Bulletin of the *Transilvania* University of Braşov • Vol 11(60), No. 2 - 2018
Series III: Mathematics, Informatics, Physics, 89-98

ON THE SOLUTIONS OF A CAPUTO-KATUGAMPOLA FRACTIONALl INTEGRO-DIFFERENTIAL INCLUSION

Aurelian CERNEA[1]

Communicated to:
International Conference on Mathematics and Computer Science ,
June 14-16, 2018, Braşov, Romania, 3rd Edition - MACOS 2018

Abstract

We consider a Cauchy problem associated to a integro-differential inclusion of fractional order defined by Caputo-Katugampola derivative and by a set-valued map with nonconvex values and we prove that the set of selections corresponding to the solutions of the problem considered is a retract of the space of integrable functions on unbounded interval.

2000 *Mathematics Subject Classification*: 34A60, 26A33, 34B15.
Key words: differential inclusion, fractional derivative, initial value problem.

1 Introduction

In the last years one may see a strong development of the theory of differential equations and inclusions of fractional order ([2, 7, 9, 10, 11] etc.). The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena.

Recently, a generalized Caputo-Katugampola fractional derivative was proposed in [8] by Katugampola and further he proved the existence of solutions for fractional differential equations defined by this derivative. This Caputo-Katugampola fractional derivative extends the well known Caputo and Caputo-Hadamard fractional derivatives. Also, in some recent papers [1, 12], several qualitative properties of solutions of fractional differential equations defined by Caputo-Katugampola derivative were obtained.

This paper is devoted to the following Cauchy problem

$$D_c^{\alpha,\rho}x(t) \in F(t,x(t),V(x)(t)) \quad a.e.\ ([0,\infty)), \quad x(0)=x_0, \tag{1}$$

[1]Faculty of Mathematics and Computer Science, University of Bucharest, Academiei 14, 010014 Bucharest, Romania, e-mail: acernea@fmi.unibuc.ro

where $\alpha \in (0,1]$, $\rho > 0$, $D_c^{\alpha,\rho}$ is the Caputo-Katugampola fractional derivative, $F : [0,\infty) \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map, $V : C([0,\infty),\mathbf{R}) \to C([0,\infty),\mathbf{R})$ is a nonlinear Volterra integral operator defined by $V(x)(t) = \int_0^t k(t,s,x(s))ds$ with $k(.,.,.) : [0,\infty) \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ a given function and $x_0 \in \mathbf{R}$.

The aim of the present paper is to prove that the set of selections of the multifunction F that correspond to the solutions of problem (1) is a retract of $L^1_{loc}([0,\infty),\mathbf{R})$. Our main hypothesis is that the multifunction is Lipschitz with respect to the second and third variable and the proof uses a well known selection theorem due to Bressan and Colombo ([3]) which gives continuous selections for multifunctions that are lower semicontinuous and with decomposable values.

We note that a similar result for a fractional differential inclusion defined by the classical Caputo fractional derivative may be found in our previous paper [4]. Afterwards, this result was generalized to fractional integro-differential inclusions defined by the same Caputo derivative in [6]. The present paper extends and unifies all these results in the case of the more general problem (1).

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our result.

2 Preliminaries

In what follows $I \subset \mathbf{R}$ is a given interval, $\mathcal{L}(I)$ is the σ-algebra of all Lebesgue measurable subsets of I and $(X,|.|)$ is a real separable Banach. $C(I,X)$ denotes the space of continuous functions $x : I \to X$ with the norm $|x|_C = \sup_{t\in I}|x(t)|$ and $L^1(I,X)$ denotes the space of integrable functions $x : I \to X$ with the norm $|x|_1 = \int_0^T |x(t)|dt$.

The distance between a point $x \in X$ and a subset $A \subset X$ is defined by $d(x,A) = \inf\{|x-a|; a \in A\}$ and Pompeiu-Hausdorff distance between the closed subsets $A, B \subset X$ is defined by $d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}$, $d^*(A,B) = \sup\{d(a,B);\ a \in A\}$.

$\mathcal{P}(X)$ denotes the family of all nonempty subsets of X and with $\mathcal{B}(X)$ the family of all Borel subsets of X. For $A \subset I$ with $\chi_A(.) : I \to \{0,1\}$ we describe the characteristic function of A. Finally, for any $A \subset X$ $cl(A)$ is its closure.

By definition a subset $D \subset L^1(I,X)$ is *decomposable* if for any $u, v \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I\backslash A$.

We use the notation $\mathcal{D}(I,X)$ for the family of all decomposable closed subsets of $L^1(I,X)$.

In the next two results (S,d) is a separable metric space. By definition a set-valued map $H : S \to \mathcal{P}(X)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $G \subset X$, the subset $\{s \in S;\ H(s) \subset G\}$ is closed. The next two lemmas are proved in [3].

