Bulletin of the *Transilvania* University of Braşov • Vol 11(60), No. 2 -2018
Series III: Mathematics, Informatics, Physics, 149-164

LINEAR WEINGARTEN REVOLUTION SURFACES IN THREE-DIMENSIONAL PSEUDO-GALILEAN SPACE

Mohamd Saleem LONE [1]

Abstract

In this paper, we classify the revolution surfaces in the pseudo-Galilean space G_3^1 having a linear relationship between the Gaussian (K) curvature and the mean(H) curvature i.e., $aH + bK = c$, where a, b, c are constants. In special cases, we classify the revolution surfaces having null Gaussian curvature and null mean curvature. Further, we study the revolution surfaces satisfying $\Delta x_i = \lambda_i x_i$, where Δ is the Laplacian operator with respect to first fundamental form, $\lambda_i' s$ are the eigenvalue values and $x_i' s$ are the coordinate functions of the given surface.

2000 *Mathematics Subject Classification:* 53A35, 53B30, 53C50.
Key words: Laplacian operator, pseudo-Galilean space, revolution surface.

1 Introduction

In addition to the Euclidean geometry, various type of geometries have been developed in the last two centuries. One natural possible extension is to define in projective manner, where one expresses metric properties through projective relations. For this purpose A. Cayley and F. Klein fixed conic (called absolute) at infinity and all metric relations are considered in projective relations with respect to the absolute. Due to the nature of the absolute various geometries are possible. Galilean and pseudo-Galilean geometry are among the nine Cayley-Klein geometries. These are the ambient spaces in which we study the nature of the surfaces. The detailed development can be found in [23]. In Galilean and pseudo-Galilean spaces ruled surfaces and tubular surfaces have been studied in [16, 21]. One of the important problems in classical differential geometry is to classify the surfaces with null Gaussian and null mean curvature. In particular, a surface is called as developable if its Gaussian curvature becomes zero. In this case, we say that this surface is topologically similar to a flat surface, i.e., it can be flattened onto a plane

[1]Department of Mathematics, Central University of Jammu, 181143, Jammu and Kashmir, India: e-mail: saleemraja2008@gmail.com

without distortions. Null curvature surfaces have applications in microeconomics in the way when production function graph has vanishing Gaussian curvature, one can predict an efficient analysis of isoquants by projections, without losing important information about their geometry [3].
A surface is called a Weingarten surface if there is a smooth relation $\sigma(\kappa_1, \kappa_2) = 0$ between its principal curvatures κ_1 and κ_2. This relation implies that there exists a functional relation $\varphi(H, K) = 0$, where H and K are the mean and the Gaussian curvatures, respectively. The existence of a functional relation is equivalent to the vanishing of the corresponding Jacobian determinant given by $\left|\frac{\partial(H,K)}{\partial(u_1,u_2)}\right| = 0$ [24]. The trivial case is when $\varphi = aH + bK - c$, where a, b, c are constants with $a^2 + b^2 \neq 0$, then the surface is called linear Weingarten(L.W.) surface. When the constant $a = 0$, the L.W. surface reduces to surface with constant Gaussian curvature. When the constant $b = 0$, the L.W. surface reduces to surface with constant mean curvature. In such a sense a L.W. surface can be regarded as a natural generalisation of surfaces with constant Gaussian and constant mean curvature.
The study of Weingarten surfaces was initiated by J. Weingarten in 1861 [27] followed by E. Beltrami [6], Darboux [9], S. Lie [20] and many others. W. Kühnel studied the ruled Weingarten surfaces in $\mathbb{R}^3$ [18] and Minkowski space $\mathbb{E}^3_1$ [10]. When the ambient space is pseudo-Galilean space, tubular and ruled surfaces were studied in [16] and [25]. C.W. Lee studied the linear Weingarten rotation surfaces in three dimensional pseudo-Galilean space [19]. M. E Aydin *et.al* obtained the conditions for factorable surfaces to be minimal and developable in pseudo-Galilean space [2]. D.W. Yoon [28] obtained some of the classification results for the revolution surfaces in three dimensional pseudo-Galilean spaces. For more study of linear Weingarten surfaces we refer [14, 17, 29].

