Bulletin of the *Transilvania* University of Braşov • Vol 11(60), No. 2 - 2018
Series III: Mathematics, Informatics, Physics, 125-138

THE APPLICATIONS OF THE UNIVERSAL MORPHISMS OF CF-TOP THE CATEGORY OF ALL FUZZY TOPOLOGICAL SPACES

Farhan ISMAIL [1] **and Abdelkrim LATRECHE*** [2]

Abstract

In the present work, we built a category of fuzzy topological spaces from Chang's definition of Fuzzy TOPological space, that we denoted CF-TOP. Firstly, we collected universal morphisms of TOP category, listed by Sander Mac Lane [6], then, we studied universal morphisms of CF-TOP. This study shows that these morphisms are just generalizations of TOP category morphisms, which confirms that Chang's fuzziness to topological space is weak. At the end of this work, we prove that TOP and CF-TOP are not isomorphic.

2000 *Mathematics Subject Classification:* 03E72, 18B30, 18A30.
Key words: category, functor, fuzzy topological space, TOP-category, universal morphisms.

1 Introduction

In the early 1940's Samuel Eilenberg and Saunders Mac Lane [7] invented Category theory, with the aim of bridging what may appear to be two quite different fields: Topology and Algebra. Later, it was propagated by Alexander Grothendieck in 1960's. From another side, L. A. Zadeh [10] introduced the fuzzy set in 1965, since then many researchers have used this tool to generalize different concepts of Mathematics.
General topology is considered to be one of the first branches of pure mathematics that appeared at the end of the 19th century. However, the fuzzification of topological space is defined by C. L. Chang [3] in 1968, that is, three years after Zadeh's paper.
Regarding the importance of fuzzy applications and category theory, it seems more interesting to join both. This leads us to speaking about the applications of the universal morphisms of the fuzzy category.

[1]Department of Mechatronics, *Sakarya University* , Sakarya, Turkey, e-mail: farhanismail@sakarya.edu.tr
[2]*Corresponding author, Department of Mathematics, *20 Aout 1955 University*, Skikda, Algeria, e-mail: a.latreche@univ-skikda.dz

The present work is organized as follows: in the next section, we recall some of the basic definitions (fuzzy set and operations on it, the fuzzy topological space, fuzzy continuous application, fuzzy topological space, universal morphisms, ...). Then, we collect the universal morphisms of TOPological spaces category (TOP). In the 3rd section, we study the universal morphisms of fuzzy topological spaces category (CF-TOP). And finally, we chose category functor (simple)[8] for clarifying the relation between TOP and CF-TOP categories and we proved that this functor is not isomorphic.

2 PRELIMINARY NOTIONS

Let X be a set. A fuzzy set A in X is characterized by a membership function $\mu_A(x)$ from X into $[0,1]$. [3]-[9]-[10]

Definition 1. *[3]-[10] Let A and B be fuzzy sets in X. Then:*

1. $A = B \Longleftrightarrow \mu_A(x) = \mu_B(x),$ *for all* $x \in X$.
2. $A \subset B \Longleftrightarrow \mu_A(x) \leq \mu_B(x),$ *for all* $x \in X$.
3. $C = A \cup B \Longleftrightarrow \mu_C(x) = max\{\mu_A(x), \mu_B(x)\}$ *for all* $x \in X$.
4. $D = A \cap B \Longleftrightarrow \mu_D(x) = min\{\mu_A(x), \mu_B(x)\}$ *for all* $x \in X$.

More generally, for a family of fuzzy sets, $A = \{A_i,\ i \in I\}$, the union, $C = \cup_I A_i$, and the intersection $D = \cap_I A_i$, are defined by:

$$\mu_C(x) = sup_I\{\mu_{A_i}(x)\} \qquad for\,all\ x \in X.$$

$$\mu_D(x) = inf_I\{\mu_{A_i}(x)\} \qquad for\,all\ x \in X.$$

The symbol $\varnothing$ will be used to denote an empty fuzzy set ($\mu_\varnothing(x) = 0$ for all $x \in X$). For X, we have by definition $\mu_X(x) = 1$, for all $x \in X$.

Definition 2. *[3] Let f be a function from X to Y. Let B be a fuzzy set in Y with membership function $\mu_B(y)$. Then the inverse of B, written as $f^{-1}(B)$, is a fuzzy set in X whose membership function is defined by:*

$$\mu_{f^{-1}(B)}(x) = \mu_B(f(x)) \quad for\,all\ \ x \in X.$$

Definition 3. *[3] A fuzzy topology is a family T of fuzzy sets in X which satisfies the following conditions:*

1. $\varnothing, X \in T$.
2. *Si* $A_1, A_2 \in T$, *then* $A_1 \cap A_2 \in T$.
3. *Si* $A_i \in T$ *four all* $i \in I$, *then* $\cup_I A_i \in T$.

T is called a fuzzy topology for X, and the pair (X,T) is a fuzzy topological space or (F-TOP) in short. Every member of T is called a T-open fuzzy set.

Definition 4. *[3] A function f from an F-TOP (X,T) to an F-TOP (Y,U) is fuzzy continuous (F-continuous) iff the inverse of each U-open fuzzy set is T-open.*

Definition 5. *[9]*

(a) Let T be a fuzzy topology. A subfamily B of T is a base for T iff each member of T can be expressed as the union of some members of B.

(b) A subfamily S of B is a sub-base for T iff the family of finite intersections of members of S forms a base for T.

