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ON A PROBLEM OF MATHEMATICAL PHYSICS EQUATIONS

Irina MEGHEA*[1] and Cristina Stefania STAMIN[2]

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Abstract

An improved result of critical points for nondifferentiable functionals has been generalized at a nonlinear boundary value problem involving the p-Laplacian and the p-pseudo-Laplacian. An application for these statements has been also proposed.

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1 Introduction

Taking into account the importance of p-Laplacian and p-pseudo-Laplacian in mathematical models for fluid mechanics such as Newtonian, pseudoplastic or dilatant fluids, study of flow through porous media, glacial sliding, Helle-Shaw flow and also for solid mechanics such as mathematical models of torsional creep (elastic or plastic) and quantum mechanics, the problems involving these operators are always up to date.

Obtaining and / or characterizing of weak solutions for problems of mathematical physics equations involving p-Laplacian and p-pseudo-Laplacian is a subject

[1]* *Corresponding author*, Faculty of Applied Sciences, University POLITEHNICA of Bucharest, Romania, e-mail: irina.meghea@upb.ro
[2]Faculty of Applied Sciences, University POLITEHNICA of Bucharest, Romania, e-mail: criset@yahoo.com

matter previously discussed by the first author through several approach methods in [7]-[13].

Some problems of critical points for nondifferentiable functionals have been analized by the author in [7] and [13] regarding the nonlinear boundary value problem which follows and / or its generalizations at p-Laplacian and p-pseudo-Laplacian.

Let Ω be a bounded domain of $\mathbb{R}^N$ with the smooth boundary $\partial\Omega$ (topological bounda-ry). Consider the nonlinear boundary value problem

$$\begin{cases} -\Delta u = f(x,u), \ x \in \Omega \\ Bu = 0 \text{ on } \partial\Omega, \end{cases}$$

where B designates the boundary condition Dirichlet or Neumann and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function with *subcritical growth*, i.e.

$$|f(x,s)| \leq a + b|s|^{\sigma} \ \ \forall s \in \mathbb{R}, x \in \Omega \ a.e., \tag{1}$$

where $a, b > 0$, $0 \leq \sigma < \frac{N+2}{N-2}$ for $N > 2$ and $\sigma \in [0, \infty)$ for $N = 1$ or $N = 2$.

In a section of [13], among other statements, some notions from [2] have been used together with improved and generalized versions of some results from [4] which are involved in characterization of weak solutions for Dirichlet problems for p-Laplacian and p-pseudo-Laplacian.

In this paper we continue to improve and generalize some results on Neumann problem using conditions of Palais-Smale type suggested by Ekeland principle and also we propose an application.

2 Fixing problem and involved notions

Let Ω be a bounded domain of $\mathbb{R}^N$ with the smooth boundary $\partial\Omega$ (topological boundary). Consider the nonlinear boundary value problems

$$\begin{cases} -\Delta_p u = f(x,u), \ x \in \Omega \\ Bu = 0 \text{ on } \partial\Omega, \end{cases} \tag{2}$$

and

$$\begin{cases} -\Delta_p^s u = f(x,u), \ x \in \Omega \\ Bu = 0 \text{ on } \partial\Omega, \end{cases} \tag{3}$$

where B designates the Dirichlet or Neumann boundary condition and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function with *subcritical growth* as above.

Set (following [2])

$$\underline{f}(x,t) = \varliminf_{s \to t} f(x,s), \ \overline{f}(x,t) = \varlimsup_{s \to t} f(x,s).$$

Suppose

$$\underline{f}, \overline{f} : \Omega \times \mathbb{R} \to \mathbb{R} \text{ are } N \text{ - measurable (i.e. with respect to } x). \tag{4}$$

We emphasize that (4) is verified in the following two cases:

1. f is independent of x;
2. f is Baire measurable and $s \to f(x,s)$ is decreasing $\forall x \in \Omega$, in which case we have

$$\overline{f}(x,t) = \max\{f(x,t+), f(x,t-)\}, \ \underline{f}(x,t) = \min\{f(x,t+), f(x,t-)\}.$$

Definitions. u from $W^{2,p}(\Omega)$, $p > 1$ is solution of (2) and (3) respectively if $Bu = 0$ on $\partial\Omega$ in the sense of trace[3] and

$$-\Delta_p u(x) \in \left[\underline{f}(x,u(x)), \overline{f}(x,u(x))\right] \text{ in } \Omega \text{ a.e.} \tag{5}$$

and

$$-\Delta_p^s u(x) \in \left[\underline{f}(x,u(x)), \overline{f}(x,u(x))\right] \text{ in } \Omega \text{ a.e.} \tag{6}$$

respectively.

