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CLASSES OF HARMONIC FUNCTIONS DEFINED BY SALEGEAN-TYPE $q-$ DIFFERENTIAL OPERATORS

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Abstract

We consider a complex-valued harmonic functions that are univalent can be written in the form $f = h + \overline{g}$, where h and g are analytic, in a simply connected domain $\mathbb{U}$ and sense preserving in $\mathbb{U}$, is that $|h'(z)| > |g'(z)|$ in $\mathbb{U}$. Making use of Salegean $q-$ differential operators, we define a new subclasses harmonic starlike functions and obtain sufficient coefficient bounds, distortion theorems and extreme points for f in the new function class. Moreover, we shown that these necessary coefficient bounds are also sufficient for those functions that have negative coefficients.

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1 Introduction

A continuous function $f = u + iv$ is a complex- valued harmonic function in a complex domain Ω if both u and v are real and harmonic in Ω. In any simply connected domain $\mathbb{D} \subset \Omega$ we can write $f = h + \overline{g}$ where h and g are analytic in $\mathbb{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense preserving in $\mathbb{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathbb{D}$ (see [2]).

Let $\mathcal{H}$ be the family of functions $f = h + \overline{g}$ which are harmonic univalent and sense preserving in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Such functions $f = h + \overline{g} \in \mathcal{H}$ may be expressed by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \quad |b_1| < 1. \tag{1}$$

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We note that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class $\mathcal{S}$ of normalized univalent functions if the co-analytic part of $f = h + \overline{g}$ is identically zero, that is $g \equiv 0$. We let $\overline{\mathcal{H}}$ be the subclass of $\mathcal{H}$ consisting harmonic functions of the form $f_m = h + \overline{g_m}$ where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \;\; g_m(z) = (-1)^m \overline{\sum_{n=1}^{\infty} b_n z^n} \tag{2}$$

so that $a_n \geq 0$ and $b_n \geq 0$.

We recall the notion of *q-operators* or *q-difference operators* that play vital roles in the theory of hypergeometric series, quantum physics and operator theory. The application of *q-calculus* was initiated by Jackson [4] and Kanas and Răducanu [8] who have used the fractional *q-calculus operators* in investigations of certain classes of functions which are analytic in $\mathbb{U}$. For more details on *q-calculus* and its applications one can refer to [1, 3, 4, 8] and the references cited therein.

For $0 < q < 1$ the Jackson's *q-derivative* of a function $f \in \mathcal{S}$ is given as follows [4]

$$D_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & for \quad z \neq 0, \\ f'(0) & for \quad z = 0, \end{cases} \tag{3}$$

$$D_q^2 f(z) = D_q(D_q f(z)).$$

From (3), we have $D_q f(z) = 1 + \sum\limits_{n=2}^{\infty} [n]_q a_n z^{n-1}$ where $[n]_q = \frac{1-q^n}{1-q}$ is sometimes called *the basic number n*. If $q \to 1^-$ then $[n] \to n$. For $f \in \mathcal{S}$, Govindaraj and Sivasubramanian [3] considered the Salagean *q-differential operators*

$$\begin{aligned} D_q^0 f(z) &= f(z), \\ D_q^1 f(z) &= z D_q f(z), \\ D_q^m f(z) &= z D_q^m (D_q^{m-1} f(z)), \\ D_q^m f(z) &= z + \sum_{n=2}^{\infty} [n]^m a_n z^n \quad (m \in \mathbb{N}_0, z \in \mathbb{U}). \end{aligned}$$

We note that if $\lim_q \to 1^-$ then

$$D^m f(z) = z + \sum_{n=2}^{\infty} [n]^m a_n z^n \quad (m \in \mathbb{N}_0, z \in \mathbb{U})$$

is the familiar Salagean derivative[9]. Recently Jahangiri [6] considered a generalized Salegean $q-$ differential operator for harmonic function $f = h + \overline{g} \in \mathcal{H}$ defined for $m > -1$ by

$$D_q^m f(z) = D_q^m h(z) + (-1)^m \overline{D_q^m g(z)} = z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n + (-1)^m \sum_{n=1}^{\infty} [n]_q^m \overline{b_n} \overline{z}^n. \tag{4}$$