Lemma 1. *Consider $F^* : I \times S \to \mathcal{P}(X)$ a set-valued map with closed values, $\mathcal{L}(I) \otimes \mathcal{B}(S)$-measurable and $F^*(t,.)$ is l.s.c. for any $t \in I$.*

Then the set-valued map $H : S \to \mathcal{D}(I,X)$ defined by

$$H(s) = \{f \in L^1(I,X); \quad f(t) \in F^*(t,s) \quad a.e.\ (I)\}$$

is l.s.c. with nonempty closed values if and only if there exists a continuous mapping $q : S \to L^1(I,X)$ such that

$$d(0, F^*(t,s)) \leq q(s)(t) \quad a.e.\ (I),\ \forall s \in S.$$

Lemma 2. *Let $F : S \to \mathcal{D}(I,X)$ be a l.s.c. set-valued map with closed decomposable values and let $\psi : S \to L^1(I,X)$, $\phi : S \to L^1(I,\mathbf{R})$ be continuous mappings such that the set-valued map $H : S \to \mathcal{D}(I,X)$ given by*

$$H(s) = cl\{f(.) \in F(s); \quad |f(t) - \psi(s)(t)| < \phi(s)(t) \quad a.e.\ (I)\}$$

has nonempty values.

Then H admits a continuous selection, i.e. there exists $h : S \to L^1(I,X)$ continuous with $h(s) \in H(s) \quad \forall s \in S$.

Let $\rho > 0$.

Definition 1. ([8]) *a) The generalized left-sided fractional integral of order $\alpha > 0$ of a Lebesgue integrable function $f : (0,\infty) \to \mathbf{R}$ is defined by*

$$I^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s) ds, \tag{2}$$

provided the right-hand side is pointwise defined on $(0,\infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

b) The generalized fractional derivative, corresponding to the generalized left-sided fractional integral in (2) of a function $f : [0,\infty) \to \mathbf{R}$ is defined by

$$D^{\alpha,\rho} f(t) = (t^{1-\rho} \frac{d}{dt})^n (I^{n-\alpha,\rho})(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} (t^{1-\rho} \frac{d}{dt})^n \int_0^t \frac{s^{\rho-1} f(s)}{(t^\rho - s^\rho)^{\alpha-n+1}} ds$$

if the integral exists and $n = [\alpha]$.

c) The Caputo-Katugampola generalized fractional derivative is defined by

$$D_c^{\alpha,\rho} f(t) = (D^{\alpha,\rho}[f(s) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} s^k])(t)$$

We note that if $\rho = 1$, the Caputo-Katugampola fractional derivative becames the well known Caputo fractional derivative. On the other hand, passing to the limit with $\rho \to 0+$, the above definition yields the Hadamard fractional derivative.

In what follows $\rho > 0$ and $\alpha \in [0,1]$

Lemma 3. *For a given integrable function $f(.) : [0,T] \to \mathbf{R}$, the unique solution of the initial value problem*

$$D_c^{\alpha,\rho} x(t) = f(t) \quad a.e.\ ([0,T]), \quad x(0) = x_0,$$

is given by

$$x(t) = x_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s) ds$$

For the proof of Lemma 3, see [8]; namely, Lemma 4.2.

Definition 2. *A function $x \in C([0,\infty),\mathbf{R})$ is called a solution of problem (1) if there exists a function $f \in L^1_{loc}([0,\infty),\mathbf{R})$ with $f(t) \in F(t,x(t),V(x)(t))$ a.e. ($[0,\infty)$) such that $D_c^{\alpha,\rho}x(t) = f(t)$ a.e. ($[0,\infty)$) and $x(0) = x_0$.*

In this case $(x(.), f(.))$ is called a *trajectory-selection* pair of problem (1).
Next we shall use the following notations.

$$\tilde{f}(t) = x_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t^\rho - s^\rho)^{\alpha-1}s^{\rho-1}f(s)ds, \quad f \in L^1_{loc}([0,\infty),\mathbf{R})$$
$$\mathcal{T}(x_0) = \{f \in L^1_{loc}([0,\infty),\mathbf{R}); \quad f(t) \in F(t,\tilde{f}(t),V(\tilde{f})(t)) \quad a.e.\ [0,\infty)\}.$$

3 The result

In order to prove our result we need the following assumptions.

Hypothesis. i) *The set-valued map $F(.,.) : [0,\infty)\times\mathbf{R}\times\mathbf{R} \to \mathcal{P}(\mathbf{R})$ is $\mathcal{L}([0,\infty))\otimes\mathcal{B}(\mathbf{R}\times\mathbf{R})$ measurable and has nonempty closed values.*

ii) *For almost all $t \in I$, the set-valued map $F(t,.,.)$ is $L(t)$-Lipschitz in the sense that there exists $L(.) \in L^1_{loc}([0,\infty),\mathbf{R}_+)$ with*

$$d_H(F(t,x_1,y_1),F(t,x_2,y_2)) \leq L(t)(|x_1-x_2|+|y_1-y_2|) \quad \forall\, x_1,x_2,y_1,y_2 \in \mathbf{R}.$$

iii) *There exists a locally integrable function $q(.) \in L^1_{loc}([0,\infty),\mathbf{R})$ such that*

$$d_H(\{0\},F(t,0,V(0)(t))) \leq q(t) \quad a.e.\ ([0,\infty)).$$

iv) $k(.,.,.) : [0,\infty)\times\mathbf{R}\times\mathbf{R} \to \mathbf{R}$ *is a function such that* $\forall x \in \mathbf{R}$, $(t,s) \to k(t,s,x)$ *is measurable.*

v) $|k(t,s,x) - k(t,s,y)| \leq L(t)|x-y|$ *a.e.* $(t,s) \in [0,\infty)\times[0,\infty)$, $\forall\, x,y \in \mathbf{R}$.