3 Usage of some conditions of Palais-Smale type

3.1 Previous definitions and results

3.1.1. Ekeland principle ([5], [6], [7]). Let (X,d) be a complete metric space and $\varphi : X \to (-\infty, +\infty]$ bounded from below, lower semicontinuous and proper. For any $\varepsilon > 0$ and u of X with

$$\varphi(u) \le \inf \varphi(X) + \varepsilon$$

and for any $\lambda > 0$, there exists v_ε in X such that

$$\varphi(v_\varepsilon) < \varphi(w) + \frac{\varepsilon}{\lambda} d(v_\varepsilon, w) \ \ \forall w \in X \setminus \{v_\varepsilon\}$$

and

$$\varphi(v_\varepsilon) \le \varphi(u), \ d(v_\varepsilon, u) \le \lambda.$$

Definitions. Let X be a real normed space, $E \subset X$, $f : E \to \mathbb{R}$, $x_0 \in \overset{\circ}{E}$ and $v \in X$. We set

$$f^0(x_0; v) := \overline{\lim_{\substack{x \to x_0 \\ t \to 0+}}} \frac{f(x+tv) - f(x)}{t}.$$

The upper limit exists obviously. $f^0(x_0; v)$ is by definition *Clarke derivative* (or the *generalized directional derivative*) of the function f at x_0 in the direction v.

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[3] Define $W^{1,p}(\Gamma)$ with $p \in (1, +\infty)$, Γ regular differential manifold, for instance, $\Gamma = \partial\Omega$, Ω open of C^1 class with $\partial\Omega$ bounded. In this situation, there exists a unique linear continuous operator $\gamma : W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega)$, the *trace*, such that γ is surjective and $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \Longrightarrow \gamma(u) = u\,|\partial\Omega$. This gives a sense to $u\,|\partial\Omega$ for any u in $W^{1,p}(\Omega)$. Moreover $\gamma^{-1}(0) = W_0^{1,p}(\Omega)$.

Thus by definition

$$f^0(x_0;v) = \inf_{\substack{V \in \mathcal{V}(x_0) \\ r \in (0,+\infty)}} \left(\sup_{\substack{x \in V \\ t \in (0,r)}} \frac{f(x+tv) - f(x)}{t} \right). \tag{7}$$

Let X be a real normed space, $E \subset X$, $f : E \to \mathbb{R}$ and $x_0 \in \overset{\circ}{E}$. The functional ξ from X^* is by definition *Clarke subderivative* (or *generalized gradient*) of f in x_0 if

$$f^0(x_0;v) \geq \xi(v) \;\; \forall v \in X. \tag{8}$$

The set of these generalized gradients is designated by $\partial f(x_0)$.

Some call even $\partial f(x_0)$ as the Clarke subderivative at x_0.

x_0 is a *critical poin*t (in the sense of Clarke subderivative) for the real function f if $0 \in \partial f(x_0)$. In this case $f(x_0)$ is a *critical value* (in the sense of Clarke subderivative) for f.

3.1.2. Let f be Lipschitz around x_0 with the constant L. Then

1. the function $v \to f^0(x_0;v)$ is with values in $\mathbb{R}$, positive homogeneous, subadditive on X and

$$\left| f^0(x_0;v) \right| \leq L \|v\| \;\; \forall v \in X;$$

2. $f^0(x_0;-v) = (-f)^0(x_0;v)$ $\forall v \in X$, $\lambda \geq 0 \Longrightarrow (\lambda f)^0(x_0;v) = \lambda f^0(x_0;v)$ $\forall v \in X$;

3. $v \to f^0(x_0;v)$ is Lipschitz on X with the constant L ([12]).