(c) A sub-base for the product fuzzy topology on $(X,T) = (\prod_{i\in I} X_i, \prod_{i\in I} T_i)$ is given by $S = \{\pi_i^{-1}\theta_i;\ \theta_i \in T_i,\ i \in I\}$ (π_i the projection from X onto X_i) so that a base can be taken to be

$$B = \{\cap_{j=1}^{n} \pi_{i_j}^{-1}\theta_{i_j};\ \theta_{i_j} \in T_{i_j},\ i_j \in I,\ j = 1...n,\ n \in \mathbb{N}\}.$$

Definition 6. *[6] Let D, C be two categories, $S : D \longrightarrow C$ is a functor and c an object of C, a universal arrow from c to S is a pair $< r, u >$ consisting of an object r of D and $u : c \longrightarrow Sr$ an arrow of S, such that to every pair $< d, f >$ with d an object of D and $f : c \longrightarrow Sd$ an arrow of C, there is a unique arrow $f' : r \longrightarrow d$ of D with $Sf' \circ u = f$.*

Proposition 1. *[6](THE UNIVERSAL MORPHISMS OF* <u>*TOP*</u>*)*
<u>*TOP*</u> *is the category of all topological spaces and continuous maps.*

(a) The element of Co-product of (X, τ_X) and (Y, τ_Y) in <u>*TOP*</u> *is their disjoint union.*

(b) The element of Co-equalizer of $f, g : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ in <u>*TOP*</u> *is the topological space $(Y/\sim, \tau_{Y/\sim})$, where $\sim$ is the least equivalence relation which contains all pairs $< f(x), g(x) >$, for $x \in X$.*

(c) The element of Push-out of $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$, $g : (X, \tau_X) \longrightarrow (Z, \tau_Z)$ in <u>*TOP*</u> *is the disjoint union $(Y \uplus Z, \tau_{Y \uplus Z})$ with the elements $f(x)$ and $g(x)$ identified for each $x \in X$.*

(d) The element of Product of (X, τ_X), (Y, τ_Y) in <u>*TOP*</u> *is their cartesian product.*

(e) The element of Equalizer of $f, g : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ in <u>*TOP*</u> *is the topological space (D, τ_D), where $D = \{x \in X, f(x) = g(x)\}$.*

(f) The element of Pull-back of $f : (X, \tau_X) \longrightarrow (Z, \tau_Z)$, $g : (Y, \tau_Y) \longrightarrow (Z, \tau_Z)$ in <u>*TOP*</u> *is the topological space (C, τ_C), where $C = \{(x, y) \in X \times Y, f(x) = g(y)\}$.*

3 Main results

This section is devoted to present the main results of this paper.
The fuzzy topological spaces F-TOP and fuzzy continuous mappings form a category which we denote by CF-TOP. Now, we investigate the universal morphisms of this category.

3.1 Co-product

Definition 7. *(Disjoint union of fuzzy topological spaces)*
Let $(X_1,\tau_1),(X_2,\tau_2)$ be two fuzzy topological spaces, μ and μ' denote the membership functions of the elements of τ_1 and τ_2 respectively.

The disjoint union of $(X_1,\tau_1),(X_2,\tau_2)$ is defined as:

$$(X_1,\tau_1)\uplus(X_2,\tau_2)=(X_1\uplus X_2,\tau_{X_1\uplus X_2}).$$

where

$$X_1\uplus X_2=\{X_1\times\{1\}\}\cup\{X_2\times\{2\}\}.$$

and

$$\tau_{X_1\uplus X_2}=\{\theta,\ \theta \textit{ is a fuzzy set on } X_1\uplus X_2\}.$$

The membership function of the elements of $\tau_{X_1\uplus X_2}$ is defined by:

$$\begin{aligned}(\mu\uplus\mu')_\theta : X_1\uplus X_2 &\longrightarrow [0,1]\\ (x,k)\longmapsto (\mu\uplus\mu')_\theta(x,k) &= \begin{cases}\mu_{\varphi_1^{-1}(\theta)}(x) & \textit{if } k=1.\\ \mu'_{\varphi_2^{-1}(\theta)}(x) & \textit{if } k=2.\end{cases}\end{aligned}$$

where

$$\begin{aligned}\varphi_1 : (X_1,\tau_1) &\longrightarrow (X_1\uplus X_2,\tau_{X_1\uplus X_2})\\ x &\longmapsto \varphi_1(x)=(x,1)\end{aligned}$$

and

$$\begin{aligned}\varphi_2 : (X_2,\tau_2) &\longrightarrow (X_1\uplus X_2,\tau_{X_1\uplus X_2})\\ x &\longmapsto \varphi_2(x)=(x,2)\end{aligned}$$

Proposition 2. *The disjoint union $(X_1\uplus X_2,\tau_{X_1\uplus X_2})$ is a fuzzy topological space.*

Proof. (1) We have:

$$(\mu\uplus\mu')_\varnothing(x,k)=\begin{cases}\mu_{\varphi_1^{-1}(\varnothing)}(x) & \text{if } k=1.\\ \mu'_{\varphi_2^{-1}(\varnothing)}(x) & \text{if } k=2.\end{cases}=\begin{cases}\mu_\varnothing(x) & \text{if } k=1.\\ \mu'_\varnothing(x) & \text{if } k=2.\end{cases}=0.$$

$$\begin{aligned}(\mu\uplus\mu')_{X_1\uplus X_2}(x,k) &= \begin{cases}\mu_{\varphi_1^{-1}(X_1\uplus X_2)}(x) & \text{if } k=1.\\ \mu'_{\varphi_2^{-1}(X_1\uplus X_2)}(x) & \text{if } k=2.\end{cases}\\ &= \begin{cases}\mu_{X_1}(x) & \text{if } k=1.\\ \mu'_{X_2}(x) & \text{if } k=2.\end{cases}=1.\end{aligned}$$

So $\varnothing,\ X_1\uplus X_2\in\tau_{X_1\uplus X_2}$.

