a)bj}\frac{\delta}{\delta y^i_a}+\overset{3}{\underset{01}{R}}{}^{(i)}_{(a_1a_2)bj}\frac{\partial}{\partial z^i_{a_1a_2}}$$

$$\left[\frac{\delta}{\delta t^b},\frac{\delta}{\delta y^k_c}\right]=\overset{2}{\underset{02}{R}}{}^{(i)\ (c)}_{(a)b(k)}\frac{\delta}{\delta y^i_a}+\overset{3}{\underset{02}{R}}{}^{(i)\ (c)}_{(a_1a_2)b(k)}\frac{\partial}{\partial z^i_{a_1a_2}}$$

$$\left[\frac{\delta}{\delta t^b},\frac{\partial}{\partial z^k_\gamma}\right]=\overset{2}{\underset{03}{R}}{}^{(m)\ (\gamma)}_{(f)b(k)}\frac{\delta}{\delta y^m_f}+\overset{3}{\underset{03}{R}}{}^{(m)\ (\gamma)}_{(a_1a_2)b(k)}\frac{\partial}{\partial z^i_{a_1a_2}}$$

$$\left[\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\right]=\overset{2}{\underset{11}{B}}{}^{(i)}_{(a)jk}\frac{\delta}{\delta y^i_a}+\overset{3}{\underset{11}{B}}{}^{(i)}_{(a_1a_2)jk}\frac{\partial}{\partial z^i_{a_1a_2}}$$

$$\left[\frac{\delta}{\delta x^j},\frac{\delta}{\delta y^k_c}\right]=\overset{2}{\underset{12}{B}}{}^{(i)\ (c)}_{(a)j(k)}\frac{\delta}{\delta y^i_a}+\overset{3}{\underset{12}{B}}{}^{(i)\ (c)}_{(a_1a_2)j(k)}\frac{\partial}{\partial z^i_{a_1a_2}}$$

$$\left[\frac{\delta}{\delta x^j},\frac{\partial}{\partial z^k_\gamma}\right]=\overset{2}{\underset{13}{B}}{}^{(i)\ (\gamma)}_{(a)j(k)}\frac{\delta}{\delta y^i_a}+\overset{3}{\underset{12}{B}}{}^{(i)\ (\gamma)}_{(a_1a_2)j(k)}\frac{\partial}{\partial z^i_{a_1a_2}}$$

$$\left[\frac{\delta}{\delta y^j_b},\frac{\delta}{\delta y^k_c}\right]=\overset{3}{\underset{22}{B}}{}^{(i)(b)(c)}_{(a_1a_2)(j)(k)}\frac{\partial}{\partial z^i_{a_1a_2}}$$

$$\left[\frac{\delta}{\delta y^j_b},\frac{\partial}{\partial z^k_\gamma}\right]=\overset{3}{\underset{23}{B}}{}^{(i)(b)(\gamma)}_{(a_1a_2)(j)(k)}\frac{\partial}{\partial z^i_{a_1a_2}}$$

where

$$\overset{2}{\underset{00}{R}}{}^{(i)}_{(a)bc}=\delta_c\underset{01}{N}{}^{(i)}_{(a)b}-\delta_b\underset{01}{N}{}^{(i)}_{(a)c},$$

$$\overset{3}{\underset{00}{R}}{}^{(i)}_{(a_1a_2)bc}=\delta_c\underset{02}{N}{}^{(i)}_{(a_1a_2)b}-\delta_b\underset{02}{N}{}^{(i)}_{(a_1a_2)c}+\overset{2}{\underset{00}{R}}{}^{(l)}_{(d)bc}\underset{21}{N}{}^{(i)(d)}_{(a_1a_2)(l)},$$

$$\overset{2}{\underset{01}{R}}{}^{(i)}_{(a)bj}=\delta_j\underset{01}{N}{}^{(i)}_{(a)b}-\delta_b\underset{11}{N}{}^{(i)}_{(a)j},$$

$$\overset{3}{\underset{01}{R}}{}^{(i)}_{(a_1a_2)bj}=\delta_j\underset{02}{N}{}^{(i)}_{(a_1a_2)b}-\delta_b\underset{12}{N}{}^{(i)}_{(a_1a_2)j}+\overset{2}{\underset{01}{R}}{}^{(l)}_{(d)bj}\underset{21}{N}{}^{(i)(d)}_{(a_1a_2)(l)},$$

$$\underset{02}{\overset{2}{R}}{}^{(i)\ (c)}_{(a)b(k)} = \delta_{\substack{(k)\\(c)}} \underset{01}{N}{}^{(i)}_{(a)b},$$

$$\underset{02}{\overset{3}{R}}{}^{(i)\ (c)}_{(a_1a_2)b(k)} = \delta_{\substack{(k)\\(c)}} \underset{02}{N}{}^{(i)}_{(a_1a_2)b} - \delta_b \underset{21}{N}{}^{(i)(c)}_{(a_1a_2)(k)} + \underset{02}{\overset{2}{R}}{}^{(l)\ (c)}_{(d)b(k)} \underset{21}{N}{}^{(i)(d)}_{(a_1a_2)(l)},$$

$$\underset{03}{\overset{2}{R}}{}^{(i)\ (c_1c_2)}_{\ (a)b(k)} = \dot{\partial}_{\substack{(k)\\(c_1c_2)}} \underset{01}{N}{}^{(i)}_{(a)b},$$