3.1.3. Let f be locally Lipschitz on X. The function $\Phi : X \times X \to \mathbb{R}$,

$$\Phi(x;v) = f^0(x_0;v)$$

is upper semicontinuous ([3]).

3.1.4. If f is Lipschitz around x_0, L the constant, then

1. $\partial f(x_0)$ is nonempty, convex, $*$-weak compact (for X complete) and

$$\|\xi\| \leq L \;\forall \xi \in \partial f(x_0);$$

2. $f^0(x_0;v) = \sup\limits_{\xi \in \partial f(x_0)} \xi(v)$ $\forall v \in X$ ([3]).

3.1.5. Let X be a real reflexive space and $f : X \to \mathbb{R}$ locally Lipschitz.

1. For every x_0 at X, there is ξ_0 in $\partial f(x_0)$ such that

$$\|\xi_0\| = \inf \{\|\xi\| : \xi \in \partial f(x_0)\}.$$

2. The function $\mu : X \to \mathbb{R}$

$$\mu(x) = \inf \{\|\xi\| : \xi \in \partial f(x)\}$$

is lower semicontinous ([2], p. 105).

3.1.6. Let E be a convex subset of a real Banach space and $f : E \to \mathbb{R}$ convex. If f is Lipschitz around $x_0 \in \overset{\circ}{E}$, then for every v of X we have

$$f^0\left(x_0; v\right) = f'\left(x_0; v\right)$$

and the set of the subderivatives in x_0 coincides with the set of Clarke subderivatives in x_0 ([3]).

3.1.7. Local extremum. Let f be Lipschitz around x_0. If x_0 is a point of local extremum for f we have

$$0 \in \partial f\left(x_0\right).$$

3.2 Some Palais-Smale type conditions

Present now some results following some ideas of [4], improved and generalized. These results contain conditions of Palais-Smale type suggested by Ekeland principle.

Let X be a complete metric space, $\varphi : X \to \mathbb{R}$ and $c \in \mathbb{R}$.

φ satisfies $(\mathrm{PS})^*_{\mathrm{c},+}$ *condition* when, for every sequences $(u_n)_{n\geq 1}$, $u_n \in X$, $(\varepsilon_n)_{n\geq 1}$ and $(\delta_n)_{n\geq 1}$, ε_n, $\delta_n \in \mathbb{R}_+$, $\varepsilon_n \to 0$ and $\delta_n \to 0$, if

$$\varphi\left(u_n\right) \to c \tag{9}$$

and

$$\forall u \in X \; d\left(u_n, u\right) \leq \delta_n \Longrightarrow \varphi\left(u_n\right) \leq \varphi\left(u\right) + \varepsilon_n d\left(u_n, u\right), \tag{10}$$

then

$$(u_n)_{n\geq 1} \text{ has a convergent subsequence.}$$

Changing in (10) u_n and u with each other, $(\mathrm{PS})^*_{\mathrm{c},-}$ *condition is obtained.* Finally, $(\mathrm{PS})^*_{\mathrm{c}}$ *condition* means $(\mathrm{PS})^*_{\mathrm{c},+}$+$(\mathrm{PS})^*_{\mathrm{c,-}}$.

In the case of X being a real Banach space - when the the conclusion required by the hypothesis

$$\text{"} (u_n)_{n\geq 1} \text{ has a convergent subsequence"}$$

is replaced by

$$\text{"} (u_n)_{n\geq 1} \text{ has a weakly convergent subsequence"},$$

we get respectively the conditions

$$(\mathrm{PS})^*_{\mathrm{c,w,+}},\ (\mathrm{PS})^*_{\mathrm{c,w,-}}, (\mathrm{PS})^*_{\mathrm{c,w}}.$$

Suppose X is a real Banach space and φ locally Lipschitz. φ satisfies $[\mathrm{PS}]^*_{\mathrm{c},+}$ *condition* (obvious definition for $[\mathrm{PS}]^*_{\mathrm{c,-}}$, $[\mathrm{PS}]^*_{\mathrm{c}}$) when the properties (9) and (10) imply

c is critical value of φ (for Clarke subderivative).