(2) If $\theta_1, \theta_2 \in \tau_{X_1 \uplus X_2}$, then $\theta_1 \cap \theta_2$ is a fuzzy set on $X \uplus X'$, $\lambda_{\theta_1 \cap \theta_2}$ denotes the membership function of $\theta_1 \cap \theta_2$. By definition(1) we have :

$$\begin{aligned} \lambda_{\theta_1 \cap \theta_2} : \quad & X \uplus X' \longrightarrow [0,1] \\ & (x,k) \longmapsto \lambda_{\theta_1 \cap \theta_2}(x,k) = min\{\lambda_{\theta_1}(x,k), \lambda_{\theta_2}(x,k)\} \end{aligned}$$

We have two cases:

case (1): If $k = 1$, we have:

$$\begin{aligned} \lambda_{\theta_1 \cap \theta_2}(x,1) &= min\{\lambda_{\theta_1}(x,1), \lambda_{\theta_2}(x,1)\} \\ &= min\{(\mu \uplus \mu')_{\theta_1}(x,1),\ (\mu \uplus \mu')_{\theta_2}(x,1)\} \\ &= min\{\mu_{\varphi_1^{-1}(\theta_1)}(x),\ \mu_{\varphi_1^{-1}(\theta_2)}(x)\} \\ &= \mu_{\varphi_1^{-1}(\theta_1) \cap \varphi_1^{-1}(\theta_2)}(x) = \mu_{\varphi_1^{-1}(\theta_1 \cap \theta_2)}(x). \end{aligned}$$

case (2): If $k = 2$, using the same method with $k = 1$ we prove that:

$$\lambda_{\theta_1 \cap \theta_2}(x,2) = \mu'_{\varphi_1^{-1}(\theta_1 \cap \theta_2)}(x).$$

So $\lambda_{\theta_1 \cap \theta_2}(x,k) = \begin{cases} \mu_{\varphi_1^{-1}(\theta_1 \cap \theta_2)}(x) & \text{if } k = 1. \\ \mu'_{\varphi_2^{-1}(\theta_1 \cap \theta_2)}(x) & \text{if } k = 2. \end{cases}$

then $\theta_1 \cap \theta_2 \in \tau_{X_1 \uplus X_2}$.

(3) If $\theta_h \in \tau_{X_1 \uplus X_2}$, $\forall h \in \triangle$, then $\cup_{h \in \triangle} \theta_h$ is a fuzzy set on $X \uplus X'$, $\lambda_{\cup_{h \in \triangle} \theta_h}$ denotes the membership function of $\cup_{h \in \triangle} \theta_h$. By definition(1) we have :

$$\begin{aligned} \lambda_{\cup_{h \in \triangle} \theta_h} : & X \uplus X' \longrightarrow [0,1] \\ & (x,k) \longmapsto \lambda_{\cup_{h \in \triangle} \theta_h}(x,k) = sup_{h \in \triangle}\{\lambda_{\theta_h}(x,k)\} \end{aligned}$$

We have two cases:

case (1): If $k = 1$, then:

$$\begin{aligned} \lambda_{\cup_{h \in \triangle} \theta_h}(x,1) &= sup_{h \in \triangle}\{\lambda_{\theta_h}(x,1)\} = sup_{h \in \triangle}\{(\mu \uplus \mu')_{\theta_h}(x,1)\} \\ &= \mu_{\cup_{h \in \triangle} \varphi_1^{-1}(\theta_h)}(x) = \mu_{\varphi_1^{-1}(\cup_{h \in \triangle} \theta_h)}(x). \end{aligned}$$

case (2): If $k = 2$, using the same method with $k = 1$ we prove that:

$$\lambda_{\cup_{h \in \triangle} \theta_h}(x,2) = \mu'_{\varphi_2^{-1}(\cup_{h \in \triangle} \theta_h)}(x)$$

So $\lambda_{\cup_{h \in \triangle} \theta_h}(x,k) = \begin{cases} \mu_{\varphi_1^{-1}(\cup_{h \in \triangle} \theta_h)}(x) & \text{if } k = 1. \\ \mu'_{\varphi_2^{-1}(\cup_{h \in \triangle} \theta_h)}(x) & \text{if } k = 2. \end{cases}$

then $\cup_{h \in \triangle} \theta_h \in \tau_{X_1 \uplus X_2}$.

□

Proposition 3. *The applications φ_1, φ_2 are F-continuous.*

Proof. First, let's prove that φ_1 is F-continuous.
Let $\theta \in \tau_{X_1 \uplus X_2}$, from the definition (2), the inverse of θ by φ_1 is a fuzzy set in X_1, $\lambda_{\varphi_1^{-1}(\theta)}$ denotes the membership function of $\varphi_1^{-1}(\theta)$, then:

$$\lambda_{\varphi_1^{-1}(\theta)}(x) = \lambda_\theta\, \varphi_1(x) = (\mu \uplus \mu')_\theta(x,1) = \mu_{\varphi_1^{-1}(\theta)}(x).$$

Using the same method we prove that φ_2 is F-continuous. □

Theorem 1. *Let $f : (X_1, \tau_1) \longrightarrow (C, \tau_C)$, $g : (X_2, \tau_2) \longrightarrow (C, \tau_C)$ be two F-continuous applications (μ'' denotes the membership function of the elements of τ_C), then there exists an F-continuous application $h : (X_1 \uplus X_2, \tau_{X_1 \uplus X_2}) \longrightarrow (C, \tau_C)$ such that $f = h \circ \varphi_1$ and $g = h \circ \varphi_2$.*

Proof. Let's define h by :

$$\begin{aligned} h : (X_1 \uplus X_2, \tau_{X_1 \uplus X_2}) &\longrightarrow (C, \tau_C) \\ (x,k) &\longmapsto h(x,k) = \begin{cases} f(x) & \text{if } k = 1. \\ g(x) & \text{if } k = 2. \end{cases} \end{aligned} \tag{1}$$

It is clear that : $f = h \circ \varphi_1$ *and* $g = h \circ \varphi_2$.
Let $\theta_k \in \tau_C$, by definition(2) and (1) we have:

$$\lambda_{h^{-1}(\theta_k)}(x,k) = \lambda_{\theta_k} h(x,k) = \mu''_{\theta_k} h(x,k) = \begin{cases} \mu''_{\theta_k} f(x) & \text{if } k = 1. \\ \mu''_{\theta_k} g(x) & \text{if } k = 2. \end{cases}$$