$$\underset{03}{\overset{3}{R}}{}^{(i)\ (c_1c_2)}_{(a_1a_2)b(k)} = \dot{\partial}_{\substack{(k)\\(c_1c_2)}} \underset{02}{N}{}^{(i)}_{(a_1a_2)b} + \underset{03}{\overset{2}{R}}{}^{(l)\ (c_1c_2)}_{\ (d)b(k)} \underset{21}{N}{}^{(i)(d)}_{(a_1a_2)(l)},$$

$$\underset{11}{\overset{2}{B}}{}^{(i)}_{(a)jk} = \delta_k \underset{11}{N}{}^{(i)}_{(a)j} - \delta_j \underset{11}{N}{}^{(i)}_{(a)k},$$

$$\underset{11}{\overset{3}{B}}{}^{(i)}_{(a_1a_2)jk} = \delta_k \underset{12}{N}{}^{(i)}_{(a_1a_2)j} - \delta_j \underset{02}{N}{}^{(i)}_{(a_1a_2)k} + \underset{11}{\overset{2}{B}}{}^{(l)}_{(d)jk} \underset{21}{N}{}^{(i)(d)}_{(a_1a_2)(l)},$$

$$\underset{12}{\overset{2}{B}}{}^{(i)\ (c)}_{(a)j(k)} = \delta_{\substack{(k)\\(c)}} \underset{11}{N}{}^{(i)}_{(a)j},$$

$$\underset{12}{\overset{3}{B}}{}^{(i)\ (c)}_{(a_1a_2)j(k)} = \delta_{\substack{(k)\\(c)}} \underset{12}{N}{}^{(i)}_{(a_1a_2)j} - \delta_j \underset{21}{N}{}^{(i)(c)}_{(a_1a_2)(k)} + \underset{12}{\overset{2}{B}}{}^{(l)\ (c)}_{(d)j(k)} \underset{21}{N}{}^{(i)(d)}_{(a_1a_2)(l)},$$

$$\underset{13}{\overset{2}{B}}{}^{(i)\ (c_1c_2)}_{\ (a)b(k)} = \dot{\partial}_{\substack{(k)\\(c_1c_2)}} \underset{11}{N}{}^{(i)}_{(a)j},$$

$$\underset{13}{\overset{3}{B}}{}^{(i)\ (c_1c_2)}_{(a_1a_2)j(k)} = \dot{\partial}_{\substack{(k)\\(c_1c_2)}} \underset{12}{N}{}^{(i)}_{(a_1a_2)j} + \underset{13}{\overset{2}{B}}{}^{(l)\ (c_1c_2)}_{\ (d)j(k)} \underset{21}{N}{}^{(i)(d)}_{(a_1a_2)(l)},$$

$$\underset{22}{\overset{3}{B}}{}^{(i)(b)(c_1c_2)}_{(a_1a_2)(j)(k)} = \delta_{\substack{(k)\\(c)}} \underset{21}{N}{}^{(i)(b)}_{(a_1a_2)(j)} - \delta_{\substack{(j)\\(b)}} \underset{21}{N}{}^{(i)(c)}_{(a_1a_2)(k)},$$

$$\underset{23}{\overset{3}{B}}{}^{(i)(b)(c_1c_2)}_{(a_1a_2)(j)(k)} = \dot{\partial}_{\substack{(k)\\(c_1c_2)}} \underset{21}{N}{}^{(i)(b)}_{(a_1a_2)(j)}.$$

References

[1] Asanov, G. S., *Jet extension of Finslerian gauge approach*, Fortschritte der Physik **38** (1990), no. 8, 571-610.

[2] Atanasiu, Gh., *New aspects in differential geometry of the second order*, Seminarul de Mecanică, Sisteme Dinamice Diferenţiale, Univ. de Vest din Timişoara, Facultatea de Matematică, **82** (2001), 1-81.

[3] Atanasiu, Gh., Neagu M., Oana, A., *The Geometry of Jet Multi-time Lagrange and Hamilton Spaces*, Applications in Theoretical Physics, Fair Partners Publishers, Bucharest, 2013.

[4] Balan, V., Neagu, M., *Jet Single-Time Lagrange Geometry and Its Applications*, John Wiley & Sons, Inc., Hoboken, New Jersey, 2011.

[5] Miron, R., *The geometry of higher order Lagrange spaces. Applications to mechanics and physics*, Kluwer Acd. Publ., FTPH no. 82, 1997.

[6] Miron, R., *On the Geometrical Theory of Higher-Order Hamilton Spaces*, Steps in Differential Geometry, Proc. of the Colloquim on Diff. Geom., 25-30 July, 2000, Debrecen, Hungary.

[7] Miron, R., Anastasiei, M., *The geometry of Lagrange spaces: Theory and applications*, Kluwer Academic Publishers, Dordrecht, 1994.

[8] Miron, R., Atanasiu, Gh., *Differential geometry of the k-osculator bundle*, Rev. Roumaine, Math Pures et Appl. **41** (1996), no. 3-4, 205-236.

[9] Miron, R., Hrimiuc, D., Shimada, H., Sabău, S. V., *The geometry of Hamilton and Lagrange spaces*, Dordrecht: Kluwer Academic Publishers 2001.

[10] Neagu, M., Udrişte, C., *Torsion, curvature and deflection d-tensors on* $J^1(T,M)$, Balkan Journal of Geometry and Its Applications **6** (2001), no. 1 29-45.

[11] Neagu, M., *Riemann-Lagrange geometry on* 1*-jet spaces*. Bucharest: Matrix Rom 2005.

[12] Saunders, D.J., *The geometry of jet bundles*, Cambridge University Press, New York, 1989.