The definition is, according to **3.1.4** coherent. We have

$$(\mathrm{PS})^*_{\mathrm{c},+} \Longrightarrow [\mathrm{PS}]^*_{\mathrm{c},+},$$

but the converse relation is not true (consider $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz, periodic and $c = \inf \varphi(\mathbb{R})$ or $c = \sup \varphi(\mathbb{R})$).

Finally, enounce a last version of Palais-Smale condition due to Chang ([2]). Let X be a real Banach space, $\varphi : X \to \mathbb{R}$ locally Lipschitz and $c \in \mathbb{R}$. φ satisfies $(\mathrm{PS})^{\mathrm{ch}}_{\mathrm{c}}$ condition when, for every sequence $(u_n)_{n\geq 1}$ from X, if

$$\varphi(u_n) \to c \tag{11}$$

and

$$\mu(u_n) := \inf\{\|\xi_n\| : \xi_n \in \partial\varphi(u_n)\} \to 0, \tag{12}$$

then

$$(u_n)_{n\geq 1} \ \textit{has a convergent subsequence.}$$

The definition is correct, $\partial\varphi(u_n) \neq \varnothing \ \forall n$ (**3.1.4**, see also **3.1.5**). One can state

3.2.1. Let X be a real Banach space and $\varphi : X \to \mathbb{R}$ locally Lipschitz and convex. Then

$$\varphi \text{ verifies } (\mathrm{PS})^{\mathrm{ch}}_{\mathrm{c}} \Longrightarrow \varphi \text{ verifies } (\mathrm{PS})^*_{\mathrm{c},\text{-}}$$

■ Let (u_n) be a sequence from X and (ε_n), (δ_n) sequences from $\mathbb{R}_+$, $\varepsilon_n \to 0$, $\delta_n \to 0$ such that

$$\varphi(u_n) \to c$$

and

$$\|u_n - u\| \leq \delta_n \Longrightarrow \varphi(u) \leq \varphi(u_n) + \varepsilon_n \|u_n - u\| \ \ \forall u \in X. \tag{13}$$

We must prove, by finding a convergent subsequence of (u_n),

$$\mu(u_n) := \inf\{\|\xi_n\| : \xi_n \in \partial\varphi(u_n)\} \to 0. \tag{14}$$

Take $u := u_n + tv$, $\|v\| = 1$, $0 < t \leq \delta_n$. Since $\|u_n - u\| \leq \delta_n$, (13) gives

$$\frac{\varphi(u_n + tv) - \varphi(u_n)}{t} \leq \varepsilon_n$$

and passing to the limit for $t \to 0+$ we get (**3.1.6**)

$$\varphi^0(u_n; v) = \varphi'(u_n; v) \leq \varepsilon_n.$$

Consequently $\xi(v) \leq \varepsilon_n \ \forall \xi \in \partial\varphi(u_n)$ (**3.1.4**) and hence, changing v in $-v$, one gets

$$\|\xi\| \leq \varepsilon_n \ \forall \xi \in \partial\varphi(u_n). \tag{15}$$

Let ξ_n be in $\partial\varphi(u_n)$ such that $\|\xi_n\| = \mu(u_n)$ (see **3.1.4**). Then, taking (15) into account, obtain

$$\mu(u_n) \leq \varepsilon_n,$$

which yields (10) by passing to the limit. ■

Remark 1. *The last statement represents that the author recovered from Proposition 3, [4], p.475. The proof of this ([4], p.483) contains among other things the implicit statement that $\upsilon \to \Phi^0(u_0;\upsilon)$ is not a subnorm, but even a linear functional.*

We are now going to some propositions of [4].

3.2.2. Let X be a complete metric space, $\varphi : X \to \mathbb{R}$ lower bounded, lower semiconti-nuous and $c := \inf \varphi(X)$. c is attained when φ verifies $(\mathrm{PS})^*_{\mathrm{c},+}$.