As f, g are F-continuous then:

$$\begin{aligned} \lambda_{h^{-1}(\theta_k)}(x,k) &= \begin{cases} \mu_{f^{-1}(\theta_k)}(x) & \text{if } k = 1. \\ \mu'_{g^{-1}(\theta_k)}(x) & \text{if } k = 2. \end{cases} \\ &= \begin{cases} \mu_{(h\circ\varphi_1)^{-1}(\theta_k)}(x) & \text{if } k = 1. \\ \mu'_{(h\circ\varphi_2)^{-1}(\theta_k)}(x) & \text{if } k = 2. \end{cases} \\ &= \begin{cases} \mu_{\varphi_1^{-1}(h^{-1}(\theta_k))}(x) & \text{if } k = 1. \\ \mu'_{\varphi_2^{-1}(h^{-1}(\theta_k))}(x) & \text{if } k = 2. \end{cases} \end{aligned}$$

which gives $h^{-1}(\theta_k) \in \tau_{X_1 \uplus X_2}$, so h is F-continuous. □

Corollary 1. *The element of Co-product of $(X_1, \tau_1), (X_2, \tau_2) \in$ CF-TOP is a fuzzy topological space $(X_1, \tau_1) \uplus (X_2, \tau_2)$ (defined above).*

Proof. By proposition(2) $(X_1 \uplus X_2, \tau_{X_1 \uplus X_2}) \in$ CF-TOP. Also by proposition(3) φ_1, φ_2 are F- continuous.
By theorem (1), if $f : (X_1, \tau_1) \longrightarrow (C, \tau_C)$, $g : (X_2, \tau_2) \longrightarrow (C, \tau_C)$ are F-continuous applications, then there exists an F-continuous application h defined by (1), that verifies $f = h \circ \varphi_1$, $g = h \circ \varphi_2$.
Let $h' : (X_1 \uplus X_2, \tau_{X_1 \uplus X_2}) \longrightarrow (C, \tau_C)$ be another F-continuous application where $f = h' \circ \varphi_1$ *and* $g = h' \circ \varphi_2$. We have:

$$(h' \circ \varphi_1)(x) = h'(\varphi_1(x)) = h'(x,1) = f(x).$$

and

$$(h' \circ \varphi_2)(x) = h'(\varphi_2(x)) = h'(x, 2) = g(x).$$

therefore h is unique. □

3.2 Co-equalizer

Definition 8. *Let (X, τ_X) be a fuzzy topological space, μ denotes the membership function of the elements of τ_X, $\sim$ is the equivalence relation on X and $P : X \longrightarrow X/\sim$ is the natural projection map, we define $\tau_{X/\sim}$ by:*

$$\tau_{X/\sim} = \{\theta,\ \theta \text{ is a fuzzy set on } X/\sim\}.$$

The membership function of the elements of $\tau_{X/\sim}$ is defined by:

$$\begin{aligned} \overline{\mu_\theta} : X/\sim &\longrightarrow [0,1] \\ \overline{x} &\longmapsto \overline{\mu_\theta}(\overline{x}) = \mu_{P^{-1}(\theta)}(x) \end{aligned}$$

Proposition 4. *The space $(X/\sim, \tau_{X/\sim})$ is a fuzzy topological space.*

Proof. (1) We have:

$$\overline{\mu_\varnothing}(\overline{x}) = \mu_{P^{-1}(\varnothing)}(x) = \mu_\varnothing(x) = 0.$$

$$\overline{\mu_{X/\sim}}(\overline{x}) = \mu_{P^{-1}(X/\sim)}(x) = \mu_X(x) = 1.$$

So $\varnothing$, $X/\sim \in \tau_{X/\sim}$.

(2) If θ_1, $\theta_2 \in \tau_{X/\sim}$, then:

$$\begin{aligned} \lambda_{\theta_1 \cap \theta_2}(\overline{x}) &= min\{\lambda_{\theta_1}(\overline{x}),\ \lambda_{\theta_2}(\overline{x}\} = min\{\overline{\mu_{\theta_1}}(\overline{x}),\ \overline{\mu_{\theta_2}}(\overline{x})\} \\ &= min\{\mu_{P^{-1}(\theta_1)}(x),\ \mu_{P^{-1}(\theta_2)}(x)\} = \mu_{P^{-1}(\theta_1) \cap P^{-1}(\theta_2)}(x) \\ &= \mu_{P^{-1}(\theta_1 \cap \theta_2)}(x). \end{aligned}$$

So $\theta_1 \cap \theta_2 \in \tau_{X/\sim}$.

(3) If $\theta_k \in \tau_{X/\sim}$, $\forall k \in \Delta$, then:

$$\begin{aligned} \lambda_{\cup_{k\in\Delta}\theta_k}(\overline{x}) &= sup_{k\in\Delta}\{\lambda_{\theta_k}(\overline{x})\} = sup_{k\in\Delta}\{\mu_{P^{-1}(\theta_k)}(x)\ \} \\ &= \mu_{\cup_{k\in\Delta}P^{-1}(\theta_k)}(x) = \mu_{P^{-1}(\cup_{k\in\Delta}\theta_k)}(x). \end{aligned}$$

So $\cup_{k\in\Delta}\theta_k \in \tau_{X/\sim}$.

□

Proposition 5. *The application P is F-continuous.*

Proof. evident (by definition of $\tau_{X/\sim}$). □

Theorem 2. *Let $(A, \tau_A), (B, \tau_B) \in$ F-TOP, μ and μ' denote the membership functions of the elements of τ_A and τ_B respectively, $\sim$ is the equivalence relation of A and $P : A \longrightarrow A/\sim$ is the associated projection. If $h : (A, \tau_A) \longrightarrow (B, \tau_B)$ is the F-continuous application compatible with $\sim$, then there exists a unique F-continuous application h', where $h = h' \circ P$. In addition:*

$$h \text{ is F-continuous} \Longrightarrow h' \text{ is F-continuous.}$$

Proof. Let's define h' by :

$$\begin{aligned} h' : (A/\sim, \tau_{A/\sim}) &\longrightarrow (B, \tau_B) \\ \overline{x} &\longmapsto h'(\overline{x}) = h(x) \end{aligned}$$