As a generalization of the functions defined in [6], for $0 \leq \alpha < 1$, we let $\mathcal{HR}_q^m(\lambda,\alpha)$ be the subclass of $\mathcal{H}$ consisting of functions $f = h + \overline{g}$ of the form (1) so that

$$\Re\left(\frac{D_q^{m+1} f(z)}{(1-\lambda)D_q^m f(z) + \lambda D_q^{m+1} f(z)}\right) \geq \alpha \tag{5}$$

where $0 \leq \lambda < 1$, $D_q^m f$ is given by (4) and $z \in \mathbb{U}$. We also let $\overline{\mathcal{H}}\mathcal{R}_q^m(\lambda,\alpha) = \mathcal{HR}_q^m(\lambda,\alpha) \cap \overline{\mathcal{H}}$. Obviously, for $\lambda = 0$ we have $\overline{\mathcal{H}}\mathcal{R}_q^m(\lambda,\alpha) \equiv \overline{\mathcal{H}}\mathcal{R}_q^m(\alpha)$ considered in [6]. It is the aim of this paper to obtain sufficient coefficient bounds, distortion theorems and extreme points for functions in $\mathcal{HR}_q^m(\lambda,\alpha)$. Moreover we show that these necessary coefficient bounds are also sufficient for functions in $\overline{\mathcal{H}}\mathcal{R}_q^m(\lambda,\alpha)$.

2 Main Results

First we obtain a sufficient coefficient condition for functions in $\mathcal{HR}_q^m(\lambda,\alpha)$.

Theorem 1. *Let $f = h + \overline{g}$ be given by (1). If*

$$\sum_{n=1}^{\infty}[n]_q^m\left\{([n]_q - \alpha - \alpha\lambda([n]_q - 1))|a_n| + ([n]_q + \alpha - \alpha\lambda([n]_q + 1))|b_n|\right\} \leq 2(1-\alpha) \tag{6}$$

where $a_1 = 1$ and $0 \leq \alpha < 1$, then $f \in \mathcal{HR}_q^m(\lambda,\alpha)$.

Proof. We will show that if (6) holds for the coefficients of $f = h + \overline{g}$ then the required condition (5) is satisfied. We note that (5) can be rewritten as

$$\Re\left(\frac{D_q^{m+1}h(z) - (-1)^m\overline{D_q^{m+1}g(z)}}{(1-\lambda)(D_q^m h(z) + (-1)^m\overline{D_q^m g(z)}) + \lambda(D_q^{m+1}h(z) - (-1)^m\overline{D_q^{m+1}g(z)})}\right)$$
$$= \Re\,\frac{A(z)}{B(z)} \geq \alpha$$

where

$$A(z) = D_q^{m+1}h(z) - (-1)^m\overline{D_q^{m+1}g(z)} = z + \sum_{n=2}^{\infty}[n]_q^{m+1}a_n z^n - (-1)^m\sum_{n=1}^{\infty}[n]_q^{m+1}\overline{b_n}\overline{z}^n$$

and

$$\begin{aligned} B(z) &= (1-\lambda)(D_q^m h(z) + (-1)^m\overline{D_q^m g(z)}) + \lambda(D_q^{m+1}h(z) - (-1)^m\overline{D_q^{m+1}g(z)}\ ) \\ &= z + \sum_{n=2}^{\infty}[n]_q^m(1-\lambda+\lambda[n]_q)a_n z^n + (-1)^m\sum_{n=1}^{\infty}[n]_q^m(1-\lambda-\lambda[n]_q)\overline{b_n}\overline{z}^n. \end{aligned}$$

Using the fact that $\Re\{w\} \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$, it suffices to show that

