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ALMOST RICCI SOLITON AND GRADIENT ALMOST RICCI SOLITON ON 3-DIMENSIONAL LP-SASAKIAN MANIFOLDS

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Abstract

The object of the present paper is to study almost Ricci solitons and gradient almost Ricci solitons in 3-dimensional LP-Sasakian manifolds. We prove that if (g, V, λ) is an almost Ricci soliton on a 3-dimensional LP-Sasakian manifold M^3, then it reduces to a Ricci soliton and the soliton is shrinking for λ=2. Furthermore, if the scalar curvature is constant on M^3, then the potential vector field is Killing. Also, if the manifold admits a gradient almost Ricci soliton (f, ξ, λ), then the manifold is locally isometric to the unit sphere $S^n(1)$.

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1 Introduction

Ricci soliton equation on a Riemannian or pseudo-Riemannian manifold (M, g), (see Hamilton [11]) is defined by

$$\frac{1}{2}\pounds_V g + S = \lambda g, \tag{1}$$

where $\pounds_V$ is the Lie derivative operator along a vector field V, called potential vector field, λ is a real scalar and S is the Ricci tensor. Einstein manifolds satisfy the above equation, so that they are considered as trivial Ricci solitons. It will be called *shrinking*, *steady* or *expanding* according as $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. Otherwise, it will be called $indefinite$. When the vector field V is gradient of a smooth function $f : M^n \to \mathbb{R}$ then the manifold will be called gradient Ricci soliton. Ricci solitons and gradient Ricci solitons have been studied in Riemannian manifolds, Contact manifolds, Paracontact manifolds and Kähler

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manifolds by several authors. Recently, almost Ricci soliton was introduced by Pigola et. al. [17], where essentially they modified the definition of Ricci soliton by adding the condition on the parameter λ to be variable function in (1).

A general notion of Lorentzian para-Sasakian (briefly LP-Sasakian) manifold has been introduced by K. Matsumoto [12], in 1989 and several authors ([1], [13], [14], [18], [19]) have studied Lorentzian para-Sasakian manifolds. Ricci solitons for pseudo-Riemannian manifolds(in particular Lorentzian) have been studied by several authors such as ([7], [9], [15], [20]). Recently, Batat et. al [3] proved that Egorov spaces and ε-spaces have Lorentzian Ricci solitons. In a recent paper Blaga [4] studied η-Ricci solitons on Lorentzian para-Sasakian manifolds.

The object of the present paper is to study almost Ricci solitons and gradient almost Ricci solitons on 3-dimensional Lorentzian para-Sasakian manifolds. The paper is organized as follows: In section 2, we recall some fundamental formulas and properties of Lorentzian para-Sasakian manifolds. In section 3, we prove that if (g, V, λ) be an almost Ricci soliton on 3-dimensional Lorentzian para-Sasakian manifold M, then it reduces to Ricci soliton. Besides these in this section we prove that if the scalar curvature is constant on M, then the soliton is shrinking for λ=2 and the flow vector field is Killing. This section concludes with a interesting corollary. Finally in section 4, it is proved that if a 3-dimensional Lorentzian para-Sasakian manifold admits gradient almost Ricci soliton then the manifold is locally isometric to the unit sphere $S^n(1)$.

2 Preliminaries

Let M be an n-dimensional smooth manifold and ϕ, ξ, η are tensor fields on M of types (1,1), (1,0) and (0,1) respectively, such that

$$\eta(\xi) = -1, \quad \phi^2 = -I + \eta \otimes \xi. \tag{2}$$

The above equations imply that

$$\phi\xi = 0, \quad \eta \circ \phi = 0. \tag{3}$$

Then M admits a Lorentzian metric g of type (0,2) such that

$$g(X, \xi) = \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \tag{4}$$

for any vector fields X, Y. Then the structure (ϕ, ξ, η, g) is said to be Lorentzian almost para-contact structure. The manifold M equipped with a Lorentzian almost para-contact structure (ϕ, ξ, η, g) is said to be a Lorentzian almost para-contact manifold(briefly LAP-manifold).