It is clear that h' is unique and $h = h' \circ P$.
Let $B_i \in \tau_B$ then:
$\lambda_{h'^{-1}(B_i)}(\overline{x}) = \lambda_{B_i} h'(\overline{x}) = \mu'_{B_i} h(x) = \mu_{h^{-1}(B_i)}(x) = \mu_{(h' \circ P)^{-1}(B_i)}(x) = \mu_{P^{-1}(h'^{-1}(B_i))}(x)$.
So h' is F-continuous. □

Corollary 2. *The element of Co-equalizer of $f, g : (X, \tau) \longrightarrow (X', \tau')$ in CF-TOP is the fuzzy topological space $(X'/\sim, \tau_{X'/\sim})$, where $\sim$ is the least equivalence relation which contains all pairs $< f(x), g(x) >$, such that $x \in X$.*

Proof. Let $h : (X', \tau') \longrightarrow (C, \tau_C)$ be an F-continuous application where $h \circ f = h \circ g$. For the existence of a unique h', by theorem (2) it is sufficient to prove that h is compatible with $\sim$:
Let $x_1, x_2 \in X'$, $x_1 \sim x_2 \Longleftrightarrow \exists a \in X, x_1 = f(a) \ \wedge \ x_2 = g(a)$.
and $h(x_1) = h(f(a)) = (h \circ f)(a) = (h \circ g)(a) = h(g(a)) = h(x_2)$, so h is compatible with $\sim$.
Finally, h is unique by theorem (2). □

3.3 Push-out

Definition 9. *Let $(A, \tau_A), (B, \tau_B) \in$ F-TOP, μ and μ' denote the membership functions of the elements of τ_A and τ_B respectively, $\sim$ equivalence relation on $A \uplus B$ (note $X_0 = (A \uplus B)/\sim$), we define τ_{X_0} by:*

$$\tau_{X_0} = \{\theta \, , \, \theta \text{ is a fuzzy set on } X_0\}.$$

The membership function of the elements of τ_{X_0} is defined by:

$$\begin{aligned} \overline{(\mu \uplus \mu')}_\theta : X_0 &\longrightarrow [0,1] \\ \overline{(x,k)} &\longmapsto \overline{(\mu \uplus \mu')}_\theta(\overline{x,k}) = \begin{cases} \mu_{\varphi_1^{-1}(P^{-1}(\theta))}(x) & \text{if } k = 1. \\ \mu'_{\varphi_2^{-1}(P^{-1}(\theta))}(x) & \text{if } k = 2. \end{cases} \end{aligned}$$

where

$$\begin{aligned} P : A \uplus B &\longrightarrow X_0 \\ (x,k) &\longmapsto P(x,k) = \overline{(x,k)} \end{aligned}$$

$$\begin{aligned} \varphi_1 : (A, \tau_A) &\longrightarrow (A \uplus B, \tau_{A \uplus B}) \\ x &\longmapsto \varphi_1(x) = (x, 1) \end{aligned}$$

and

$$\begin{aligned} \varphi_2 : (B, \tau_B) &\longrightarrow (A \uplus B, \tau_{A \uplus B}) \\ x &\longmapsto \varphi_2(x) = (x, 2) \end{aligned}$$

Proposition 6. *The space (X_0, τ_{X_0}) is a fuzzy topological space.*

Proof. The proof is based on the proofs of proposition (2) and proposition (4). □

Proposition 7. *The following applications:*

$$\alpha : (A, \tau_A) \longrightarrow (X_0, \tau_{X_0})$$
$$x \longmapsto \alpha(x) = \overline{(x,1)}$$

$$\beta : (B, \tau_B) \longrightarrow (X_0, \tau_{X_0})$$
$$x \longmapsto \beta(x) = \overline{(x,2)}$$

are F-continuous.

Proof. First, let's prove that α is F-continuous.
For $\theta \in \tau_{X_0}$, we have:

$$\lambda_{\alpha^{-1}(\theta)}(x) = \lambda_\theta \alpha(x) = \lambda_\theta \overline{(x,1)} = \overline{(\mu \cup \mu')}_\theta \overline{(x,1)} = \mu_{\varphi_1^{-1}(P^{-1}(\theta))}(x).$$

It is clear that $\alpha = P \circ \varphi_1$, then:

$$\lambda_{\alpha^{-1}(\theta)}(x) = \mu_{(P\circ\varphi_1)^{-1}(\theta)}(x) = \mu_{\alpha^{-1}(\theta)}(x).$$

So α is F-continuous.
Using the same method we prove that β is F-continuous.

□

Theorem 3. *Let $f : (A, \tau_A) \longrightarrow (C, \tau_C), g : (A, \tau_A) \longrightarrow (B, \tau_B)$ be two F-continuous applications. The element of Push-out of $< f, g >$ is (X_0, τ_{X_0}), where $X_0 = (B \cup C)/\sim$ and $\sim$ is the least equivalence relation which contains all pairs $< (\varphi_1 \circ f)(c), (\varphi_2 \circ g)(c) >$, such that $c \in A$.*

Proof. By proposition (6) $(X_0, \tau_{X_0}) \in$ F-TOP. Also, by proposition (7) α, β are F-continuous.
Let $(Y, \tau_Y) \in$ F-TOP, and $U : (B, \tau_B) \longrightarrow (Y, \tau_Y)$, $V : (C, \tau_C) \longrightarrow (Y, \tau_Y)$ are two F-continuous applications, where $V \circ f = U \circ g$.
The proof of the existence of a unique F-continuous application
$h : (X_0, \tau_{X_0}) \longrightarrow (Y, \tau_Y)$ where $U = h \circ \alpha$, $V = h \circ \beta$ requires the following steps:

Step1: The Co-product of (B, τ_B), (C, τ_C) is a disjoint union $(B \cup C, \tau_{B\cup C})$, then for $\{\alpha, \ \beta\}$ there exists an F-continuous application $\pi : (B \cup C, \tau_{B\cup C}) \longrightarrow (X_0, \tau_{X_0})$ where $\alpha = \pi \circ \varphi_1$, $\beta = \pi \circ \varphi_2$.