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \geq 0. \tag{7}$$

Substituting for $A(z)$ and $B(z)$ in (7), we get

$$\begin{aligned}
& |A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)| \\
= \; & \Big| \; (2-\alpha)z+\sum_{n=2}^{\infty}[n]_q^m\{[n]_q+1-\alpha(1-\lambda+\lambda[n]_q)\}a_n z^n \\
& -(-1)^m\sum_{n=1}^{\infty}[n]_q^m\{[n]_q-(1-\alpha)(1-\lambda+\lambda[n]_q)\}\overline{b_n}\;\overline{z}^n \;\Big| \\
& -\Big| -\alpha z+\sum_{n=2}^{\infty}[n]_q^m\{[n]_q-(1+\alpha)(1-\lambda+\lambda[n]_q)\}a_n z^n \\
& -(-1)^m\sum_{n=1}^{\infty}[n]_q^m\{[n]_q+(1+\alpha)(1-\lambda+\lambda[n]_q)\}\overline{b_n}\overline{z}^n \Big| \\
\geq \; & (2-\alpha)|z|-\sum_{n=2}^{\infty}[n]_q^m\{[n]_q+(1-\alpha)(1-\lambda+\lambda[n]_q)\}|a_n||z|^n \\
& -\sum_{n=1}^{\infty}[n]_q^m\{[n]_q-(1-\alpha)(1-\lambda-\lambda[n]_q)\}|b_n|\;|z|^n \\
& -\alpha|z|-\sum_{n=2}^{\infty}[n]_q^m\{[n]_q-(1+\alpha)(1-\lambda+\lambda[n]_q)\}|a_n|\;|z|^n \\
& -\sum_{n=1}^{\infty}[n]_q^m\{[n]_q+(1+\alpha)(1-\lambda-\lambda[n]_q)\}|b_n|\;|z|^n \\
\geq \; & 2(1-\alpha)|z|\Big(2-\sum_{n=1}^{\infty}[n]_q^m\Big[\frac{[n]_q-\alpha-\alpha\lambda([n]_q-1)}{1-\alpha}|a_n| \\
& \qquad +\frac{[n]_q+\alpha-\alpha\lambda([n]_q+1)}{1-\alpha}|b_n|\Big]|z|^{n-1}\Big) \\
\geq \; & 2(1-\alpha)\Big(2-\sum_{n=1}^{\infty}[n]_q^m\Big[\frac{[n]_q-\alpha-\alpha\lambda([n]_q-1)}{1-\alpha}|a_n| \\
& \qquad +\frac{[n]_q+\alpha-\alpha\lambda([n]_q+1)}{1-\alpha}|b_n|\Big]\Big).
\end{aligned}$$

The above expression is non negative by (6) and so $f(z)\in\mathcal{HR}_q^m(\lambda,\alpha)$. □

For $\lambda=0$ we obtain the following corollary which is also given by Jahangiri [6].

Corollary 1. *Let $f=h+\overline{g}$ be given by (1). If*

$$\sum_{n=1}^{\infty}[n]_q^m\left\{([n]_q-\alpha)|a_n|+([n]_q+\alpha)|b_n|\right\}\leq 2(1-\alpha)$$

where $a_1=1$ and $0\leq\alpha<1$, then $f\in\mathcal{HR}_q^m(\alpha)$.

The starlikeness of the functions given in Theorem 1 follows from Theorem 1 given in [5] and noticing that

$$[n]_q - \alpha - \alpha\lambda([n]_q - 1) \leq [n]_q - \alpha \leq n - \alpha$$

and

$$[n]_q + \alpha - \alpha\lambda([n]_q + 1) \leq [n]_q + \alpha \leq n + \alpha.$$

Next we show that the coefficient bounds (6) are also sufficient for functions in $\overline{\mathcal{H}\mathcal{R}}_q^m(\lambda,\alpha)$.