■ Let $(\upsilon_n)_{n\geq 1}$, $\upsilon_n \in X$ be a minimizing sequence for φ such that, for every n, $\varepsilon_n := \varphi(\upsilon_n) - c > 0$, hence $\varepsilon_n \to 0$. Apply Ekeland principle **3.1.1** with $\varepsilon = \varepsilon_n$, $\lambda = 1$, one finds $(u_n)_{n\geq 1}$ a sequence in X with the properties

$$\varphi(u_n) \leq \varphi(\upsilon_n),$$

$$\varphi(u_n) \leq \varphi(u) + \varepsilon_n d(u_n, u) \ \ \forall u \in X.$$

Since $c \leq \varphi(u_n)$, we have $\varphi(u_n) \to c$; apply $(\mathrm{PS})^*_{\mathrm{c},+}$ and let $(u_{k_n})_{n\geq 1}$ be a convergent subsequence, $u_{k_n} \to u_0$. But $\varphi(u_0) \leq \varliminf\limits_{n\to\infty} \varphi(u_{k_n}) = c$, this imposes $\varphi(u_0) = c$. ■

3.2.3. Let X be a real Banach space, $\varphi : X \to \mathbb{R}$ lower bounded, locally Lipschitz and $c := \inf \varphi(X)$.

If φ satisfies $(\mathrm{PS})^*_{\mathrm{c},+}$, then φ has critical points (for Clarke subderivative).

■ Apply **3.2.2** combined with **3.1.7.** ■

3.2.4. Let X be a real Banach space, $\varphi : X \to \mathbb{R}$ lower bounded, locally Lipschitz and $c := \inf \varphi(X)$.

If φ satisfies $[\mathrm{PS}]^*_{\mathrm{c},+}$, then c is a critical value of φ (for Clarke subderivative).

■ Let be $(\varepsilon_n)_{n\geq 1}$, $\varepsilon_n > 0$, $\varepsilon_n \to 0$. For every ε_n take υ_n such that $\varphi(\upsilon_n) \leq c + \varepsilon_n$ and apply Ekeland principle **3.1.1** with $\lambda = 1$. $\exists u_n$ such that $\varphi(u_n) \leq \varphi(\upsilon_n)$, and

$$\varphi(u_n) \leq \varphi(u) + \varepsilon_n \|u_n - u\|, \ \forall u \in X. \tag{16}$$

Since $\varphi(u_n) \to c$, (16) allows us to apply the hypothesis $[\mathrm{PS}]^*_{\mathrm{c},+}$, hence c is a critical value. ■

As an application we continue with the problems (2) and (3). But firstly

3.2.5. Let $X := W^{1,p}(\Omega)$ and $\Phi : X \to \mathbb{R}$, $\Phi(u) = \frac{1}{p}\|u\|^p_{1,p} - \int\limits_\Omega G(u)\,dx - \int\limits_\Omega hu$ dx and $\Phi(u) = \frac{1}{p} \mathbin{\mid} u \mathbin{\mid}^p_{1,p} - \int\limits_\Omega G(u)\,dx - \int\limits_\Omega hu\ dx$, respectively, where $G : \mathbb{R} \to \mathbb{R}$ is with the period T and Lipschitz, $h \in L^{p'}(\Omega)$ and $\int\limits_\Omega h\ dx = 0$.

Then, for every c from $\mathbb{R}$, Φ verifies $[\mathrm{PS}]^*_{\mathrm{c}}$.

Clarification. On $X = W^{1,p}(\Omega)$ we can consider for this statement the following norms: $\|u\|_{1,p} = \left(\|u\|^p_{L^p(\Omega)} + \sum\limits_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}$, which is equivalent to the norm $u \to \|u\|_{L^p(\Omega)} + \sum\limits_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}$ for (2). For the second case (3), equip the same vector space with the norm $u \to \|u\|_{1,p} = \left(\sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}$, which is equivalent to $u \to |u|_{1,p} = \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}$.