If we denote $\Phi(X, Y) = g(X, \phi Y)$, then we have [12]

$$\Phi(X, Y) = g(X, \phi Y) = g(\phi X, Y) = \Phi(Y, X), \tag{5}$$

where X, Y are any vector fields.
An LAP-manifold M equipped with the structure (ϕ, ξ, η, g) is called a Lorentzian para-contact manifold(briefly LP-manifold) if

$$\Phi(X,Y) = \frac{1}{2}\{(\nabla_X \eta)Y + (\nabla_Y \eta)X\}, \tag{6}$$

where Φ is defined by (5) and ∇ denotes the covariant differentiation operator with respect to the Lorentzian metric g . A Lorentzian almost para-contact manifold M is called Lorentzian para-Sasakian manifold(briefly LP-Sasakian) if it satisfies

$$(\nabla_X \phi)Y = \eta(Y)X + g(X,Y)\xi + 2\eta(X)\eta(Y)\xi. \tag{7}$$

Also since the vector field η is closed in an LP-Sasakian manifold we have

$$(\nabla_X \eta)Y = \Phi(X,Y) = g(X, \phi Y), \quad \Phi(X, \xi) = 0, \quad \nabla_X \xi = \phi X. \tag{8}$$

Moreover, the eigen values of ϕ are -1, 0 and 1; and multiplicity of 0 is one. Let k and l be the multiplicities of -1 and 1 respectively. Then $trace(\phi) = l - k$. So, if $(trace(\phi))^2 = (n-1)$, then either l=0 or k=0. In this case we call the structure a trivial LP-Sasakian structure.

Also in an LP-Sasakian manifold, the following relations hold ([1], [12], [19]):

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{9}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{10}$$

$$R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X, \tag{11}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{12}$$

$$\nabla_\xi \eta = 0, \tag{13}$$

for any vector fields X, Y, Z where R is the Riemannian curvature tensor, S is the Ricci tensor and ∇ is the Levi-Civita connection associated to the metric g.

Throughout this paper we assume that $trace(\phi) \neq 0$, i.e., ξ is not harmonic.

3 Almost Ricci soliton

The well-known Riemannain curvature tensor of a three dimensional Riemannian manifold is given by

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &\quad -\frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{14}$$

for any vector fields X, Y, Z where Q is the Ricci operator, i.e., $g(QX, Y) = S(X, Y)$ and r is the scalar curvature of the manifold. Replacing $Y{=}Z{=}\xi$ in the above equation and using (10) and (12) we obtain(see [19])

$$QX = \frac{1}{2}[(r-2)X + (r-6)\eta(X)\xi]. \tag{15}$$

In view of (15) the Ricci tensor is written as

$$S(X, Y) = \frac{1}{2}[(r-2)g(X, Y) + (r-6)\eta(X)\eta(Y)]. \tag{16}$$

Using (15) and (16) in (14), we deduce

$$\begin{aligned} R(X, Y)Z &= \frac{(r-4)}{2}\{g(Y, Z)X - g(X, Z)Y\} \\ &+ \frac{(r-6)}{2}\{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}. \end{aligned} \tag{17}$$

Now before introducing the detailed proof of our main theorem, we first prove the following result:

Lemma 3.1. *Let $M(\phi, \xi, \eta, g)$ be a 3-dimensional LP-Sasakian manifold. Then we have*

$$\xi r = -(r-6)trace(\phi) \tag{18}$$

where r denotes the scalar curvature of M.

