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VORONOVSKAJA THEOREM FOR COMBINATIONS OF BETA OPERATORS

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Abstract

In this paper we investigate the Voronovskaya type theorem for the Boolean sum of Beta operators

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1 Introduction

A general Voronovskaja type theorem for a sequence of linear positive operators $(L_n)_n$, is a limit of the form:

$$\lim_{n\longrightarrow\infty} \alpha_n(L_n(f)(x) - f(x)) = E(x, f'(x), f''(x), ...) \tag{1}$$

where $(\alpha_n)_n, \alpha_n \longrightarrow \infty$. For classical operators of approximation the usual value for α_n is $\alpha_n = n$.

Using combinations of linear positive operators it is possible the improve the order of approximation of the original operators. One of the methods is to use the Boolean sums. By Boolean sum of two operators S and T we means operator $S + T - S \circ T$. The purpose of the work is to use Boolean sums of Beta operators in order to obtain an improvement in Voronovskaja type theorem. Namely, there is obtained a sequence of positive linear operators $(L_n)_n,$, $L_n : C[0, 1] \longrightarrow C[0, 1]$ for which the following conditions:

1) $\lim_{n\longrightarrow\infty} L_n f = f$ uniformly on $[0, 1]$

2) $\exists \alpha_n$ cu $\dfrac{\alpha_n}{n} \longrightarrow \infty$ for which it takes place (1).

are satisfied.

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2 Beta operator and combinations of Beta operators

The operators Beta, introduced by A Lupaş [4], are defined in this way:

$$\mathbb{B}_n(f)(x) = \int_0^1 \frac{t^{nx}(1-t)^{n(1-x)}}{B(nx+1, n+1-nx)} f(t)dt, \ f \in C[0,1], \ x \in [0,1], \qquad (2)$$

where $B(u,v)$ is Beta function.

For the Beta operators, the following result is known:

Theorem B *We have:*

1) $\lim_{n \longrightarrow \infty} \mathbb{B}_n(f)(x) = f(x)$ *uniformly* $\forall f \in C[0,1]$,

2) If $f \in C^{IV}[0,1]$ *and for each* $x \in [0,1]$ *we have:*

$$\lim_{n \longrightarrow \infty} n[\mathbb{B}_n(f)(x) - f(x)] = (1-2x)f'(x) + \frac{1}{2}x(1-x)f''(x). \qquad (3)$$

Therefore the Beta operators satisfy Voronovskaja theorem with $\alpha_n = n$, as well. In order to a obtain a better approximation order in Voronovskaja theorem, we consider the sequence of operators, $(L_n)_n$, defined as follows:

$$L_n := 2\mathbb{B}_n - \mathbb{B}_n \circ \mathbb{B}_n. \qquad (4)$$

For the study of operators L_n, we need calculating the moments of the operators $\mathbb{B}_n$. Let be e_k, the monomial applications $e_k(t) = t^k$. We obtain:

$$(\mathbb{B}_n e_p)(x) = \frac{(nx+p)(nx+p-1)\cdots(nx+1)}{(n+2)(n+3)\cdots(n+p+1)}.$$

Afterwards, if $k = 0, 1, 2, \ldots$, $n \geq 1$, $x \in [0,1]$, we denote:

$$m_{n,k}(x) = \mathbb{B}_n((e_1 - xe_0)^k)(x).$$

Using the formulas above, and the function $\Psi(x) = x(1-x)$ we get:

Lemma 1. *We have:*

$$m_{n,1}(x) = A_{n,1}\Psi'(x) \qquad (5)$$
$$m_{n,2}(x) = A_{n,2}\Psi(x) + B_{n,2} \qquad (6)$$
$$m_{n,3}(x) = A_{n,3}\Psi(x)\Psi'(x) + B_{n,3}\Psi'(x) \qquad (7)$$
$$m_{n,4}(x) = A_{n,4}\Psi(x)^2 + B_{n,4}\Psi(x) + C_{n,4}, \qquad (8)$$

where

$$A_{n,1} = \frac{1}{n+2},\ A_{n,2} = \frac{n-6}{(n+2)(n+3)},\ B_{n,2} = \frac{2}{(n+2)(n+3)} \tag{9}$$