■ It is sufficient to prove this for $[\mathrm{PS}]_{c,+}^{*}$. Let $(u_n)_{n\geq 1}$ be a sequence from X, $(\varepsilon_n)_{n\geq 1}$ and $(\delta_n)_{n\geq 1}$ sequences from $\mathbb{R}_+$ convergent to 0. Suppose $\Phi(u_n) \to c$ and

$$\|u_n - u\| \leq \delta_n \Longrightarrow \Phi(u_n) \leq \Phi(u) + \varepsilon_n \|u_n - u\|. \tag{17}$$

Decompose X in direct sum

$$X = X_0 \oplus X_1, \tag{18}$$

X_1 the vector space of the constant functions, $X_0 = X_1^{\perp}$, the vector space of $W^{1,p}$ - functions having the mean value equal to 0. Let $u_n = \upsilon_n + c_n$, $\upsilon_n \in X_0$, $c_n \in \mathbb{R}$ and $|G(s)| \leq M$ on $\mathbb{R}$. Hence

$$\left| \int_\Omega G(u_n)\, dx \right| \leq \int_\Omega |G(u_n)|\, dx \leq \int_\Omega M dx = M\mu(\Omega)$$

and since $\int_\Omega c_n h\; dx = 0$, we have in the first case

$$\Phi(u_n) = \frac{1}{p} \|u_n\|_{1,p}^p - \int_\Omega G(u_n)\, dx - \int_\Omega h u_n dx =$$

$$= \frac{1}{p} \|\upsilon_n + c_n\|_{1,p}^p - \int_\Omega G(\upsilon_n + c_n)\, dx - \int_\Omega h(\upsilon_n + c_n)\, dx \geq$$

$$\geq \frac{1}{p} \|\upsilon_n + c_n\|_p^p + \frac{1}{p} \sum_{i=1}^{N} \left\| \frac{\partial \upsilon_n}{\partial x_i} \right\|_p^p - M\mu(\Omega) - \int_\Omega h\upsilon_n dx - c_n \int_\Omega h dx \geq$$

$$\geq \frac{1}{p} \|\upsilon_n\|_p^p + \frac{1}{p} \sum_{i=1}^{N} \left\| \frac{\partial \upsilon_n}{\partial x_i} \right\|_p^p - M\mu(\Omega) - \|h\|_{p'} \|\upsilon_n\|_{1,p} =$$

$$= \frac{1}{p} \|\upsilon_n\|_{1,p}^p - M\mu(\Omega) - \|h\|_{p'} \|\upsilon_n\|_{1,p} \Longrightarrow$$

$$\Phi(u_n) \geq \|\upsilon_n\|_{1,p} \left(\frac{1}{p} \|\upsilon_n\|_{1,p}^{p-1} - \|h\|_{p'} - M\mu(\Omega) \right) \tag{19}$$

since

$$\int_\Omega h\upsilon_n dx \leq \left| \int_\Omega h\upsilon_n dx \right| \leq \|h\|_{p'} \|\upsilon_n\|_p \leq \|h\|_{p'} \|\upsilon_n\|_{1,p},$$

and, similarly, for the second case,

$$\Phi(u_n) = \frac{1}{p} \|u_n\|_{1,p}^p - \int_\Omega G(u_n)\, dx - \int_\Omega h u_n dx =$$

$$= \frac{1}{p} \Vert v_n + c_n \Vert_{1,p}^{p} - \int_\Omega G(v_n + c_n)\,dx - \int_\Omega h(v_n + c_n)\,dx \geq$$

$$\geq \frac{1}{p} \Vert v_n + c_n \Vert_{1,p}^{p} - M\mu(\Omega) - \int_\Omega h v_n dx - c_n \int_\Omega h dx \geq$$

$$\geq \frac{1}{p} \Vert v_n \Vert_{1,p}^{p} - M\mu(\Omega) - \|h\|_{p'} \|v_n\|_p = \frac{1}{p} \Vert v_n \Vert_{1,p}^{p} - M\mu(\Omega) - \alpha \Vert v_n \Vert_{1,p} \Longrightarrow$$

$$\Phi(u_n) \geq \Vert v_n \Vert_{1,p} \left(\frac{1}{p} \Vert v_n \Vert_{1,p}^{p-1} - \alpha - M\mu(\Omega) \right) \tag{20}$$

since

$$\int_\Omega h v_n dx \leq \left| \int_\Omega h v_n dx \right| \leq \|h\|_{p'} \|v_n\|_p \leq \alpha \Vert v_n \Vert_{1,p} .$$

As $(\Phi(u_n))_{n\geq 1}$ is bounded, (19) and (20) impose: $\left(\int\limits_\Omega |\nabla v_n|^2 dx \right)_{n\geq 1}$ bounded and hence also $\left(\|v_n\|_{1,p} \right)_{n\geq 1}$ and $\left(\Vert v_n \Vert_{1,p} \right)_{n\geq 1}$ respectively, bounded.