Proof: The equation (15) can be rewritten as:

$$QY = \frac{1}{2}[(r-2)Y + (r-6)\eta(Y)\xi].$$

Taking covariant derivative of the above equation with respect to an arbitrary vector field X and recalling (8) we write

$$\begin{aligned} (\nabla_X Q)Y &= \frac{(Xr)}{2}Y + \frac{(Xr)}{2}\eta(Y)\xi + \frac{(r-6)}{2}g(X, \phi Y)\xi \\ &+ \frac{(r-6)}{2}\eta(Y)\phi X. \end{aligned} \tag{19}$$

Taking inner product with respect to an arbitrary vector field Z in the above equation, we have

$$\begin{aligned} g((\nabla_X Q)Y, Z) &= \frac{(Xr)}{2}g(Y, Z) + \frac{(Xr)}{2}\eta(Y)\eta(Z) + \frac{(r-6)}{2}g(X, \phi Y)\eta(Z) \\ & \frac{(r-6)}{2}\eta(Y)g(\phi X, Z). \end{aligned} \tag{20}$$

Putting $X = Z = e_i$ (where $\{e_i\}$ is an orthonormal basis for the tangent space of M and taking $\sum_i$, $1 \leq i \leq 3$) in the above equation and using the well-known formula of Riemannian manifolds $divQ = \frac{1}{2} grad\ r$, we obtain

$$(\xi r)\eta(Y) = -(r-6)\eta(Y)trace(\phi). \tag{21}$$

Substituting $Y = \xi$ in the above equation we have the required result. This completes the proof.

We consider a 3-dimensional LP-Sasakian manifold M admitting an almost Ricci soliton defined by(1). Using (16) in (1) we write

$$(\pounds_V g)(Y,Z) = (2\lambda - r + 2)g(Y,Z) - (r-6)\eta(Y)\eta(Z). \tag{22}$$

Differentiating the above equation with respect to X and making use (8) we obtain

$$\begin{aligned}(\nabla_X \pounds_V g)(Y,Z) &= [2(X\lambda) - (Xr)]g(Y,Z) - (Xr)\eta(Y)\eta(Z) \\ &\quad -(r-6)\{g(X,\phi Y)\eta(Z) + \eta(Y)g(X,\phi Z)\}. \end{aligned} \tag{23}$$

Now we recall the following well-known formula(Yano [21]):

$$(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z) = -g((\pounds_V \nabla)(X,Y),Z) - g((\pounds_V \nabla)(X,Z),Y),$$

for any vector fields X, Y, Z on M. From this we can easily deduce:

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(X,Z),Y). \tag{24}$$

Since $\pounds_V \nabla$ is symmetric tensor of type (1,2), it follows from (24) that

$$\begin{aligned} &g((\pounds_V \nabla)(X,Y),Z) \\ &= \frac{1}{2}(\nabla_X \pounds_V g)(Y,Z) + \frac{1}{2}(\nabla_Y \pounds_V g)(X,Z) - \frac{1}{2}(\nabla_Z \pounds_V g)(X,Y). \end{aligned} \tag{25}$$

Using (23) in (25) we get

$$\begin{aligned} 2g((\pounds_V \nabla)(X,Y),Z) &= [2(X\lambda) - (Xr)]g(Y,Z) - (Xr)\eta(Y)\eta(Z) \\ &\quad +[2(Y\lambda) - (Yr)]g(X,Z) - (Yr)\eta(X)\eta(Z) \\ &\quad -[2(Z\lambda) - (Zr)]g(X,Y) + (Zr)\eta(X)\eta(Y) \\ &\quad -2(r-6)g(X,\phi Y)\eta(Z). \end{aligned} \tag{26}$$

After substituting $X = Y = e_i$ in the above equation and removing Z from both sides, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking $\sum_i$, $1 \leq i \leq 3$, we have

$$(\pounds_V \nabla)(e_i, e_i) = -D\lambda - (\xi r)\xi - 2(r-6)trace(\phi)\xi, \tag{27}$$

where $X\alpha = g(D\alpha, X)$, D denotes the gradient operator with respect to g.
Now differentiating(1) and using it in (24) we can easily determine

$$g((\pounds_V \nabla)(X,Y),Z) = (\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z). \tag{28}$$