2 Preliminaries

The pseudo-Galilean geometry is one of the Cayley-Klein geometries of projective signature $(0, 0, +, -)$. The absolute of the pseudo-Galilean geometry is an ordered triplet (ω, f, I), where ω is the absolute plane in the three dimensional projective space $P_3(\mathbb{R})$, f is the absolute line in ω and I is the fixed hyperbolic involution of the points of f. The geometry of a pseudo-Galilean space G^1_3 can be found in the dissertation [12] and the theory of curves and surfaces are described in [11] and [13], respectively. Homogeneous coordinates of G^1_3 are written in such a way that the absolute plane ω is given by $x_0 = 0$, the absolute line f by $x_0 = x_1 = 0$ and the hyperbolic involution by $(0 : 0 : x_2 : x_3) \longmapsto (0 : 0 : x_3 : x_2)$, which is equivalent to the requirement that the conic $x_2^2 - x_3^2 = 0$ is the absolute conic. The metric relations are introduced with respect to the absolute figure. In affine coordinates defined by $(x_0 : x_1 : x_2 : x_3) = (1 : x : y : z)$. The distance between the points $P=(x_1,x_2,x_3)$ and $Q=(y_1,y_2,y_3)$ is defined by

$$d(P,Q)=\begin{cases} |y_1-x_1| & if \quad x_1\neq y_1,\\ \sqrt{(y_2-x_2)^2-(y_3-x_3)^2} & if \quad x_1=y_1.\end{cases}$$

The scalar product of two vectors $P=(x_1,x_2,x_3)$ and $Q=(y_1,y_2,y_3)$ in pseudo-Galilean space is defined as

$$P\cdot Q=\begin{cases} x_1y_1, & if \quad x_1\neq 0 \text{ or } y_1\neq 0,\\ x_2y_2-x_3y_3 & if \quad x_1=0 \text{ and } y_1=0.\end{cases}$$

A vector $P=(x_1,x_2,x_3)$ is called isotropic or non-isotropic if $x_1=0$ or $x_1\neq 0$, respectively. All the unit non-isotropic vectors are of the form $(1,x_2,x_3)$. The isotropic vector $P=(0,x_2,x_3)$ is called spacelike, timelike and lightlike if $x_2^2-x_3^2>0$, $x_2^2-x_3^2<0$ and $x_2=\pm x_3$, respectively. The pseudo-Galilean cross product of P and Q is defined as

$$P\times Q=\begin{vmatrix} 0 & -e_2 & e_3\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix}.$$

In pseudo-Galilean space G_3^1 there are two types of rotations:
(1) Pseudo-Euclidean rotation defined by

$$\begin{cases} \overline{x}=x,\\ \overline{y}=y\cosh t+z\sinh t,\\ \overline{z}=y\sinh t+z\cosh t.\end{cases} \tag{1}$$

(2) Isotropic rotation defined as

$$\begin{cases} \overline{x}(t)=x+bt,\\ \overline{y}(t)=y+x(t)+b\frac{t^2}{2},\\ \overline{z}=z,\end{cases} \tag{2}$$

where $t\in R$ and b is some positive constant.
Let M be a C^r, $r\geqslant 1$ surface in G_3^1 parameterised by

$$\mathbf{X}(u_1,u_2)=(X(u_1,u_2),Y(u_1,u_2),Z(u_1,u_2)).$$

Denote $R_i=\frac{\partial R}{\partial u_i}$, $R=X,Y,Z$ and $i=1,2$. Then M is called admissible surface if and only if $X_i\neq 0$ for some $i=1,2$. Define a function W by

$$W=\sqrt{|(X_1Y_2-X_2Y_1)^2-(X_1Z_2-X_2Z_1)^2|}. \tag{3}$$

The unit normal vector field N of M is given by

$$N=\frac{1}{W}(0,X_1Z_2-X_2Z_1,X_1Y_2-X_2Y_1). \tag{4}$$

Since $N\cdot N=\pm 1=\epsilon$, there are two types of admissible surfaces: spacelike surfaces having timelike unit normal ($\epsilon=-1$) and timelike surface having spacelike normal

$(\epsilon = 1)$ [2].
Let us denote

$$g_i = \frac{\partial X}{\partial u_i} \text{ and } h_{ij} = (0, Y_i, Z_i) \cdot (0, Y_j, Z_j), \quad i, j = 1, 2. \tag{5}$$

The first fundamental form in matrix form M in G_3^1 is written as

$$ds^2 = \begin{pmatrix} ds_1^2 & 0 \\ 0 & ds_2^2 \end{pmatrix},$$

where $ds_1^2 = (g_1 du_1 + g_2 du_2)^2$ and $ds_2^2 = h_{11} du_1^2 + 2h_{12} du_1 du_2 + h_{22} du_2^2$. The second fundamental form coefficients of M in G_3^1 are given by

$$\left.\begin{aligned} L_{ij} &= \epsilon \tfrac{1}{g_1}(g_1(0, Y_{ij}, Z_{ij}) - g_{ij}(0, Y_1, Z_1)) \cdot N \\ &= \epsilon \tfrac{1}{g_2}(g_2(0, Y_{ij}, Z_{ij}) - g_{ij}(0, Y_2, Z_2)) \cdot N \end{aligned}\right\} \tag{6}$$

The Gaussian and the mean curvature of M is defined as

$$\begin{aligned} K &= -\epsilon \frac{L_{11}L_{22} - L_{12}^2}{W^2}, \\ H &= -\epsilon \frac{X_2^2 L_{11} - 2X_1X_2L_{12} + X_1^2 L_{22}}{2W^2}, \end{aligned}$$

respectively.