Consider the sequence $(\widetilde{u}_n)_{n\geq 1}$, $\widetilde{u}_n = v_n + \widetilde{c}_n$, where $\widetilde{c}_n \equiv c_n$ (modulo T) and $\widetilde{c}_n \in [0, T]$. Since Φ has the period T, (17) gives

$$\|\widetilde{u}_n - (u + \widetilde{c}_n - c_n)\| \leq \delta_n \Longrightarrow \Phi(\widetilde{u}_n) \leq \Phi(u) + \varepsilon_n \|(\widetilde{u}_n - u) + (c_n - \widetilde{c}_n)\| ,$$

i.e.

$$\|\widetilde{u}_n - w\| \leq \delta_n \Longrightarrow \Phi(\widetilde{u}_n) \leq \Phi(w) + \varepsilon_n \|\widetilde{u}_n - w\| . \tag{21}$$

But (v_n) and $(\widetilde{c}_n)$ are bounded, hence $(\widetilde{u}_n)$ is bounded, consequently it has a weak convergent subsequence $(\widetilde{u}_{k_n})_{n\geq 1}$, $\widetilde{u}_{k_n} \overset{weak}{\longrightarrow} \widetilde{u}$ (Eberlein-Šmulian), whence the existence of a convergent subsequence of $(\widetilde{u}_{k_n})_{n\geq 1}$, by using the same notation,

$$\widetilde{u}_{k_n} \longrightarrow \widetilde{u} \tag{22}$$

(the same proof as in Proposition 4, [4], p. 484). In (21) take $w = \widetilde{u}_{k_n} + \delta_{k_n} v$, $\|v\| = 1$, we get, $-\varepsilon_{k_n} \leq \frac{1}{\delta_{k_n}} [\Phi(\widetilde{u}_{k_n} + \delta_{k_n} v) - \Phi(\widetilde{u}_{k_n})]$ and passing to the limit we find, since $(\widetilde{u}_{k_n}, \delta_{k_n}) \overset{(22)}{\longrightarrow} (\widetilde{u}, 0)$, $0 \leq \overline{\lim\limits_{n\longrightarrow\infty}} \frac{1}{\delta_{k_n}} [\Phi(\widetilde{u}_{k_n} + \delta_{k_n} v) - \Phi(\widetilde{u}_{k_n})] \leq \Phi^0(\widetilde{u}; v)$, $\leq \Phi^0(\widetilde{u}; v)$, $\|v\| = 1$, whence $0 \leq \Phi^0(\widetilde{u}; v)$ $\forall v \in X$ $(0 \leq \Phi^0 \left(\widetilde{u}; \frac{v}{\|v\|} \right) \overset{3.1.2}{=} \frac{1}{\|v\|} \Phi^0(\widetilde{u}; v))$ (**3.1.2**), i.e. $0 \in \partial\Phi(\widetilde{u})$.

Moreover, $c = \Phi(\widetilde{u})$, since $\Phi(u_{k_n}) = \Phi(\widetilde{u}_{k_n} - \widetilde{c}_{k_n} + c_{k_n}) = \Phi(\widetilde{u}_{k_n}) \to c$ and also $\Phi(\widetilde{u}_{k_n}) \overset{(22)}{\longrightarrow} \Phi(\widetilde{u})$ (Φ is continuous being locally Lipschitz), c is a critical value for Φ. ■

And now

3.2.6. *Nonlinear Neumann problems*

$$\begin{cases} -\Delta_p u = g(u) + h(x), \ x \in \Omega \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \end{cases} \tag{23}$$

and

$$\begin{cases} -\Delta_p^s u = g(u) + h(x),\ x \in \Omega \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \end{cases} \tag{24}$$

respectively with the conditions

$$g : \mathbb{R} \to \mathbb{R} \text{ bounded measurable } T\text{-periodic, } \int_0^T g(s)\, ds = 0 \tag{25}$$

and

$$h \text{ bounded measurable, } \int_\Omega h\ dx = 0 \tag{26}$$

have a solution in $W^{1,p}(\Omega)$ in the sense of (5) and (6) respectively.