Step2: Let's define the new application $U \cup V$ by:

$$U \cup V : (B \cup C, \tau_{B\cup C}) \longrightarrow (Y, \tau_Y)$$
$$(x,k) \longmapsto (U \cup V)(x,k) = \begin{cases} U(x) & if\ k = 1. \\ V(x) & if\ k = 2. \end{cases}$$

If $U \cup V$ is compatible with $\sim$ then there exists a unique F-continuous application $h : (X_0, \tau_{X_0}) \longrightarrow (Y, \tau_Y)$ where: $U \cup V = h \circ \pi$ (theorem (2)).

Let (x,k), $(x',k') \in B \uplus C$ that:
$(x,k) \sim (x',k') \Longrightarrow \exists a \in A,\ (x,k) = (\varphi_1 \circ g)(a)$ and $(x',k') = (\varphi_2 \circ f)(a)$.

$$(U \uplus V)(x,k) = (U \uplus V)(\varphi_1 \circ g)(a) = (U \uplus V)(g(a),1) = U(g(a)) = (U \circ g)(a).$$

$$(U \uplus V)(x',k') = (U \uplus V)(\varphi_2 \circ f)(a) = (U \uplus V)(f(a),2) = V(f(a)) = (V \circ f)(a).$$

But $V \circ f = U \circ g$, then $U \uplus V$ is compatible with $\sim$.

Step3: Prove that $U = h \circ \alpha, V = h \circ \beta$.
$(h \circ \alpha)(x) = (h \circ (\pi \circ \varphi_1))(x) = (h \circ \pi)(x,1) = (U \uplus V)(x,1) = U(x),\ \forall x \in B.$
$(h \circ \beta)(x) = (h \circ (\pi \circ \varphi_2))(x) = (h \circ \pi)(x,2) = (U \uplus V)(x,2) = V(x),\ \ \forall x \in C.$

□

3.4 Product

Definition 10. *Let $(X_1,\delta_1),(X_2,\delta_2)$ be two F-TOP, μ and μ' denote the membership functions of the elements of δ_1 and δ_2 respectively. We define $\tau_{X_1\times X_2}$ by:*

$$\tau_{X_1\times X_2} = \{\theta\,,\ \theta = \cup_{i\in I}(\theta_1)_i \times (\theta_2)_i \text{ is a fuzzy set on } X_1\times X_2,\ (\theta_1)_i \in \delta_1, (\theta_2)_i \in \delta_2,\ \forall i \in I\}.$$

The membership function of the elements of $\tau_{X_1\times X_2}$ is defined by:

$$(\mu \times \mu')_\theta(x,y) = sup_{i\in I}\{min\{\mu_{(\theta_1)_i}(x), \mu'_{(\theta_2)_i}(y)\}\}, \text{ for all } (x,y) \in X_1 \times X_2.$$

Proposition 8. *[9] The space $(X_1 \times X_2, \tau_{X_1\times X_2})$ is a fuzzy topological space.*

Proposition 9. *[9] The projections P_1, P_2 are F-continuous, where :*

$$\begin{aligned} P_1 : (X_1 \times X_2, \tau_{X_1\times X_2}) &\longrightarrow (X_1,\delta_1) \\ (x,y) &\longmapsto P_1(x,y) = x \end{aligned}$$

$$\begin{aligned} P_2 : (X_1 \times X_2, \tau_{X_1\times X_2}) &\longrightarrow (X_2,\delta_2) \\ (x,y) &\longmapsto P_2(x,y) = y \end{aligned}$$

Theorem 4. *[9] Let (Y,τ_Y) be an F-TOP and let f be a function from Y to $X_1 \times X_2$. Then f is F-continuous iff $P_1 \circ f$, $P_2 \circ f$ are F-continuous.*

Corollary 3. *Let $(X_1,\delta_1),(X_2,\delta_2) \in$ CF-TOP. The element of product of (X_1,δ_1), (X_2,δ_2) are the fuzzy topological space $(X_1 \times X_2, \tau_{X_1\times X_2})$ (defined above).*

Proof. If $f : (C,\delta_3) \longrightarrow (X_1,\delta_1)$, $g : (C,\delta_3) \longrightarrow (X_2,\delta_2)$ be two F-continuous applications, then there exists a unique F-continuous application defined by:

$$\begin{aligned} h : (C,\delta_3) &\longrightarrow (X_1 \times X_2, \tau_{X_1\times X_2}) \\ x &\longmapsto h(x) = (f(x), g(x)) \end{aligned}$$

where $f = P_1 \circ h$ and $g = P_2 \circ h$.

It is clear that : $f = P_1 \circ h$ *and* $g = P_2 \circ h$.
By theorem (4), h is F-continuous.
Proof of the uniqueness of h:
Let h' be another F-continuous application where: $h' : (C, \delta) \longrightarrow (X_1 \times X_2, \tau_{X_1 \times X_2})$ and $f = P_1 \circ h'$, $g = P_2 \circ h'$.
We suppose that: $h'(x) = (a, b)$.
$a = P_1(a, b) = (P_1 \circ h')(x) = f(x)$, $b = P_2(a, b) = (P_2 \circ h')(x) = g(x)$.
Then $h'(x) = (f(x), g(x)) = h(x)$, so h is unique. □

3.5 Equalizer

Definition 11. *Let (A, τ_A), $(B, \tau_B) \in$ F-TOP, μ' denotes the membership functions of the elements of τ_B, and let $f, g : (B, \tau_B) \longrightarrow (A, \tau_A)$ be two F-continuous applications. D is a subset of B defined by: $D = \{x \in B,\ f(x) = g(x)\}$. We define τ_D by:*
$\tau_D = \{\theta\ ,\ \theta = F(D) \cap B_i$ *is a fuzzy set on* D, $B_i \in \tau_B$ *and* $F(D)$ *is a fuzzy set on* B *where* $\mu'_{F(D)}(x) = \chi_D(x)\}$.
The membership function of the elements of τ_D is defined by:

$$\mu''_\theta : D \longrightarrow [0,1]$$
$$x \longmapsto \mu''_\theta(x) = min\{\mu'_{F(D)}(x),\ \mu'_{B_i}(x)\}$$

Proposition 10. *The space (D, τ_D) is a fuzzy topological space.*

Proof. (1) We have:
We take : $(\varnothing = F(D) \cap \varnothing)$ and $(D = F(D) \cap B)$, then:
$\mu''_\varnothing(x) = min\{\mu'_{F(D)}(x)\ ,\ \mu'_\varnothing(x)\}) = min\{1\ ,\ 0\} = 0$.
$\mu''_D(x) = min\{\mu'_{F(D)}(x)\ ,\ \mu'_B(x)\} = min\{1\ ,\ 1\} = 1$.
So $\varnothing$, $D \in \tau_D$.