Taking $X = Y = e_i$ (where $\{e_i\}$ is an orthonormal frame) in (28) and summing over i we obtain

$$(\pounds_V \nabla)(e_i, e_i) = 0, \tag{29}$$

for all vector fields Z. Associating (27) and (29) yields

$$D\lambda + (\xi r)\xi + 2(r-6)trace(\phi)\xi = 0. \tag{30}$$

Using (18) in the above equation we obtain

$$D\lambda = 0. \tag{31}$$

This implies that λ is constant. This leads to the the following theorem:

Theorem 1. *An almost Ricci soliton on 3-dimensional LP-Sasakian manifolds reduces to Ricci soliton.*

Following the above theorem and removing Z from both sides of (26) yields

$$\begin{aligned} 2(\pounds_V \nabla)(X,Y) &= -(Xr)Y - (Xr)\eta(Y)\xi - (Yr)X - (Yr)\eta(X)\xi \\ &\quad + g(X,Y)Dr + \eta(X)\eta(Y)Dr - 2(r-6)g(X,\phi Y)\xi. \end{aligned} \tag{32}$$

Setting $Y = \xi$ in the above equation and using (18) we obtain

$$2(\pounds_V \nabla)(X,\xi) = (r-6)trace(\phi)(X + \eta(X)\xi). \tag{33}$$

Taking covariant derivative of (33) along an arbitrary vector field Y we get

$$\begin{aligned} 2(\nabla_Y \pounds_V \nabla)(X,\xi) &+ 2(\pounds_V \nabla)(X,\phi Y) = (Yr)trace(\phi)(X + \eta(X)\xi) \\ &+ (r-6)trace(\phi)\{(\nabla_Y \eta)(X)\xi + \eta(X)\phi Y\}. \end{aligned} \tag{34}$$

If, we apply the following formula:

$$(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z)$$

in the above equation we have

$$\begin{aligned} 2(\pounds_V R)(X,Y)\xi &= (Xr)trace(\phi)(Y + \eta(Y)\xi) - (Yr)trace(\phi)(X + \eta(X)\xi) \\ &\quad + (r-6)trace(\phi)\{\eta(Y)\phi X - \eta(X)\phi Y\}. \end{aligned} \tag{35}$$

Taking Lie derivative of (10) along V and using (22) we obtain

$$\begin{aligned} (\pounds_V R)(X,Y)\xi + R(X,Y)\pounds_V \xi &= (2\lambda - 4)\{\eta(Y)X - \eta(X)Y\} \\ &\quad + g(Y, \pounds_V \xi)X - g(X, \pounds_V \xi)Y. \end{aligned} \tag{36}$$

Now combining(35) with (36) and contracting over X we write

$$\begin{aligned} &trace(\phi)\{(Yr) + (\xi r)\eta(Y)\} - 2trace(\phi)(Yr) \\ &+ (trace(\phi))^2(r-6)\eta(Y) + 2S(Y, \pounds_V \xi) \\ &= 8(\lambda - 2)\eta(Y) + 4g(Y, \pounds_V \xi). \end{aligned} \tag{37}$$

Putting $Y = \xi$ in (37) and making use of (18) we have

$$-3(r-6)(trace(\phi))^2 = 8(\lambda - 2). \tag{38}$$

If r=constant, then from Lemma(3.1) we obtain r=6.
Using $r = 6$ in (38) we have $\lambda = 2$. Thus we can state the following:

Theorem 2. *If a 3-dimensional LP-Sasakian manifold M admitting almost Ricci solitons has constant scalar curvature, then the soliton is shrinking for $\lambda = 2$.*

Moreover, using r=6 and $\lambda = 2$ in (22) we get $(\pounds_V g)(Y, Z) = 0$ which implies that the potential vector field V is a Killing vector field. Also putting the value $r = 6$ in (17) we find that the manifold is of constant curvature 1. Consequently the space is locally isometric to the unit Sphere $S^n(1)$ (see O'Neill [16]).