Theorem 2. *Let $f_m = h + \overline{g_m}$ given by (2)is $\in \overline{\mathcal{H}\mathcal{R}}_q^m(\lambda,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}[n]_q^m \left\{([n]_q - \alpha - \alpha\lambda([n]_q - 1))a_n + ([n]_q + \alpha - \alpha\lambda([n]_q + 1))b_n\right\} \leq 2(1-\alpha) \tag{8}$$

where $a_1 = 1$ and $0 \leq \alpha < 1$.

Proof. Since $\overline{\mathcal{H}\mathcal{R}}_q^m(\lambda,\alpha) \subset \mathcal{H}\mathcal{R}_q^m(\lambda,\alpha)$, we only need to prove the "only if" part of the theorem. To this end, for functions $f_m = h + \overline{g_m}$ in $\overline{\mathcal{H}\mathcal{R}}_q^m(\lambda,\alpha)$ we must have

$$\Re\left(\frac{D_q^{m+1} f_m(z)}{(1-\lambda)D_q^m f_m(z) + \lambda D_q^{m+1} f_m(z)}\right) \geq \alpha$$

or equivalently,

$$\Re\left(\frac{(1-\alpha)z - \sum\limits_{n=2}^{\infty}[n]_q^m\{[n]_q - \alpha - \alpha\lambda([n]_q - 1)\}a_n z^n}{z - \sum\limits_{n=2}^{\infty}[n]_q^m(1-\lambda+\lambda[n]_q)a_n z^n + (-1)^{2m}\sum\limits_{n=1}^{\infty}[n]_q^m(1-\lambda-\lambda[n]_q)b_n \overline{z}^n}\right)$$

$$-\Re\left(\frac{(-1)^{2m}\sum\limits_{n=1}^{\infty}[n]_q^m\{[n]_q + \alpha - \alpha\lambda([n]_q + 1)\}b_n \overline{z}^n}{z - \sum\limits_{n=2}^{\infty}[n]_q^m(1-\lambda+\lambda[n]_q)a_n z^n + (-1)^{2m}\sum\limits_{n=1}^{\infty}[n]_q^m(1-\lambda-\lambda[n]_q)b_n \overline{z}^n}\right) \geq 0.$$

The above condition must hold for all values of z in $\mathbb{U}$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$\Big((1-\alpha) - \sum_{n=2}^{\infty}[n]_q^m\{[n]_q - \alpha - \alpha\lambda([n]_q - 1)\}a_n r^{n-1}$$
$$- \sum_{n=1}^{\infty}[n]_q^m\{[n]_q + \alpha - \alpha\lambda([n]_q + 1)\}b_n r^{n-1}\Big) \times$$
$$\times\Big(1 - \sum_{n=2}^{\infty}[n]_q^m(1-\lambda+[n]_q\lambda)a_n r^{n-1} + \sum_{n=1}^{\infty}[n]_q^m(1-\lambda-[n]_q\lambda)b_n r^{n-1}\Big)^{-1}$$
$$\geq 0.$$

If the condition (8) does not hold, then the numerator in the above inequality is negative for r sufficiently close to 1. Hence, there exists $z_0 = r_0$ in (0,1) for which the left hand side of the above inequality is negative. This contradicts the required condition for $f(z) \in \overline{\mathcal{H}}\mathcal{R}_q^m(\lambda, \alpha)$ and so the proof is complete. □

Next we determine the extreme points of closed convex hulls of $\overline{\mathcal{H}}\mathcal{R}_q^m(\lambda, \alpha)$ denoted by $clco\overline{\mathcal{H}}\mathcal{R}_q^m(\lambda, \alpha)$.