(2) If $\theta_1, \theta_2 \in \tau_D$ where $\theta_1 = F(D) \cap B_1$, $B_1 \in \tau_B$ and $\theta_2 = F(D) \cap B_2$, $B_2 \in \tau_B$:

$$\begin{aligned}\lambda_{\theta_1 \cap \theta_2}(x) &= \lambda_{F(D) \cap B_1 \cap F(D) \cap B_2}(x) = \lambda_{F(D) \cap (B_1 \cap B_2)}(x) \\ &= min\{\mu'_{F(D)}(x), \mu'_{(B_1 \cap B_2)}(x)\}.\end{aligned}$$

Then $\theta_1 \cap \theta_2 \in \tau_D$ (as $B_1 \cap B_2 \in \tau_B$).

(3) If $\theta_i \in \tau_D$, $\forall i \in I$, where: $\theta_i = F(D) \cap B_i$, $B_i \in \tau_D$, then:

$$\lambda_{\cup_{i \in I}(F(D) \cap B_i)}(x) = \lambda_{F(D) \cap (\cup_{i \in I} B_i)}(x) = min\{\mu'_D(x),\ \mu'_{(\cup_{i \in I} B_i)}(x)\}.$$

Then $\cup_{i \in I} \theta_i \in \tau_D$ (as $\cup_{i \in I} B_i \in \tau_B$).

□

Proposition 11. *e is F-continuous, where:*

$$e : (D, \tau_D) \longrightarrow (B, \tau_B)$$
$$x \longmapsto e(x) = x$$

Proof. Clear. □

Corollary 4. *The element of Equalizer of* $f, g : (B, \tau_B) \longrightarrow (A, \tau_A)$ *in* <u>*CF-TOP*</u> *is the fuzzy topological space* (D, τ_D) *(defined above).*

Proof. Let $(C, \tau_C) \in$ F-TOP, for $h : (C, \tau_C) \longrightarrow (B, \tau_B)$ the F-continuous application where: $f \circ h = g \circ h$, then there exists a unique F-continuous application h' defined by:

$$h' : (C, \tau_C) \longrightarrow (D, \tau_D)$$
$$x \longmapsto h'(x) = h(x)$$

h' is F-continuous since h is F-continuous.
Let $x \in C$: $(e \circ h')(x) = e(h'(x)) = e(h(x)) = h(x)$ then $e \circ h' = h$.
Proof of the uniqueness of h':
Let $h'' : (C, \tau_C) \longrightarrow (D, \tau_D)$ be another F-continuous application where $e \circ h'' = h$
$(e \circ h'')(x) = (e \circ h')(x) \Longrightarrow e(h''(x)) = e(h'(x)) \Longrightarrow h''(x) = h'(x), \ \forall x \in C.$ □

3.6 Pull-back

Definition 12. *Let* (A, τ_A), (B, τ_B), $(D, \tau_D) \in$ *F-TOP,* μ *and* μ' *denote the membership functions of the elements of* τ_A *and* τ_B *respectively, and*
$f : (B, \tau_B) \longrightarrow (A, \tau_A)$*,* $g : (D, \tau_D) \longrightarrow (A, \tau_A)$ *in* <u>*CF-TOP*</u>*.* C *is a subset of* $B \times D$ *defined by:* $C = \{(x, y) \in B \times D,\ f(x) = g(y)\} \subseteq B \times D$*. We define* τ_C *by:*
$\tau_C = \{\theta,\ \theta = F(C) \cap \theta'$ *is a fuzzy set on* C, $\theta' \in \tau_{B \times D}$ *and* $F(C)$ *is a the fuzzy set on* $B \times D$, *where*: $(\mu \times \mu')_{F(C)}(x, y) = \chi_C(x, y)\}$.
The membership function of the elements of τ_C *is defined by:*

$$\Gamma_\theta(x, y) = min\{(\mu \times \mu')_{F(C)}(x, y),\ sup_{i \in I}\{min\{\mu_{B_i}(x),\ \mu'_{D_i}(y)\}\}$$
$$\theta' = \cup_{i \in I}(B_i \times D_i),\ \forall (x, y) \in C.$$

Proposition 12. *The space* (C, τ_C) *is a fuzzy topological space.*

Proof. The proof is based on the proofs of proposition (8) and proposition (10). □

Proposition 13. *The projections* p, q *are F-continuous, where :*

$$p : (C, \tau_C) \longrightarrow (B, \tau_B)$$
$$(x, y) \longmapsto p(x, y) = x$$

$$q : (C, \tau_C) \longrightarrow (D, \tau_D)$$
$$(x, y) \longmapsto q(x, y) = y$$

Proof. First, let's prove that p is F-continuous.
Let $B_i \in \tau_B$: $\lambda_{p^{-1}(B_i)}(x, y) = \mu'_{B_i} p(x, y) = \mu'_{B_i}(x) = min\{(\mu \times \mu')_{F(C)}(x, y),$ $min\{\mu'_{B_i}(x),\ \mu'_D(y)\}\}$.
Then $p^{-1}(B_i) \in \tau_C$, so p is F-continuous.
Using the same method we prove q is F-continuous.
□

Theorem 5. *Let $f:(B,\tau_B)\longrightarrow(A,\tau_A)$, $g:(D,\tau_D)\longrightarrow(A,\tau_A)$ be two F-continuous applications and $(E,\tau_E)\in$ F-TOP, μ'' denotes the membership function of the elements of τ_E, and $h:(E,\tau_E)\longrightarrow(B,\tau_B)$, $k:(E,\tau_E)\longrightarrow(D,\tau_D)$ are two F-continuous applications where $f\circ h=g\circ k$, r an application defined by:*

$$\begin{array}{rcl} r:(E,\tau_E) & \longrightarrow & (C,\tau_C) \\ x & \longmapsto & r(x)=(h(x),k(x)). \end{array} \tag{2}$$

then: h, k are F-continuous $\Longrightarrow r$ is F-continuous.