A surface M is called as a finite Chen-type if its coordinate functions can be written finitely as a sum of eigenfunctions of its Laplacian [7]. Afterward, various authors have classified the various finite type surfaces in Euclidean space $\mathbb{E}^3$ and in other spaces. In [26] Takahashi states that minimal surfaces and spheres are the only surfaces in $\mathbb{E}^3$ satisfying

$$\Delta \mathbf{X} = \lambda \mathbf{X}.$$

For more studies of finite type of surfaces, we refer [4, 5, 15]. Let (u_1, u_2) be a local coordinate system of M, then the Laplacian of the first fundamental form on M is given by [22]

$$\Delta = -\frac{1}{\sqrt{\mathcal{D}}} \sum_{i,j=1}^{2} \frac{\partial}{\partial u_i} \left(\sqrt{\mathcal{D}} g^{ij} \frac{\partial}{\partial u_j} \right), \tag{7}$$

where g_{ij} are the components of ds^2, $\mathcal{D} = det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$.
In this paper, we investigate the linear Weingarten revolution surfaces i.e., the revolution surfaces satisfying the relation $aH + bK = C$. In particular, we classify the NGC(Null Gaussian curvature) revolution surfaces and the revolution surfaces with vanishing mean curvature. Further, we study the finite type revolution surfaces in G_3^1, i.e., M satisfies

$$\Delta \mathbf{X}_i = \lambda_i \mathbf{X}_i. \tag{8}$$

We furnish the study by providing some figures also.

3 Revolution surfaces of type I and type II in G_3^1

Rotating a non-isotropic curve $(f(u), g(u), 0)$, $g > 0$, around the x-axis by pseudo-Euclidean rotation, we obtain a surface

$$x(u_1, u_2) = (f(u_1), g(u_1)\cosh u_2, g(u_1)\sinh u_2). \tag{9}$$

Again rotating a non-isotropic curve $(f(u), 0, g(u))$, $g > 0$, around the x-axis we obtain a surface

$$x(u_1, u_2) = (f(u_1), g(u_1)\sinh u_2, g(u_1)\cosh u_2). \tag{10}$$

We call (9) and (10) as a revolution surface of type I and type II, respectively.
Now suppose M is a revolution surface of type I. Then from (5), we get
$g_1 = f', \quad g_2 = 0,$
$h_{11} = 0, \quad h_{12} = 0, \quad h_{22} = -g^2,$
The components of the first fundamental form ds^2 on M are given by

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = -g^2. \tag{11}$$

The unit normal vector is given by

$$N = \frac{1}{W}(0, f'g\sinh u_2, f'g\cosh u_2),$$

where $W = f'g$.
Also the second fundamental coefficients are given as
$L_{11} = \frac{\epsilon}{Wf_1}(f'^2 gg'' - f'f''gg'), \quad L_{12} = 0, \quad L_{22} = \frac{\epsilon}{w}f'g^2.$

Theorem 1. *The revolution surfaces of type I are of Weingarten type.*

Proof. It can be easily seen that the Gaussian curvature and the mean curvature of M is

$$K = -\epsilon\frac{f'g'' - g'f''}{f'} \cdot \frac{1}{f'^2 g}, \tag{12}$$

$$H = \frac{-1}{2g}, \tag{13}$$

respectively. Equations (12) and (13) yield that K and H depend on the coordinate u_1 implying $\left|\frac{\partial(K,H)}{\partial(u_1,u_2)}\right| = 0$, which implies that the surface is of Weingarten type. □

Theorem 2. *Let M be a linear Weingarten revolution surface of type I in G_3^1. Then M is of the form*

(1) *When M is timelike and $c = 0$*

$$\begin{cases} x(u_1, u_2) = (u_1, g(u_1)\cosh u_2, g(u_1)\sinh u_2); \\ g(u_1) = -\frac{mu_1^2}{2} + c_1u_1 + c_2, \quad c_1, c_2 \in R. \end{cases} \tag{14}$$

