Bulletin of the *Transilvania* University of Braşov • Vol 11(60), No. 2 - 2018
Series III: Mathematics, Informatics, Physics, 109-118

ON A CLASS OF THREE DIMENSIONAL f-KENMOTSU MANIFOLDS

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Abstract

The curvature properties of three-dimensional f-Kenmotsu manifolds have been studied.

2000 *Mathematics Subject Classification:* 53C20, 53C25.
Key words: f-Kenmotsu manifolds, projective curvature tensor, ϕ-semisymmetric f-Kenmotsu manifolds, Ricci solitons.

1 Introduction

In 1972, K. Kenmotsu [9] introduced and studied a new class of almost contact metric manifolds, later known as Kenmotsu manifolds. Z. Olszak and R. Rosca [11] have studied f-Kenmotsu manifolds, an almost contact metric manifold which is normal and locally conformal almost cosymplectic. Further, they gave a geometric interpretation of f-Kenmotsu manifold and proved that a Ricci symmetric f-Kenmotsu manifold is an Einstein manifold. Recently, f-Kenmotsu manifolds have been studied by many authors in several ways to a different extent such as [12, 14, 15].

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a $(2n+1)$-dimensional Riemannian manifold. If there exists a one to one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes, the projective curvature tensor is defined by [1, 13]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{1}$$

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where $X, Y, Z \in \chi(M)$, R is the curvature tensor and S is the Ricci tensor with respect to the Levi-Civita connection.

A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that [5, 7, 10]

$$(\pounds_V g + 2S + 2\lambda g)(X, Y) = 0, \tag{2}$$

where S is the Ricci tensor, $\pounds_V$ is the Lie derivative operator along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive, respectively. Ricci solitons, in the context of general relativity, have been studied by M. Ali and Z. Ahsan $[2-4]$.

Motivated by the above studies, in this paper we study some curvature properties of 3-dimensional f-Kenmotsu manifolds. The paper is organized as follows: In Section 2, we give a brief account of an f-Kenmotsu manifold. In Section 3, we show that a projectively flat 3-dimensional f-Kenmotsu manifold is an Einstein manifold of constant curvature $-(f^2 + f')$. Section 4 is devoted to study ϕ-projectively semisymmetric 3-dimensional f-Kenmotsu manifolds. In Section 5, we discuss projectively semisymmetric 3-dimensional f-Kenmotsu manifolds. In Section 6, we show that a 3-dimensional f-Kenmotsu manifold satisfying the condition $P \cdot S = 0$ is an Einstein manifold. Moreover, the fact that a 3-dimensional f-Kenmotsu manifold satisfying the condition $S \cdot R = 0$ is an η-Einstein manifold is shown in Section 7. In Section 8, we show that a 3-dimensional f-Kenmotsu manifold admitting Ricci soliton is an η-Einstein manifold and the Ricci soliton is shrinking, steady and expanding if $r + 2f > 0$, $r + 2f = 0$ and $r + 2f < 0$, respectively. Finally, we give an example of 3-dimensional f-Kenmotsu manifold.

2 f-Kenmotsu manifolds

Let M be a real $(2n+1)$-dimensional differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g) which satisfies

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{3}$$

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi), \tag{4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{5}$$

for all vector fields X, $Y \in \chi(M)$, where I is the identity of the tangent bundle TM, ϕ is a tensor field of $(1, 1)$ type, η is a 1-form, ξ is a vector field and g is a metric tensor field. We say that (M, ϕ, ξ, η, g) is an f-Kenmotsu manifold if the Levi-Civita connection of g satisfies

$$(\nabla_X \phi)(Y) = f[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{6}$$

where $f \in C^\infty(M)$ is strictly positive and $df \wedge \eta = 0$. If $f = 0$, then the manifold is cosymplectic [8]. An f-Kenmotsu manifold is said to be regular if $f^2 + f' \neq 0$,

where $f' = \xi f$.
In an f-Kenmotsu manifold, from (6) we have

$$\nabla_X \xi = f[X - \eta(X)\xi]. \tag{7}$$

The condition $df \wedge \eta = 0$ holds if dim $M \geq 5$. This does not hold in general if dim $M = 3$ [14]

$$(\nabla_X \eta)Y = f[g(X, Y) - \eta(X)\eta(Y)]. \tag{8}$$

In a 3-dimensional Riemannian manifold, we have

$$R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y \tag{9}$$
$$-\frac{r}{2}[g(Y, Z)X - g(X, Z)Y].$$