Theorem 3. *A function $f_m(z) \in \overline{\mathcal{H}}\mathcal{R}_q^m(\lambda, \alpha)$ if and only if*

$$f_m(z) = \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_{n_m}(z))$$

where $h_1(z) = z$, $h_n(z) = z - \frac{1-\alpha}{[n]_q^m\{[n]_q - \alpha - \alpha\lambda([n]_q - 1)\}} z^n$; $(n \geq 2)$, *and* $g_{n_m}(z) = z + \frac{(-1)^m(1-\alpha)}{[n]_q^m\{[n]_q + \alpha - \alpha\lambda([n]_q + 1)\}} \overline{z}^n; (n \geq 2)$, $\sum\limits_{n=1}^{\infty}(X_n + Y_n) = 1$, $X_n \geq 0$ *and* $Y_n \geq 0$. *In particular, the extreme points of $\overline{\mathcal{H}}\mathcal{R}_q^m(\lambda, \alpha)$ are $\{h_n\}$ and $\{g_{n_m}\}$.*

Proof. First, we note that for f_m as given in the theorem, we may write

$$\begin{aligned} f_m(z) &= \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_{n_m}(z)) \\ &= \sum_{n=1}^{\infty}(X_n + Y_n)z - \sum_{n=2}^{\infty} \frac{1-\alpha}{[n]_q^m\{[n]_q - \alpha - \alpha\lambda([n]_q - 1)\}} X_n z^n \\ &\quad +(-1)^m \sum_{n=1}^{\infty} \frac{1-\alpha}{[n]_q^m\{[n]_q + \alpha - \alpha\lambda([n]_q + 1)\}} Y_n \overline{z}^n \\ &= z - \sum_{n=2}^{\infty} A_n z^n + (-1)^m \sum_{n=1}^{\infty} B_n \overline{z}^n, \end{aligned}$$

where

$$\begin{aligned} A_n &= \frac{1-\alpha}{[n]_q^m\{[n]_q - \alpha - \alpha\lambda([n]_q - 1)\}} X_n, \\ B_n &= \frac{1-\alpha}{[n]_q^m\{[n]_q + \alpha - \alpha\lambda([n]_q + 1)\}} Y_n. \end{aligned}$$

Therefore

$$\sum_{n=2}^{\infty} \frac{[n]_q^m\{[n]_q - \alpha - \alpha\lambda([n]_q - 1)\}}{1-\alpha} A_n + \sum_{n=1}^{\infty} \frac{[n]_q^m\{[n]_q + \alpha - \alpha\lambda([n]_q + 1)\}}{1-\alpha} B_n$$
$$= \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 - X_1 \leq 1,$$

and hence $f_m(z) \in clco\overline{\mathcal{H}}\mathcal{R}_q^m(\lambda,\alpha)$. Conversely, suppose $f_m(z) \in clco\overline{\mathcal{H}}\mathcal{R}_q^m(\lambda,\alpha)$. Set $X_n = \frac{[n]_q^m\{[n]_q-\alpha-\alpha\lambda([n]_q-1)\}}{1-\alpha}A_n$ and $Y_n = \frac{[n]_q^m\{[n]_q+\alpha-\alpha\lambda([n]_q-1)\}}{1-\alpha}B_n$, where $\sum\limits_{n=1}^{\infty}(X_n+Y_n)=1$. Then

$$\begin{aligned}
f_m(z) &= z-\sum_{n=2}^{\infty}a_nz^n+(-1)^m\sum_{n=1}^{\infty}b_n\overline{z}^n\\
&= z-\sum_{n=2}^{\infty}\frac{1-\alpha}{[n]_q^m\{[n]_q-\alpha-\alpha\lambda([n]_q-1)\}}X_nz^n\\
&\quad+(-1)^m\sum_{n=1}^{\infty}\frac{1-\alpha}{[n]_q^m\{[n]_q+\alpha-\alpha\lambda([n]_q-1)\}}Y_n\overline{z}^n\\
&= z-\sum_{n=2}^{\infty}(h_n(z)-z)X_n+\sum_{n=1}^{\infty}(g_{n_m}(z)-z)Y_n\\
&= \sum_{n=1}^{\infty}(X_nh_n(z)+Y_ng_{n_m}(z))
\end{aligned}$$

as required. □

Next we give distortion bounds and a covering result for the class $\overline{\mathcal{H}}\mathcal{R}_q^m(\lambda,\alpha)$.