$$A_{n,3} = \frac{5n-12}{(n+2)(n+3)(n+4)},\ B_{n,3} = \frac{6}{(n+2)(n+3)(n+4)} \tag{10}$$

$$A_{n,4} = \frac{3n^2-86n+120}{(n+2)(n+3)(n+4)(n+5)} \tag{11}$$

$$B_{n,4} = \frac{26n-120}{(n+2)(n+3)(n+4)(n+5)} \tag{12}$$

$$C_{n,4} = \frac{24}{(n+2)(n+3)(n+4)(n+5)}. \tag{13}$$

Moreover, $m_{n,6}(x) = O\Big(\frac{1}{n^3}\Big)$.

The main result is the following:

Theorem 1. *Operators L_n satisfy the following conditions:*

i) If $f \in C[0,1]$, then:

$$\lim_{n\to\infty} L_n(f)(x) = f(x),\ \textit{unifomly for}\ x \in [0,1].$$

ii) If $f \in C^4[0,1]$ and $x \in [0,1]$, then:

$$\lim_{n\to\infty} n^2[L_n(f)(x) - f(x)] = 2f'(x)\Psi'(x) + f''(x)\Big[-\frac{3}{2} + 8\Psi(x)\Big] - 2f'''(x)\Psi(x)\Psi'(x) - \frac{1}{2}f^{IV}(x)\Psi^2(x).$$

Proof. i) Let be $f \in C[0,1]$, $n \geq 1$. We write:

$$L_n(f) - f = (B_n f - f) - (B_n(B_n f - f)).$$

From Theorem A-i) we obtain:

$$\lim_{n\longrightarrow\infty} \mathbb{B}_n f = f,\ \text{uniformly}.$$

Then we take into account that $\|\mathbb{B}_n\| = 1$.

ii) If $g \in C^k[0,1]$, $k \geq 0$, $t, x \in [0,1]$, we denote by $R_k[g](t,x)$, the remainder of order k in Taylor polynomial around the point x of function g at point t, namely:

$$g(t) = \sum_{j=0}^{k} \frac{g^j(x)}{j!}(t-x)^j + R_k[g](t,x).$$

We can write $R_k[g](t,x) = (t-x)^k \omega_k[g](t,x)$, where

$$\lim_{t\to x} \omega_k[g](t,x) = 0.$$

Further, we evaluate the remainders. From point i) of theorem, we have:

$$\lim_{n\to\infty}(\mathbb{B}(|\omega_k[g](\cdot,x)|^p)(x) = 0,\ k \geq 0,\ p > 0, \tag{14}$$

because $\lim_{t\to x}\omega_k[g](t,x) = 0$ and $\omega_k[g](x,x) = 0$. We have the following limits:

a) If $g \in C[0,1]$, we have directly from (14):

$$\lim_{n\to\infty}\mathbb{B}_n(R_0[g](\cdot,x))(x) = 0. \tag{15}$$

b) If $g \in C^2[0,1]$, we have:

$$|\mathbb{B}_n(R_2[g](\cdot,x))(x)| \leq \mathbb{B}_n(|R_2[g](\cdot,x)|)(x) = \mathbb{B}_n((e_1 - xe_0)^2|\omega_2[g](\cdot,x))|)(x).$$

Afterwards, from Cauchy-Schwartz inequality:

$$\mathbb{B}_n((e_1 - xe_0)^2|\omega_2[g](\cdot,x)|)(x) \leq [\mathbb{B}_n((e_1 - xe_0)^4)(x)]^{\frac{1}{2}}(\mathbb{B}_n(|\omega_2[g](\cdot,x)|^2)(x))^{\frac{1}{2}}$$
$$= (m_{n,4}(x))^{\frac{1}{2}}[\mathbb{B}(|\omega_2[g](\cdot,x)|^2)(x)]^{\frac{1}{2}}$$