(2) *When M is spacelike and $c=0$*

$$\begin{cases} x(u_1,u_2)=(u_1,g(u_1)\cosh u_2,g(u_1)\sinh u_2); \\ g(u_1)=\frac{mu_1^2}{2}+c_1u_1+c_2, \quad c_1,c_2\in R. \end{cases} \tag{15}$$

(3) *When M is timelike and $c\neq 0$*

$$\begin{cases} x(u_1,u_2)=(u_1,g(u_1)\cosh u_2,g(u_1)\sinh u_2); \\ g(u_1)=-\frac{m}{n}+c_1\cos(\sqrt{n}u_1)+c_2\sin(\sqrt{n}u_1), \quad c_1,c_2\in R. \end{cases} \tag{16}$$

(4) *When M is spacelike $c\neq 0$*

$$\begin{cases} x(u_1,u_2)=(u_1,g(u_1)\cosh u_2,g(u_1)\sinh u_2); \\ g(u_1)=-\frac{m}{n}+c_1e^{\sqrt{n}u_1}+c_2e^{-\sqrt{n}u_1}, \quad c_1,c_2\in R. \end{cases} \tag{17}$$

Proof. Since M is a linear Weingarten revolution surface M of type I, i.e., the Gaussian and the mean curvature of M satisfies a relation

$$aH+bK=c, \quad a,b,c\in R \text{ and } (a,b,c)\neq(0,0,0). \tag{18}$$

Without loss of generality assuming $b\neq 0$. Dividing both sides of (18) by b, we get

$$2mH+K=n, \quad \text{where } \frac{a}{b}=2m,\ \frac{c}{b}=n. \tag{19}$$

Since f,g are C^r, $r\geqslant 1$ arbitrary functions of u_1, assuming $f(u_1)=u_1$. Substituting (12) and (13) in (18), we get

$$-\frac{m}{g}-\epsilon\left(\frac{g''}{g}\right)=n,$$

or

$$\epsilon g''+ng+m=0. \tag{20}$$

Now when $c=0$ i.e., $n=0$, from (20), we obtain

$$\epsilon g''+m=0.$$

Integrating the above equation twice for $\epsilon=1,-1$, we obtain (14) and (15), respectively.
For $c\neq 0$ and $\epsilon=1,-1$, from (20), we obtain (16) and (17), respectively. □

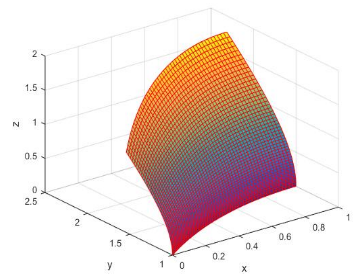

Figure 1: Timelike L.W. revolution for $c = 0$.

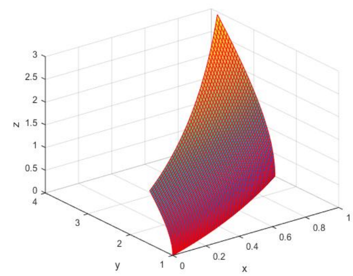

Figure 2: Spacelike L.W. revolution for $c = 0$.

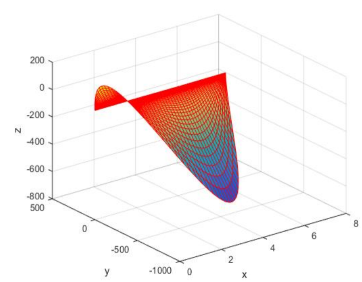

Figure 3: Timelike L.W. revolution for $c \neq 0$.

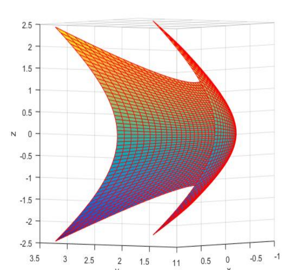

Figure 4: Spacelike L.W. revolution for $c \neq 0$

Corollary 1. *Let M be a NGC revolution surface of type I in G_3^1, then M is of the form*

$$\begin{cases} x(u_1,u_2) = (u_1, g(u_1)\cosh u_2, g(u_1)\sinh u_2); \\ g(u_1) = \epsilon(c_1 u_1 + c_2), \quad c_1, c_2 \in R. \end{cases} \tag{21}$$

Proof. For M to be a NGC revolution surface, using $a = 0$ and $c = 0$ in (20), which implies that $n = 0$ and $m = 0$, i.e., we get

$$\epsilon g'' = 0.$$

Integrating the above equation twice, we obtain (21). □

Corollary 2. *Let M be a revolution surface of type I of constant Gaussian curvature in G_3^1, then M is of the form*
(1) When M is timelike

$$\begin{cases} x(u_1,u_2) = (u_1, g(u_1)\cosh u_2, g(u_1)\sinh u_2); \\ g(u_1) = c_1\cos(\sqrt{n}u_1) + c_2\sin(\sqrt{n}u_1), \quad c_1, c_2 \in R. \end{cases} \tag{22}$$