Theorem 4. *Let $f_m \in \overline{\mathcal{H}}\mathcal{R}_q^m(\lambda,\alpha)$. Then for $|z|=r<1$, we have*

$$\begin{aligned}
&(1-b_1)r-\frac{1}{[2]_q^m}\left(\frac{1-\alpha}{[2]_q-\alpha-\alpha\lambda}-\frac{1+\alpha}{[2]_q-\alpha-\alpha\lambda}b_1\right)r^2\le|f_m(z)|\\
&\le(1+b_1)r+\frac{1}{[2]_q^m}\left(\frac{1-\alpha}{[2]_q-\alpha-\alpha\lambda}-\frac{1+\alpha}{[2]_q-\alpha-\alpha\lambda}b_1\right)r^2.
\end{aligned}$$

Proof. We only prove the right hand inequality. Taking the absolute value of $f_m(z)$, we obtain

$$\begin{aligned}
|f_m(z)| &= \left|z+\sum_{n=2}^{\infty}a_nz^n+(-1)^m\sum_{n=1}^{\infty}b_n\overline{z}^n\right|\\
&\le (1+b_1)|z|+\sum_{n=2}^{\infty}(a_n+b_n)|z|^n\\
&\le (1+b_1)r+\sum_{n=2}^{\infty}(a_n+b_n)r^2
\end{aligned}$$

$$
\begin{aligned}
&\leq (1+b_1)r + \frac{1-\alpha}{[2]_q^m([2]_q-\alpha-\alpha\lambda)} \\
&\quad \sum_{n=2}^{\infty}\left(\frac{[2]_q^m([2]_q-\alpha-\alpha\lambda)}{1-\alpha}a_n + \frac{[2]_q^m([2]_q-\alpha-\alpha\lambda)}{1-\alpha}b_n\right)r^2 \\
&\leq (1+b_1)r + \frac{1-\alpha}{[2]_q^m([2]_q-\alpha-\alpha\lambda)}\left(1-\frac{1+\alpha}{1-\alpha}b_1\right)r^2 \\
&\leq (1+b_1)r + \frac{1}{[2]_q^m}\left(\frac{1-\alpha}{[2]_q-\alpha-\alpha\lambda} - \frac{1+\alpha}{[2]_q-\alpha-\alpha\lambda}b_1\right)r^2.
\end{aligned}
$$

The proof of the left hand inequality is similar and is omitted. □

As a consequence of Theorem 4 we obtain the following corollary.

Corollary 2. *Let* $f_m(z) \in \overline{\mathcal{H}}\mathcal{R}_q^m(\lambda,\alpha)$. *Then*

$$
\left\{w : |w| < \frac{[2]_q^m([2]_q-\alpha-\alpha\lambda)-1+\alpha}{[2]_q^m([2]_q-\alpha-\alpha\lambda)} - \frac{[2]_q^m([2]_q-\alpha-\alpha\lambda)-(1+\alpha)}{[2]_q^m([2]_q-\alpha-\alpha\lambda)}b_1\right\} \subset f_m(\mathbb{U}).
$$

Proof. For completeness, we provide a brief justification. Using the left hand inequality of Theorem 4 and letting $r \to 1$, it follows that