Proof. Let $\theta\in\tau_C$ then $\theta=F(C)\cap\theta'$ and $\theta'=\cup_{i\in I}(B_i\times D_i)\in\tau_{B\times D}$, we have:

$$\begin{aligned}\lambda_{r^{-1}(F(C)\cap(\cup_{i\in I}(B_i\times D_i))}(x) &= \Gamma_{F(C)\cap(\cup_{i\in I}(B_i\times D_i))}(h(x),k(x)) \\ &= min\{(\mu\times\mu')_{F(C)}(h(x),k(x)), \\ &\qquad \sup_{i\in I}\{min\{\mu_{B_i}h(x),\ \mu'_{D_i}k(x)\}\}\}\end{aligned}$$

As $(\mu\times\mu')_{F(C)}(h(x),k(x))=1$, then:

$$\begin{aligned}\lambda_{r^{-1}(F(C)\cap(\cup_{i\in I}(B_i\times D_i))}(x) &= \Gamma_{F(C)\cap(\cup_{i\in I}(B_i\times D_i))}(h(x),k(x)) \\ &= sup_{i\in I}\{min\{\mu_{B_i}h(x),\ \mu'_{D_i}k(x)\}\} \\ &= sup_{i\in I}\{min\{\mu''_{h^{-1}(B_i)}(x),\ \mu''_{k^{-1}(D_i)}(x)\}\} \\ &= \mu''_{\cup_{i\in I}\{k^{-1}(D_i)\cap h^{-1}(B_i)\}}(x).\end{aligned}$$

.

.

Then r is F-continuous. □

Corollary 5. *Let $f:(B,\tau_B)\longrightarrow(A,\tau_A), g:(D,\tau_D)\longrightarrow(A,\tau_A)$ in CF-TOP. The element of Pull-back of $\langle f,g\rangle$ is a fuzzy topological space (C,τ_C) (defined above).*

Proof. For the projections p, q it is clear that $f\circ p=g\circ q$.
By theorem (5), if $h:(E,\tau_E)\longrightarrow(B,\tau_B)$, $k:(E,\tau_E)\longrightarrow(D,\tau_D)$ two F-continuous applications where $f\circ h=g\circ k$, then there exists a unique F-continuous application r defined by (2).
It is clear that $k=q\circ r$ and $h=p\circ r$.
Proof of the uniqueness of r:
If r' is another F-continuous application, where $r':(E,\tau_E)\longrightarrow(C,\tau_C)$ and $k=q\circ r'$, $h=p\circ r'$.
Suppose that $r'(x)=(a,b)$ therefore $a=p(a,b)=(p\circ r')(x)=h(x)$ and
$b=q(a,b)=(q\circ r')(x)=k(x)$. So $r=r'$.

□

4 Interrelation between the category TOP and CF-TOP

Many TOP and CF-TOP functors are built [1]-[2]-[4]-[5] and we choose those that suit better our work.
The natural inclusion functor [8]:

Identifying, as usual, subsets of a given set with the corresponding characteristic functions, we can treat a topological space (X,T) as an object of CF-TOP. In this way an inclusion functor $e:\underline{TOP}\longrightarrow$ CF-TOP arises.
This functor is not isomorphic since it is not surjective. Indeed:
Suppose that e is surjective, let $(X,\tau)\in$ CF-TOP, where $X=\{a,b\}$ and $\tau=\{X,\varnothing,\theta\}$ where:

$$\begin{cases}\mu_X(a)=1.\\ \mu_X(b)=1.\end{cases},\begin{cases}\mu_\varnothing(a)=0.\\ \mu_\varnothing(b)=0.\end{cases},\begin{cases}\mu_\theta(a)=0.8.\\ \mu_\theta(b)=0.7.\end{cases}$$

Posed $T=\{X,\varnothing,B\}\in\underline{TOP}$, where $e(X,T)=(X,F(T))$, but $F(T)\neq\tau$ (as $\mu_{F(B)}=\chi_B\neq\mu_\theta$).

References

[1] Azad, K.K., *Fuzzy Hausdorff spaces and fuzzy perfect mappings*, J. Math. **82** (1981), 297-305.

[2] Bourbaki, N., *Topologie gènèrale. Ch. I. Structures topologiques*, Hermann, Paris, 1961.

[3] Chang, C.L., *Fuzzy topological spaces*, J. Math. **24** (1968), 182-190.

[4] Lowen, R., *Fuzzy topological spaces and fuzzy compactness*, J. Math. **56** (1976), 182-190.

[5] Rodabaugh, S.E., *The Hausdorff separation axiom for fuzzy topological spaces*, J. Math. **11** (1980), 319-334.

[6] Saunders, M.L., *Category for the Working Mathematician*, University of Chicago, USA, 1991.

[7] Saunder, M.L., Samuel, E., *General Theory of Natural Equivalences*, J. Math. **58** (1945), 231-294.

[8] Shostak, A.P., *Two decades of fuzzy topology: basic ideas, notions, and results*, Russian Math. Surveys, 1989, Uspekhi Mat. Nauk, 1989.

[9] Wong, C.K., *Fuzzy Topology: Product and Quotient Theorems*, J. Math. **45** (1974), 512-521.

[10] Zadeh, L.A., *Fuzzy sets*, J. Math. **8** (1965), 338-353.