(2) When M is spacelike

$$\begin{cases} x(u_1,u_2) = (u_1, g(u_1)\cosh u_2, g(u_1)\sinh u_2); \\ g(u_1) = c_1 e^{\sqrt{n}u_1} + c_2 e^{-\sqrt{n}u_1}, \quad c_1, c_2 \in R. \end{cases} \tag{23}$$

Proof. Since M is a revolution surface of constant Gaussian curvature, i.e., $a = 0$ in (18), which implies from (19) $m = 0$,
Thus, from (20), we have

$$\epsilon g'' + ng = 0. \tag{24}$$

Substituting $\epsilon = 1$ i.e., M is timelike and $\epsilon = -1$ i.e., M is spacelike in (24), we obtain (22) and (23), respectively. □

Corollary 3. *There are no minimal revolution surfaces of type I in G_3^1.*

Proof. The proof follows easily from (13). □

Remark 1. *The same results hold for the revolution surfaces of type II.*

4 Revolution surfaces of type III in G_3^1

Definition 1. *Using the isotropic rotation to rotate the isotropic curve $(0, f(u_1), g(u_1))$ about the z-axis, we obtain a revolution surface of type III given by:*

$$x(u_1,u_2) = \left(u_2, f(u_1) + \frac{u_2^2}{2b}, g(u_1)\right), \quad b \neq 0. \tag{25}$$

Theorem 3. *The revolution surfaces of type III are of Weingarten type.*

Proof. Assuming that the curve is parameterised by $f'^2 - g'^2 = -\epsilon,\ \epsilon = \pm 1$. One can easily deduce that

$$K = -\frac{f''}{b} \tag{26}$$

and

$$H = -\frac{\epsilon f''}{2\sqrt{f'^2 + \epsilon}}. \tag{27}$$

From (26) and (27), both K and H are independent of u_2, which implies that $\left|\frac{\partial(H,K)}{\partial(u_1,u_2)}\right| = 0$. Hence the result follows. □

Theorem 4. *Let M be a linear Weingarten revolution surface of type III, then M is of the form*

$$\begin{cases} x(u_1,u_2) = \left(u_2, f(u_1) + \frac{u_2^2}{2b}, g(u_1)\right), \\ f(u_1) = \pm\sqrt{p\left(\frac{n}{H} - 2m\right)^2 - \epsilon}\ u_1, \\ g(u_1) = \pm p\left(\frac{n}{H} - 2m\right) u_1 + d_2, \quad \text{where } d_2 \text{ is some constant.} \end{cases} \tag{28}$$

Proof. Using (26) and (27), we have

$$\begin{aligned} H &= -\frac{\epsilon f''}{2\sqrt{f'^2 + \epsilon}} \\ &= \frac{\epsilon b K}{2\sqrt{f'^2 + \epsilon}} \\ &= \frac{\epsilon p K}{\sqrt{f'^2 + \epsilon}}, \quad \text{where } \frac{b}{2} = p. \end{aligned} \tag{29}$$

From (29), we get

$$f' = \pm\sqrt{\left(\frac{pK}{H}\right)^2 - \epsilon}. \tag{30}$$

Integrating (30), we have

$$f = \pm\sqrt{\left(\frac{pK}{H}\right)^2 - \epsilon}\ u_1 + d_1, \quad \text{where } d_1 \text{ is a constant of intergration.}$$

Now assuming that M is a linear Weingarten revolution surface of type III, from (19), we obtain

$$f = \pm\sqrt{p^2\left(\frac{n}{H} - 2m\right)^2 - \epsilon}\ u_1 + d_1. \tag{31}$$

Choosing $d_1 = 0$ and using the parametrization equation, we can easily deduce

$$g = \pm p\left(\frac{n}{H} - 2m\right) u_1 + d_2, \quad \text{where } d_2 \text{ is some constant.} \tag{32}$$