$$
\begin{aligned}
&(1-b_1) - \frac{1}{[2]_q^m}\left(\frac{1-\alpha}{[2]_q-\alpha-\alpha\lambda} - \frac{1+\alpha}{[2]_q-\alpha-\alpha\lambda}b_1\right) \\
&= (1-b_1) - \frac{1}{[2]_q^m([2]_q-\alpha-\alpha\lambda)}[1-\alpha-(1+\alpha)b_1] \\
&= \frac{(1-b_1)[2]_q^m([2]_q-\alpha-\alpha\lambda)-(1-\alpha)+(1+\alpha)b_1}{[2]_q^m([2]_q-\alpha-\alpha\lambda)} \\
&= \frac{[2]_q^m([2]_q-\alpha-\alpha\lambda)-[2]_q^m([2]_q-\alpha-\alpha\lambda)b_1-(1-\alpha)+(1+\alpha)b_1}{[2]_q^m([2]_q-\alpha-\alpha\lambda)} \\
&= \frac{[2]_q^m([2]_q-\alpha-\alpha\lambda)-1+\alpha-[[2]_q^m([2]_q-\alpha-\alpha\lambda)-(1+\alpha)]b_1}{[2]_q^m([2]_q-\alpha-\alpha\lambda)} \\
&= \frac{[2]_q^m([2]_q-\alpha-\alpha\lambda)-1+\alpha}{[2]_q^m([2]_q-\alpha-\alpha\lambda)} - \frac{[2]_q^m([2]_q-\alpha-\alpha\lambda)-(1+\alpha)}{[2]_q^m([2]_q-\alpha-\alpha\lambda)}b_1 \subset f_m(\mathbb{U}).
\end{aligned}
$$

□

Finally we show that class $\overline{\mathcal{H}}\mathcal{R}_q^m(\lambda,\alpha)$ is closed under convex combinations.

Theorem 5. *The family* $\overline{\mathcal{H}}\mathcal{R}_q^m(\lambda,\alpha)$ *is closed under convex combinations.*

Proof. For $i = 1, 2, \ldots$, suppose that $f_{m_i} \in \overline{\mathcal{H}}\mathcal{R}_q^m(\lambda,\alpha)$ where

$$
f_{m_i}(z) = z - \sum_{n=2}^{\infty} a_{i,n}z^n + (-1)^m \sum_{n=2}^{\infty} \overline{b}_{i,n}\overline{z}^n.
$$

Then, by Theorem 2

$$\sum_{n=2}^{\infty} \frac{[n]_q^m\{[n]_q-\alpha-\alpha\lambda([n]_q-1)\}}{1-\alpha} a_{i,n} + \sum_{n=1}^{\infty} \frac{[n]_q^m\{[n]_q+\alpha-\alpha\lambda([n]_q+1)\}}{1-\alpha} b_{i,n} \leq 1.$$

For $\sum\limits_{i=1}^{\infty} t_i$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_{m_i}(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{i,n}\right) z^n + (-1)^m \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{i,n}\right) \overline{z}^n.$$

Using the inequality (8), we obtain

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{[n]_q^m\{[n]_q-\alpha-\alpha\lambda([n]_q-1)\}}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i a_{i,n}\right) \\ &\qquad + \sum_{n=1}^{\infty} \frac{[n]_q^m\{[n]_q+\alpha-\alpha\lambda([n]_q+1)\}}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i b_{i,n}\right) \\ &= \sum_{i=1}^{\infty} t_i \Big(\sum_{n=2}^{\infty} \frac{[n]_q^m\{[n]_q-\alpha-\alpha\lambda([n]_q-1)\}}{1-\alpha} a_{i,n} \\ &\qquad + \sum_{n=1}^{\infty} \frac{[n]_q^m\{[n]_q+\alpha-\alpha\lambda([n]_q+1)\}}{1-\alpha} b_{i,n} \Big) \\ &\leq \sum_{i=1}^{\infty} t_i = 1, \end{aligned}$$

and therefore $\sum\limits_{i=1}^{\infty} t_i f_{m_i} \in \overline{\mathcal{H}\mathcal{R}}_q^m(\lambda, \alpha)$. □

Concluding Remarks: The results of this paper for the special case $\lambda = 0$ yield analogous results obtained in [6]. Furthermore, by letting $\lim_{q\to 1^-}$ and taking $\lambda = 0$ and $m = 0$ we obtain the analogous results for the classes studied in [7] and [5], respectively. Moreover, if we let $\alpha = 0$ we obtain the results given in [10].

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