In a 3-dimensional f-Kenmotsu manifold, we have

$$R(X, Y)Z = (\frac{r}{2} + 2f^2 + 2f')[g(Y, Z)X - g(X, Z)Y] \tag{10}$$

$$-(\frac{r}{2} + 3f^2 + 3f')[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y],$$

$$S(X, Y) = (\frac{r}{2} + f^2 + f')g(X, Y) - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\eta(Y), \tag{11}$$

where R, S, Q and r are the Riemann curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature, respectively.
Now from (10), we find

$$R(X, Y)\xi = -(f^2 + f')[\eta(Y)X - \eta(X)Y], \tag{12}$$

$$R(\xi, X)Y = -(f^2 + f')[g(X, Y)\xi - \eta(Y)X], \tag{13}$$

$$R(X, \xi)\xi = -(f^2 + f')[X - \eta(X)\xi], \tag{14}$$

$$\eta(R(X, Y)Z) = -(f^2 + f')[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \tag{15}$$

And from (11), we get

$$S(X, \xi) = -2(f^2 + f')\eta(X), \tag{16}$$

$$Q\xi = -2(f^2 + f')\xi. \tag{17}$$

Definition 1. *An f-Kenmotsu manifold is said to be an η-Einstein manifold if the Ricci tensor S of type $(0, 2)$ is of the form*

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{18}$$

where a and b are smooth functions on M. In particular, if $b = 0$, then the manifold is said to be an Einstein manifold.

3 Projectively flat 3-dimensional f-Kenmotsu manifolds

Let M be a projectively flat 3-dimensional f-Kenmotsu manifold, that is, $P = 0$. Then from (1), it follows that

$$R(X,Y)Z = \frac{1}{2}[S(Y,Z)X - S(X,Z)Y]. \tag{19}$$

Taking inner product of (19) with ξ and using (4), we have

$$g[R(X,Y)Z,\xi] = \frac{1}{2}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)]. \tag{20}$$

Putting $X = \xi$ in (20) and using (3), (13) and (16), we get

$$S(Y,Z) = -2(f^2 + f')g(Y,Z). \tag{21}$$

Now using (21) in (19), we obtain

$$R(X,Y)Z = -(f^2 + f')[g(Y,Z)X - g(X,Z)Y] \tag{22}$$

which can be written as

$$g[R(X,Y)Z,U] = -(f^2 + f')[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \tag{23}$$

Thus we can state the following:

Theorem 1. *A projectively flat 3-dimensional f-Kenmotsu manifold is an Einstein manifold of constant curvature $-(f^2 + f')$ and consequently it is locally isometric to the Hyperbolic space $H^3[-(f^2 + f')]$.*

4 ϕ-projectively semisymmetric 3-dimensional f-Kenmotsu manifolds

Definition 2. *A 3-dimensional f-Kenmotsu manifold is said to be ϕ-projectively semisymmetric if* [6]

$$P(X,Y)\cdot\phi = 0$$

for all $X, Y \in \chi(M)$.

Let M be a ϕ-projectively semisymmetric 3-dimensional f-Kenmotsu manifold. Therefore $P(X,Y)\cdot\phi = 0$ turns into

$$(P(X,Y)\cdot\phi)Z = P(X,Y)\phi Z - \phi P(X,Y)Z = 0 \tag{24}$$

for any vector fields $X, Y, Z \in \chi(M)$. From (1), we write

$$P(X,Y)\phi Z = R(X,Y)\phi Z - \frac{1}{2}[S(Y,\phi Z)X - S(X,\phi Z)Y] \tag{25}$$

and

$$\phi P(X,Y)Z = \phi R(X,Y)Z - \frac{1}{2}[S(Y,Z)\phi X - S(X,Z)\phi Y]. \tag{26}$$

Now combining (24), (25) and (26), we have

$$R(X,Y)\phi Z - \phi R(X,Y)Z - \frac{1}{2}[S(Y,\phi Z)X - S(X,\phi Z)Y] \tag{27}$$

$$+\frac{1}{2}[S(Y,Z)\phi X - S(X,Z)\phi Y] = 0.$$

Taking $X = \xi$ in (27) and then using (4), (11), (13) and (16), we get

$$r = -6(f^2 + f'), \quad g(Y,\phi Z) \neq 0. \tag{28}$$

Using this value of r in (11), we obtain

$$S(Y,Z) = -2(f^2 + f')g(Y,Z). \tag{29}$$

Thus in view of (10), (28) and (29), we have the following:

Theorem 2. *In a 3-dimensional f-Kenmotsu manifold M, the following conditions are equivalent:*