As V is KIlling, we also conclude that $\pounds_V \xi = 0$. Finally, Lie-differentiating the equation $\eta(X) = g(X, \xi)$ along V and since Lie-derivation commutes with exterior derivation, we conclude $\pounds_V \phi = 0$. Thus, V is an infinitesimal automorphism of the contact metric structure on M. Hence we can state the following:

Corollary 3.1. *If a 3-dimensional LP-Sasakian manifold M admitting almost Ricci solitons has constant scalar curvature, then the flow vector V is Killing and also V is an infinitesimal automorphism of the contact metric structure on M. Moreover the manifold is locally isometric to the unit Sphere $S^n(1)$.*

4 Gradient almost Ricci soliton

If the vector field V is the gradient of a potential function $-f$, then g is called a gradient almost Ricci soliton. Then (1) takes the form

$$\nabla\nabla f + S = \lambda g.$$

This reduces to

$$\nabla_Y Df = -QY + \lambda Y. \tag{39}$$

where D denotes the gradient operator of g.
Differentiating (39) covariantly in the direction of X yields

$$\nabla_X \nabla_Y Df = -\nabla_X QY + (X\lambda)Y + \lambda \nabla_X Y. \tag{40}$$

Similarly we get

$$\nabla_Y \nabla_X Df = -\nabla_Y QX + (Y\lambda)X + \lambda \nabla_Y X, \tag{41}$$

and

$$\nabla_{[X,Y]} Df = -Q[X, Y] + \lambda[X, Y]. \tag{42}$$

In view of (40),(41) and (42) we have

$$R(X,Y)Df = \nabla_X\nabla_Y Df - \nabla_Y\nabla_X Df - \nabla_{[X,Y]}Df$$
$$= -(\nabla_X Q)Y + (\nabla_Y Q)X + (X\lambda)Y - (Y\lambda)X. \quad (43)$$

In view of (19) we obtain

$$R(X,Y)Df = \frac{(Yr)}{2}X - \frac{(Xr)}{2}Y + \frac{(Yr)}{2}\eta(X)\xi - \frac{(Xr)}{2}\eta(Y)\xi + \frac{(r-6)}{2}\eta(X)\phi Y - \frac{(r-6)}{2}\eta(Y)\phi X + (X\lambda)Y - (Y\lambda)X. \quad (44)$$

This reduces to

$$g(R(X,Y)\xi, Df) = (Y\lambda)\eta(X) - (X\lambda)\eta(Y). \quad (45)$$

Using (10) in the above equation we obtain

$$\eta(Y)(Xf) - \eta(X)(Yf) = (Y\lambda)\eta(X) - (X\lambda)\eta(Y). \quad (46)$$

Putting $Y = \xi$ in (46) we have

$$d(f + \lambda) = -\xi(f + \lambda)\eta. \quad (47)$$

Operating (47) by d and using Poincare lemma $d^2 \equiv 0$, we obtain

$$d[\xi(f + \lambda)]\eta \wedge d\eta = 0. \quad (48)$$

Since in a 3-dimensional LP-Sasakian manifold $\eta \wedge d\eta \neq 0$, we have

$$f + \lambda = constant. \quad (49)$$

Now contracting Y in (44) and using (18) we obtain

$$S(X, Df) = \frac{1}{2}(Xr) - 2(X\lambda). \quad (50)$$

Comparing (16) and (50) we have

$$\frac{1}{2}(Xr) - 2(X\lambda) = \frac{(r-2)}{2}(Xf) + \frac{(r-6)}{2}\eta(X)(\xi f). \quad (51)$$

Substituting $X = \xi$ and using (18) in (51) we obtain

$$d(f + \lambda) = \frac{(r-6)}{4}trace(\phi)\eta. \quad (52)$$

In view of (49) and (52) we get r=6. Moreover, using r=6 in (17) we easily find that the manifold is of constant curvature 1. Consequently the space is locally isometric to the unit sphere $S^n(1)$. Hence we can state the following:

Theorem 3. *If a 3-dimensional LP-Sasakian manifold admits a gradient almost Ricci soliton (f, ξ, λ), then the manifold is locally isometric to the unit sphere $S^n(1)$.*

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