From (31) and (32), the result follows. □

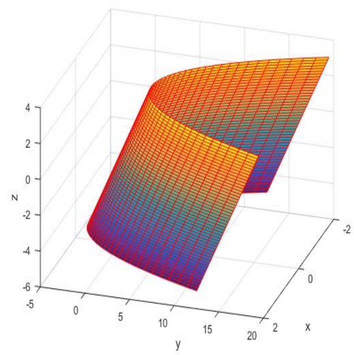

Figure 5: L.W. Revolution surface of type III

Theorem 5. *Let M be a linear Weingarten revolution surface of type III in G_3^1, then M is of the form*

$$\begin{cases} x(u_1,u_2) = \left(u_2, f(u_1) + \frac{u_2^2}{2b}, u_1\right); \\ f(u_1) = f(u_1) = \frac{\pm(1\pm 2mp)Exp(\mp\frac{-2np}{\epsilon(1\pm 2mp)u_1} - c_1)}{2pn} \pm \frac{\epsilon(1\pm 2mp)Exp(\pm\frac{-2np}{\epsilon(1\pm 2mp)u_1} + c_1)}{8pn} + c_2. \end{cases}$$

Proof. Since f, g are arbitrary functions C^r, $r \geq 1$ functions of u_1. Assuming $g(u_1) = u_1$. Then for $d_2 = 0$, from (32), we have

$$u_1 = \pm p\left(\frac{n}{H} - 2m\right)u_1,$$

or

$$H \pm 2mpH = \pm pn. \tag{33}$$

Equation (33) is equivalent to

$$\epsilon(1 \pm 2mp)f'' \pm 2pn\sqrt{f'^2 + \epsilon} = 0. \tag{34}$$

From (34), we obtain

$$f(u_1) = \frac{\pm(1 \pm 2mp)Exp(\mp\frac{-2np}{\epsilon(1\pm 2mp)u_1} - c_1)}{2pn} \pm \frac{\epsilon(1 \pm 2mp)Exp(\pm\frac{-2np}{\epsilon(1\pm 2mp)u_1} + c_1)}{8pn} + c_2,$$

where $c_1, c_2 \in \mathbb{R}$.

Hence, the result follows. □

4.1 Amalgamatic curvature

For hypersurfaces in Euclidean n-spaces, a new kind of curvature called as amalgamatic curvature was defined in [8]. In particular when $n = 3$, the amalgamatic curvature is the harmonic ratio of the principal curvatures, i.e., the ratio of the Gaussian and the mean curvature. Indeed in the above results, we see that for the cases $c = 0$, the results reduce to amalgamatic case.

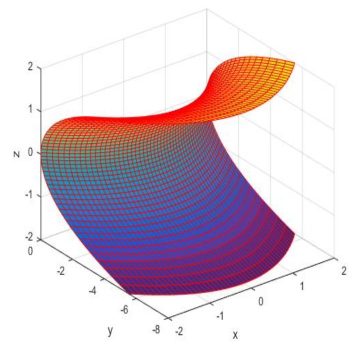

Figure 6: L.W. revolution surface of type III

Corollary 4. *Let M be a revolution surface of type III in the pseudo-Galilean space G_3^1 with the ratio $\frac{K}{H} = c_1 \neq 0$, where c_1 is a constant, then M is of the form*

$$\begin{cases} x(u_1, u_2) = \left(u_2, f(u_1) + \frac{u_2^2}{2b}, g(u_1)\right), \\ f(u_1) = \sqrt{p^2c_1^2 - \epsilon}\; u_1 + c_2, \\ g(u_1) = \pm pc_1 + c_3, \end{cases} \tag{35}$$

where c_2 and c_3 are constants.

Proof. From (29), we have

$$H = \frac{\epsilon p K}{\sqrt{f'^2 + \epsilon}}$$

or,

$$\frac{\sqrt{f'^2 + \epsilon}}{\epsilon p} = \frac{K}{H} = c_1$$

or,

$$f' = \sqrt{p^2c_1^2 - \epsilon}. \tag{36}$$

Integrating (36), we obtain

$$f = \sqrt{p^2c_1^2 - \epsilon}\; u_1 + c_2,$$

where c_2 is a constant.
Now since $f'^2 - g'^2 = -\epsilon$, we get

$$g = \pm pc_1 + c_3,$$

where c_3 is a constant. □

5 Finite type of revolution surface in G_3^1

Theorem 6. *Let M be a revolution surface of type I satisfying $\Delta x_i = \lambda_i x_i$, then g is of the form*

(1) $\pm\dfrac{u_1}{\sqrt{2}} + c,$

(2) $\dfrac{1}{\sqrt{\lambda_1 - \lambda_2}} \sinh\left[\dfrac{1}{2}\left(\pm\sqrt{2}\sqrt{\lambda_1 - \lambda_2}\; u_1 + 2\sqrt{\lambda_1 - \lambda_2}\; c\right)\right],$
where $\lambda_1 - \lambda_2 > 0$ and c is a constant.