(*a*) *ϕ-projectively semisymmetric,*

(*b*) *the scalar curvature $r = -6(f^2 + f')$,*

(*c*) *the manifold M is of constant curvature,*

(*d*) *M is an Einstein manifold.*

5 Projectively semisymmetric 3-dimensional f-Kenmotsu manifolds

In this section, we suppose that a 3-dimensional f-Kenmotsu manifold is projectively semisymmetric, that is,

$$(R(X,Y)\cdot P)(U,V)W = 0$$

for any vector fields X, Y, U, V and $W \in \chi(M)$. This implies that

$$R(X,Y)P(U,V)W - P(R(X,Y)U,V)W - P(U,R(X,Y)V)W \tag{30}$$

$$-P(U,V)R(X,Y)W = 0.$$

Putting $U = W = Y = \xi$ in (30), we have

$$R(X,\xi)P(\xi,V)\xi - P(R(X,\xi)\xi,V)\xi - P(\xi,R(X,\xi)V)\xi - P(\xi,V)R(X,\xi)\xi = 0$$

which in view of (1), (13) and (14) reduces to

$$P(\xi, V)R(X, \xi)\xi = 0$$

which by using (14) gives

$$P(\xi, V)X = 0, \quad as \quad f^2 + f' \neq 0.$$

This implies that

$$R(\xi, V)X - \frac{1}{2}[S(V, X)\xi - S(\xi, X)V] = 0. \tag{31}$$

By virtue of (11), (13) and (16), (31) takes the form

$$(\frac{r}{2} + 3f^2 + 3f')[g(V, X)\xi - \eta(X)\eta(V)\xi] = 0. \tag{32}$$

Now by replacing X by ϕX, V by ϕV in (32) and using (5), we get

$$r = -6(f^2 + f'). \tag{33}$$

Using this value of r in (11), we obtain

$$S(Y, Z) = -2(f^2 + f')g(Y, Z). \tag{34}$$

Thus in view of (10), (33) and (34), we have the following:

Theorem 3. *In a 3-dimensional f-Kenmotsu manifold M, the following conditions are equivalent:*

(a) *projectively semisymmetric,*

(b) *the scalar curvature $r = -6(f^2 + f')$,*

(c) *the manifold M is of constant curvature,*

(d) *M is an Einstein manifold.*

6 3-dimensional f-Kenmotsu manifolds satisfying $P \cdot S = 0$

In this section, we study a 3-dimensional f-Kenmotsu manifold satisfying the condition $P \cdot S = 0$. Therefore we have

$$(P(X, Y) \cdot S)(U, V) = 0$$

for any vector fields X, Y, U and $V \in \chi(M)$. This implies that

$$S(P(X, Y)U, V) + S(U, P(X, Y)V) = 0. \tag{35}$$

Putting $U = \xi$ in (35), we have

$$S(P(X,Y)\xi, V) + S(\xi, P(X,Y)V) = 0$$

which by using the fact that $P(X,Y)\xi = 0$ reduces to

$$S(\xi, P(X,Y)V) = 0. \tag{36}$$

In view of (16), (36) becomes

$$g[R(X,Y)V,\xi] - \frac{1}{2}[S(Y,V)\eta(X) - S(X,V)\eta(Y)] = 0. \tag{37}$$

Taking $Y = \xi$ in (37) and using (3), (13) and (16), we obtain

$$S(Y,Z) = -2(f^2 + f')g(Y,Z).$$

Thus we have the following:

Theorem 4. *A 3-dimensional f-Kenmotsu manifold satisfying $P \cdot S = 0$ is an Einstein manifold.*

7 3-dimensional f-Kenmotsu manifolds satisfying $S \cdot R = 0$

In this section, we study a 3-dimensional f-Kenmotsu manifold satisfying the condition

$$(S(X,Y) \cdot R)(U,V)W = 0$$

for any vector fields X, Y, U, V and $W \in \chi(M)$. Therefore we have

$$(X_{\wedge S}Y)R(U,V)W + R((X_{\wedge S}Y)U,V)W + R(U,(X_{\wedge S}Y)V)W \tag{38}$$

$$+R(U,V)(X_{\wedge S}Y)W = 0,$$

where the endomorphism $X_{\wedge S}Y$ is defined by

$$(X_{\wedge S}Y)W = S(Y,W)X - S(X,W)Y. \tag{39}$$

Taking $Y = \xi$ in (38) and using (39), we have

$$2(f^2 + f')[\eta(R(U,V)W)X + \eta(U)R(X,V)W + \eta(V)R(U,X)W$$

$$+\eta(W)R(U,V)X] + S[X,R(U,V)W]\xi + S(X,U)R(\xi,V)W$$

$$+S(X,V)R(U,\xi)W + S(X,W)R(U,V)\xi = 0$$

which by taking inner product with ξ and using (4) takes the form

$$2(f^2 + f')[\eta(R(U,V)W)\eta(X) + \eta(U)\eta(R(X,V)W) + \eta(V)\eta(R(U,X)W) \tag{40}$$

$$+\eta(W)\eta(R(U,V)X)] + S[X, R(U,V)W] + S(X,U)\eta(R(\xi,V)W)$$
$$+S(X,V)\eta(R(U,\xi)W) + S(X,W)\eta(R(U,V)\xi) = 0.$$