Since $m_{n,4} = O\Big(\frac{1}{n^2}\Big)$ and using (14), we obtain:

$$\lim_{n\to\infty} n\mathbb{B}_n(R_2[g](\cdot,x))(x) = 0. \tag{16}$$

c) If $g \in C^4[0,1]$, we have:

$$|\mathbb{B}_n(R_4[g](\cdot,x))(x)| \leq \mathbb{B}_n(|R_4[g](\cdot,x)|)(x) = \mathbb{B}_n((e_1 - xe_0)^4|\omega_4[g](\cdot,x)|)(x)$$

From inequality of Hölder:

$$\mathbb{B}_n((e_1 - x)^4|\omega_k[g](\cdot,x)|)(x) \leq [\mathbb{B}_n((e_1 - xe_0)^{4\cdot\frac{3}{2}})(x)]^{\frac{2}{3}}(\mathbb{B}_n(|\omega_4[g](\cdot,x)|^3)(x))^{\frac{1}{3}}$$
$$= (m_{n,6}(x))^{\frac{2}{3}}[(\mathbb{B}|\omega_k[g](\cdot,x)|^3)(x)]^{\frac{1}{3}}$$

Since $m_{n,6} = O\Big(\frac{1}{n^3}\Big)$ and using (14),

$$\lim_{n\to\infty} n^2\mathbb{B}_n(R_4[g](\cdot,x))(x) = 0. \tag{17}$$

We have:

$$\begin{aligned}
\mathbb{B}_n(f)(x) &= \sum_{j=0}^{4}\frac{f^{(j)}(x)}{j!}m_{n,j}(x) + o\Big(\frac{1}{n^2}\Big) \\
&= f(x) + f'(x)[A_{n,1}\Psi'(x)] + \frac{f''(x)}{2}[A_{n,2}\Psi(x) + B_{n,2}(x)] \\
&\quad + \frac{f'''(x)}{3!}[A_{n,3}\Psi(x)\Psi'(x) + B_{n,3}\Psi'(x)] \\
&\quad + \frac{f^{IV}(x)}{4!}[A_{n,4}\Psi^2(x) + B_{n,4}\Psi(x) + C_{n,4}] + o\Big(\frac{1}{n^2}\Big) \\
&= f(x) + f'(x)[A_{n,1}\Psi'(x)] + \frac{f''(x)}{2}[A_{n,2}\Psi(x) + B_{n,2}(x)] \\
&\quad + \frac{f'''(x)}{3!}[A_{n,3}\Psi(x)\Psi'(x)] + \frac{f^{IV}(x)}{4!}[A_{n,4}\Psi^2(x)] + o\Big(\frac{1}{n^2}\Big).
\end{aligned}$$

Then we get:

$$\begin{aligned} L_n(f)(x) &= \sum_{j=0}^{4} \frac{f^{(j)}(x)}{j!} m_{n,j}(x) - \mathbb{B}_n\Big((f'\Psi')(x)A_{n,1} + \frac{f''(x)}{2!}[A_{n,2}\Psi(x) + B_{n,2}] \\ &\quad + \frac{f'''(x)}{3!}[A_{n,3}(\Psi\Psi')(x)] + \frac{f^{IV}(x)}{4!}[A_{n,4}\Psi^2(x)]\Big) + o\Big(\frac{1}{n^2}\Big). \\ &= f(x) + f'(x)[A_{n,1}\Psi'(x)] + \frac{f''(x)}{2}[A_{n,2}\Psi(x) + B_{n,2}(x)] \\ &\quad + \frac{f'''(x)}{3!}[A_{n,3}\Psi(x)\Psi'(x)] + \frac{f^{IV}(x)}{4!}[A_{n,4}\Psi^2(x)] + o\Big(\frac{1}{n^2}\Big) \\ &\quad - \mathbb{B}_n\Big((f'\Psi')(x)A_{n,1} + \frac{f''(x)}{2!}[A_{n,2}\Psi(x) + B_{n,2}] \\ &\quad + \frac{f'''(x)}{3!}[A_{n,3}(\Psi\Psi')(x)] + \frac{f^{IV}(x)}{4!}[A_{n,4}\Psi^2(x)]\Big) + o\Big(\frac{1}{n^2}\Big). \end{aligned}$$