■ We are in the presence of a problem of type (2) and (3) respectively with $f(x,u) = g(u) + h(x)$. The conditions (25) and (26) imply (1) ($\sigma = 0$) and (4). The associate functional is

$$\Phi(u) = \frac{1}{p}\|u\|_{1,p}^p - \int_\Omega G(u)\ dx - \int_\Omega hu\ dx,\ u \in W^{1,p}(\Omega),$$

and

$$\Phi(u) = \frac{1}{p} \mid u \mid_{1,p}^p - \int_\Omega G(u)\ dx - \int_\Omega hu\ dx,\ u \in W^{1,p}(\Omega),$$

respectively, where $G(u(x)) = \int_0^{u(x)} g(t)\, dt$. G is Lipschitz and has the period T (use (25)). Since (1) and (4) are satisfied, any critical point of Φ is, according to Proposition 16 from [13], a solution to the problems (23) and (24) respectively. But Φ verifies $[\mathrm{PS}]_c^*$ for every c in $\mathbb{R}$ in particular for $c = \inf_{u \in W^{1,p}(\Omega)} \Phi(u)$. This is correct since Φ is lower bounded (justification as for (19) and (20) respectively). It only remains to apply **3.2.5**, hence c is a critical value, $c = \Phi(u_0)$, u_0 a critical point, u_0 is a solution for (23) or (24) respectively. ■

4 Application in modelling of injection mould filling

Using the last result, we give another solution to the problem studied in [1].

The physical problem according to [1] is the following:

The polymer is injected, over a period of time $t_0 < t < t_1$ at some point $x_1 \in \Omega$. The basic domain Ω needs not be simply connected.

Some further notations:

Ω_t = the part of Ω which is filled by fluid at time t;

φ = pressure;

υ = fluid velocity (averaged over -h$\leq z \leq h$);

$\Gamma_0 = \partial\Omega_t \cap \Omega$, and $\Gamma_1 = \partial\Omega_t \cap \partial\Omega$.

Clearly, Γ_0 is the flow front. It is assumed that $\partial\Omega$ is solid (except for air vents). The equation for φ is:

$$\Delta_p \varphi = 0 \text{ on } \Omega_t \setminus \{x_1\}$$

where $p = \frac{1}{n} + 1$, n a material constant.

Further, $\varphi =$ constant (chosen 0) on Γ_0 and $\frac{\partial\varphi}{\partial n} = 0$ on Γ_1.

It follows that the fluid front Γ_0 meets $\partial\Omega$ at right angles (provided that φ is smooth up to the boundary and $\nabla\varphi \neq 0$). Note also that φ must have a singularity at x_1.

In the above approach, the development with time of the domain Ω_t filled by fluid, is controlled by φ, which is determined from an elliptic partial differential equation. Note that, in this physically oriented description, all considered curves, functions and vector fields are assumed to be 'smooth', so that each of the crucial expressions has a well-defined pointwise meaning.

The mathematical problem:

The desired solution φ of the instantaneous flow problem can be obtained as the solution φ^* of a convex extremum problem [1] by an appropriate and rather obvious choice of that problem.

First a remark about smoothness and uniqueness. The function φ^* satisfies *a priori* the p-harmonic equation and the boundary conditions only in a generalized (weak) sense, whereas one would like to have a classical (smooth) solution to the physical problem. Now a classical solution is also a weak solution of the boundary-value problem, and the weak solution is unique, since *the problem* has a unique solution. Therefore the function φ^*, defined by this problem, is what we are looking for. We must, however, refrain from a complete discussion of the smoothness of φ^*.

Taking Ω_t instead of Ω in (23), we are placed under the conditions of Proposition **3.2.6** and we provided, *via* this result, another proof of the existence of the problem mentioned.

This is a preliminary study for the solution of such a problem.

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