Proof. Let M be a revolution surface given by

$$M = (u_1, g(u_1)\cosh u_2, g(u_1)\sinh u_2). \tag{37}$$

From (11), we have

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & -g^2 \end{pmatrix} \text{ and } (g_{ij})^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{-1}{g^2} \end{pmatrix}. \tag{38}$$

Using (7) and (38), we can easily find

$$\Delta = -\frac{\partial^2}{\partial u_1^2} - \frac{g'}{g}\frac{\partial}{\partial u_1} + \frac{1}{g^2}\frac{\partial^2}{\partial u_2^2}. \tag{39}$$

Since M satisfies (8), from (37) and (39), we obtain

$$\begin{cases} \lambda_1 u_1 = -\frac{g'}{g}, \\ \lambda_2 g \cosh u_2 = \left(-g'' - \frac{g'^2}{g} + \frac{1}{g}\right)\cosh u_2, \\ \lambda_3 g \sinh u_2 = \left(-g'' - \frac{g'^2}{g} + \frac{1}{g}\right)\sinh u_2. \end{cases} \tag{40}$$

From (40), we see that M is of at most two types. Rewriting (40), we have

$$\begin{cases} \lambda_1 u_1 = -\frac{g'}{g}, \\ \lambda_2 g = \left(-g'' - \frac{g'^2}{g} + \frac{1}{g}\right). \end{cases} \tag{41}$$

Now, we classify (37) according to the different possibilities of λ_1 and λ_2.
Case1: If $\lambda_1 = \lambda_2 = 0$, from (41), we arrive at a contradiction. So there exist no revolution surfaces in this case.
Differentiating the first equation of (41) w.r.t. u_1, we get

$$\lambda_1 = -\frac{gg'' - g'^2}{g^2}. \tag{42}$$

Using (42) in the second equation of (41), we obtain

$$2g'^2 + (\lambda_2 - \lambda_1)g^2 - 1 = 0. \tag{43}$$

Case2: If $\lambda_1 = \lambda_2 \neq 0$, from (43), we have

$$g = \pm\frac{u_1}{\sqrt{2}} + c.$$

Case3: If $\lambda_1 = 0, \lambda_2 \neq 0$, from (43), we have

$$g = \frac{1}{\sqrt{\lambda_2}} \sinh\left[\frac{1}{2}\left(\pm\sqrt{2}\sqrt{\lambda_2}\, u_1 + 2\sqrt{\lambda_2}\, c\right)\right].$$

Case4: If $\lambda_1 \neq 0, \lambda_2 = 0$, from (43), we obtain

$$g = \frac{1}{\sqrt{\lambda_1}} \sinh\left[\frac{1}{2}\left(\pm\sqrt{2}\sqrt{\lambda_1}\, u_1 + 2\sqrt{\lambda_1}\, c\right)\right].$$

Case5: If $\lambda_1 \neq \lambda_2 \neq 0$, from (43), we get

$$g = \frac{1}{\sqrt{\lambda_1 - \lambda_2}} \sinh\left[\frac{1}{2}\left(\pm\sqrt{2}\sqrt{\lambda_1 - \lambda_2}\, u_1 + 2\sqrt{\lambda_1 - \lambda_2}\, c\right)\right],\ \lambda_1 - \lambda_2 > 0$$

and c is a constant. From case 5, we see that cases 2, 3, 4 follow from case 5 by fixing up the values of λ_1 and λ_2.
Hence the result follows. □

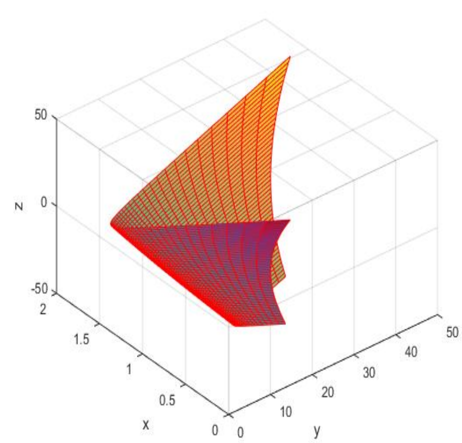

Figure 7: Finite type of revolution surface

Acknowledgement

The author is thankful to the anonymous referees for the valuable comments and suggestions.

References

[1] L. Arkady, L. and Sulanke, R., *Projective and Cayley-Klein geometries*, Springer-verlag Berlin Heidelberg, 2006.

[2] Aydin, M. E. , Öğrenmiş, A. and Erğut, M., *Classification of factorable surfaces in the pseudo-Galilean space*, Glas. Mat., **50**(**70**) (2015), 441-451.