Also, we obtain:

$$\begin{aligned} &\mathbb{B}_n(f'\Psi')(x) = (f'\Psi')(x) + A_{n,1}[(f''(\Psi')^2)(x) - 2(f'\Psi')(x)] \\ &\quad + \frac{1}{2!}A_{n,2}[(f'''\Psi'\Psi)(x) - 2(f''\Psi)(x)] + \frac{1}{2!}B_{n,2}[(f'''\Psi')(x) - 2f''(x)] \\ &\quad + o\Big(\frac{1}{n^2}\Big), \\ &\mathbb{B}_n\Big(\frac{1}{2!}f''\Psi\Big)(x) = \frac{1}{2}(f''\Psi)(x) + \frac{1}{2}A_{n,1}[(f'''\Psi\Psi')(x) + (f''(\Psi')^2)(x)] \\ &\quad + \frac{1}{2}\Big[(f^{IV}\Psi)(x) + 2(f'''\Psi')(x) - 2f''(x))\Big] \cdot [A_{n,2}\Psi(x) + B_{n,2}] \\ &\quad + o\Big(\frac{1}{n^2}\Big), \\ &\mathbb{B}_n\Big(\frac{f''(x)}{2!}\Big) = \frac{f''(x)}{2!} + o\Big(\frac{1}{n^2}\Big), \\ &\mathbb{B}_n\Big(\frac{f'''(x)}{3!}\Psi(x)\Psi'(x)\Big) = \frac{1}{3!}\Big[(f'''\Psi\Psi')(x)\Big] + o\Big(\frac{1}{n^2}\Big), \\ &\mathbb{B}_n\Big(\frac{f^{IV}}{4!}\Psi^2(x)\Big) = \frac{1}{4!}(f^{IV}\Psi^2)(x) + o\Big(\frac{1}{n^2}\Big). \end{aligned}$$

Summing, making reductions in terms and using relations (5)-(13) and $(\Psi')^2(x) = 1 - 4\Psi(x)$, we obtain that:

$$
\begin{aligned}
&L_n(f)(x) - f(x)\\
&\quad = A_{n,1}(f'\Psi')(x) + \frac{A_{n,2}}{2!}(f''\Psi)(x) + \frac{B_{n,2}}{2!}f''(x) + \frac{A_{n,3}}{3!}(f'''\Psi\Psi')(x)\\
&\quad + \frac{A_{n,4}}{4!}(f^{IV}\Psi^2)(x) - A_{n,1}\Big[(f'\Psi')(x) + A_{n,1}[(f''(\Psi')^2)(x) - 2(f'\Psi')(x)]\\
&\quad + \frac{1}{2}A_{n,2}[(f'''\Psi'\Psi)(x) - 2(f''\Psi)(x)] + \frac{1}{2!}B_{n,2}[(f'''\Psi')(x) - 2f''(x)]\Big]\\
&\quad - A_{n,2}\Big[\frac{1}{2}(f''\Psi)(x) + \frac{1}{2}A_{n,1}[(f'''\Psi\Psi')(x) + (f''(\Psi')^2)(x)]\\
&\quad + \frac{1}{2}\Big[(f^{IV}\Psi)(x) + 2(f'''\Psi')(x) - 2f''(x))\Big]\cdot[A_{n,2}\Psi(x) + B_{n,2}]\Big]\\
&\quad - B_{n,2}\frac{f''(x)}{2!} - \frac{A_{n,3}}{3!}\Big[(f'''\Psi\Psi')(x)\Big] - \frac{A_{n,4}}{4!}(f^{IV}\Psi^2)(x) + o\Big(\frac{1}{n^2}\Big).
\end{aligned}
$$

Finally, we obtain:

$$
\begin{aligned}
\lim_{n\to\infty} n^2[L_n(f)(x) - f(x)] &= 2f'(x)\Psi'(x) + f''(x)\Big[-\frac{3}{2} + 8\Psi(x)\Big]\\
&\quad -2f'''(x)\Psi(x)\Psi'(x) - \frac{1}{2}f^{IV}(x)\Psi^2(x).
\end{aligned}
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