Now taking $U = W = \xi$ in (40) and using (3), (12), (13) and (16), we get

$$S(X,V) = 2(f^2 + f')g(X,V) - 4(f^2 + f')\eta(X)\eta(V). \tag{41}$$

Contracting (41) over X and V, we obtain

$$r = 2(f^2 + f').$$

Thus we have the following:

Theorem 5. *A 3-dimensional f-Kenmotsu manifold satisfying $S \cdot R = 0$ is an η-Einstein manifold with the scalar curvature $2(f^2 + f')$.*

8 Ricci solitons in 3-dimensional f-Kenmotsu manifolds

Suppose that a 3-dimensional f-Kenmotsu manifold admits a Ricci soliton. Then

$$(\pounds_V g + 2S + 2\lambda g)(X,Y) = 0$$

which implies that

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \tag{42}$$

By using (7) in (42), we have

$$S(X,Y) + (\lambda + f)g(X,Y) - f\eta(X)\eta(Y) = 0. \tag{43}$$

Contracting (43) over X and Y yields

$$\lambda = -\frac{r + 2f}{3}. \tag{44}$$

Putting this value of λ in (43), we obtain

$$S(X,Y) = (\frac{r - f}{3})g(X,Y) + f\eta(X)\eta(Y). \tag{45}$$

Thus we can state the following:

Theorem 6. *If a 3-dimensional f-Kenmotsu manifold admits a Ricci soliton, then the manifold is an η-Einstein manifold and its Ricci soliton is shrinking, steady or expanding accordingly as $r + 2f > 0$, $r + 2f = 0$ or $r + 2f < 0$, respectively.*

Example of a 3-dimensional f-Kenmotsu manifold. We consider the 3-dimensional manifold $M = \{(x, y, z) \in R^3\}$, where (x, y, z) are the standard coordinates in R^3. Let e_1, e_2 and e_3 be the vector fields on M given by

$$e_1 = e^{-2z}\frac{\partial}{\partial x}, \quad e_2 = e^{-2z}\frac{\partial}{\partial y}, \quad e_3 = e^{-z}\frac{\partial}{\partial z} = \xi,$$

which are linearly independent at each point of M and hence form a basis of T_pM. Let g be the Riemannian metric defined by

$$g(e_1, e_1) = \quad g(e_2, e_2) = \quad g(e_3, e_3) = 1, \quad g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$

Let η be the 1-form on M defined as $\eta(X) = g(X, e_3) = g(X, \xi)$ for all $X \in \chi(M)$, and let ϕ be the $(1, 1)$ tensor field on M defined as

$$\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0.$$

By applying linearity of ϕ and g, we have

$$\eta(\xi) = g(\xi, \xi) = 1, \quad \phi^2 X = -X + \eta(X)\xi, \quad \eta(\phi X) = 0,$$

$$g(X, \xi) = \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all $X, Y \in \chi(M)$.
Now, by direct computations we obtain

$$[e_1, e_2] = 0, \quad [e_3, e_1] = -2e^{-z}e_1, \quad [e_2, e_3] = 2e^{-z}e_2.$$

The Riemannian connection ∇ of the metric tensor g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [Z, X]) \\ &\quad + g(Z, [X, Y]), \end{aligned}$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate

$$\nabla_{e_1} e_1 = -2e^{-z}e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 2e^{-z}e_1, \quad \nabla_{e_2} e_1 = 0,$$

$$\nabla_{e_2} e_2 = -2e^{-z}e_3, \quad \nabla_{e_2} e_3 = 2e^{-z}e_2, \quad \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

$$\text{Let} \qquad X = \sum_{i=1}^{3} X^i e_i = X^1 e_1 + X^2 e_2 + X^3 e_3 \in \chi(M).$$

It can be easily verified that the manifold satisfies

$$\nabla_X \xi = f[X - \eta(X)\xi] \quad \text{and} \quad (\nabla_X \phi)Y = f[g(\phi X, Y)\xi - \eta(Y)\phi X]$$

for $\xi = e_3$, where $f = 2e^{-z}$.
Hence we conclude that M is a 3-dimensional f-Kenmotsu manifold. Also $f^2 + f' \neq 0$. Hence M is a regular f-Kenmotsu manifold.

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