[3] Aydin, M. E. and Mihai, A., *Classifications of quasi-sum production functions with Allen determinants,* Filomat **29** (6) (2015), 1351-1359.

[4] Baba-Hamed, C. and Bekkar, M., *On the Gauss map of surfaces of Revolution in the three dimensional minkowski space,* Tsukuba J. Math., **36** (2) (2013), 193-215.

[5] Bekkar, M. and Senoussi, B., *Factorable surfaces in three dimensional Euclidean and Lorentzian spaces satisfying* $\Delta r_i = \lambda_i r_i$, J. Geom., **103** (1) (2012), 17-29.

[6] Beltrami, E., *Risoluzione di un Problema Relativo alla Teoria delle Superficie Gobbe,* Ann. Mat. Pura Appl., **7** (1865/1866), 139-150.

[7] Chen, B. Y., *Total mean curvature and submanifolds of finite type,* World Scientific Publisher, 1984.

[8] Conley, C. T. R., Etnyre, R. Gardener, B., Odom, L. H. and Suceava, B. D. *New curvature inequalities for hypersurfaces in the Euclidean ambient space,* Taiwanese J. Math. **17** (3) (2013), 885-895.

[9] Darboux, G., *Leçcons sur la Théorie général des surfaces et les applications géométriques du calcul infnitésimal,* Gauthier-Villars, Paris,**3** (1894).

[10] Dillen, F. and Kühnel, W., *Ruled Weingarten surfaces inMinkowski 3-space,* Manuscripta Math. **98** (1999), 307-320.

[11] Divjak, B., *Curves in pseudo-Galilean geometry,* Annales Univ. Sci. Budapest **41** (1998), 117-128.

[12] Divjak, B., *Geometrija pseudogalilean prostora,* Ph.D. Thesis, University of Zagreb, 1997.

[13] Divjak, B., *Some special surfaces in pseudo-Galilean space,* Acta Math. Hungar. **118** (2008), 209-226.

[14] Galv́ez, J. A.., Martińez, A. and Milań, F., *Linear Weingarten surfaces in* $\mathbb{R}^3$, Monatsh. Math., **138** (2003), 133-144.

[15] Kaimakamis, G. and Papantoniou, B. J, *Surfaces of revolution in the 3-dimensional Lorentz-Minkowski space satisfying* $\Delta^{II}\vec{r} = A\vec{r}$, J. Geom. **81** (2004), 81-92.

[16] Karacan, M. K. and Tuncer, Y., *Tubular surfaces of Weingarten types in Galilean and pseudo-Galilean,* Bull. Math. Analysis Appl. **5** (2013), 87-100.

[17] Kim, Y. H. and Yoon, D. W., *Classifcation of ruled surfaces in Minkowski 3-spaces,* J. Geom. Phys. **49** (2004), 89-100.

[18] . Kühnel, W., *Ruled W-surfaces,* Arch. Math. **62** (1994), 475-480.

[19] C.W. Lee: *Linear Weingarten Rotational Surfaces in Pseudo-Galilean 3-Space*, Inter. J. Math. Anal., **9** (2015), 2469-2483.

[20] Lie, S., *Über Flächen deren Krümmungsradien durch eine Relation verknüpft sind,* Arch. Math. **4** (1880), 507-512.

[21] Milin *Šipŭ*, *Ž.*, *Ruled Weingarten surfaces in the Galilean space,* Period. Math. Hungar. **56** (2008), 213-225.

[22] O'Neill, B., *Semi Riemannian deometry and its applications to Relativity,* Academic Press, New York, 1983.

[23] Pierort, J., *The history of mathematics in the nineteenth century,* Bull. Amer. Math. Soc. **37** (1999), 9-24.

[24] Ro, J.S. and Yoon, D.W., *Tubes of Weingarten types in a Euclidean-3 space*, Journal of Chungcheong Mathematical Society, **22**(3), (2009), 359-366.

[25] Šipuš, Ž. M., *Ruled Weingarten surfaces in the Galilean space,* Period. Math. Hungar. **56** (2008), 213-225.

[26] Takahashi, T., *Minimal immersions of Riemannian manifolds,* J. Math. Soc. Japan, **18** (1966), 380-385.

[27] Weingarten, J., *Über eine Klasse auf einander abwickelbarer Flachen,* J. Reine Angew. Math. (**59**) (1861), 382-393.

[28] Yoon, D. W., *Surfaces of revolution in the three dimensional pseudo-Galilean space,* Glas. Mat., **48** (2013), 415-428.

[29] Yoon, D. W., Tuncer, Y. and Karacan, M. K., *Non-degenerate quadric surfaces of Weingarten type,* Annales Polonici Math. **107** (2013), 59-69.