We use next the following notations

$$M(t) := L(t)(1+\int_0^t L(u)du),\ t \in I, \quad I^{\alpha,\rho}M := \sup_{t\in[0,\infty)}|I^{\alpha,\rho}M(t)|. \tag{3}$$

$$q_0(h)(t) = |h(t)| + q(t) + L(t)|(|\tilde{h}(t)| + \int_0^t L(s)|\tilde{h}(s)|ds), \quad t \in I \tag{4}$$

Let us note that

$$d(h(t),F(t,\tilde{h}(t),V(\tilde{h})(t)) \leq q_0(h)(t) \quad a.e.\ (I) \tag{5}$$

and for any $u_1,u_2 \in L^1(I,\mathbf{R})$

$$|q_0(h_1) - q_0(h_2)|_1 \leq (1+|I^{\alpha,\rho}M(T)|)|h_1-h_2|_1;$$

therefore, the mapping $q_0 : L^1(I,\mathbf{R}) \to L^1(I,\mathbf{R})$ is continuous.

Also define

$$\mathcal{T}_I(x_0)=\{h\in L^1(I,\mathbf{R});\quad h(t)\in F(t,\tilde{h}(t),V(\tilde{h})(t))\quad a.e.\ (I)\}.$$

$$I_k=[0,k],\quad k\geq 1,\quad |h|_{1,k}=\int_0^k|h(t)|dt,\quad h\in L^1(I_k,\mathbf{R}).$$

The proof of the next result may be found in [4].

Lemma 4. *Suppose that Hypothesis is verified and consider $\phi : L^1(I,\mathbf{R})\to L^1(I,\mathbf{R})$ a continuous function with $\phi(h)=h$ for all $h\in\mathcal{T}_I(x_0)$. If $h\in L^1(I,\mathbf{R})$, we put*

$$\Psi(h)=\{h\in L^1(I,\mathbf{R});\quad h(t)\in F(t,\widetilde{\phi(h)}(t),V(\widetilde{\phi(h)})(t))\quad a.e.\ (I)\},$$

$$\Phi(h)=\begin{cases}\{h\} & \text{if } h\in\mathcal{T}_I(x_0),\\ \Psi(h) & \text{otherwise.}\end{cases}$$

Then the set-valued map $\Phi : L^1(I,\mathbf{R})\to\mathcal{P}(L^1(I,\mathbf{R}))$ is l.s.c. with nonempty closed and decomposable values.

Theorem 1. *Assume that Hypothesis is satisfied, $I^{\alpha,\rho}M<1$ and $x_0\in\mathbf{R}$.*
Then there exists $G : L^1_{loc}([0,\infty),\mathbf{R})\to L^1_{loc}([0,\infty),\mathbf{R})$ continuous with the properties
(i) $G(h)\in\mathcal{T}(x_0),\quad\forall h\in L^1_{loc}([0,\infty),\mathbf{R})$,
(ii) $G(h)=h,\quad\forall h\in\mathcal{T}(x_0)$.

Proof. The idea of the proof consists in the construction, for every $k\geq 1$, of a sequence of continuous functions $g^k : L^1(I_k,\mathbf{R})\to L^1(I_k,\mathbf{R})$ satisfying the following conditions
(I) $g^k(h)=h,\quad\forall h\in\mathcal{T}_{I_k}(x_0)$
(II) $g^k(h)\in\mathcal{T}_{I_k}(x_0),\quad\forall h\in L^1(I_k,\mathbf{R})$
(III) $g^k(h)(t)=g^{k-1}(h|_{I_{k-1}})(t),\quad t\in I_{k-1}$
If this construction is realized, we introduce $G : L^1_{loc}([0,\infty),\mathbf{R})\to L^1_{loc}([0,\infty),\mathbf{R})$ with

$$G(h)(t)=g^k(h|_{I_k})(t),\quad k\geq 1.$$

The continuity of $g^k(.)$ and (III) allows to deduce that $G(.)$ is continuous. Taking into account (II), for each $h\in L^1_{loc}([0,\infty),\mathbf{R})$, we get

$$G(h)|_{I_k}(t)=g^k(h|_{I_k})(t)\in\mathcal{T}_{I_k}(x_0),\quad\forall k\geq 1,$$

which shows that $G(h)\in\mathcal{T}(x_0)$.

Consider $\varepsilon>0$ and $m\geq 0$. We define $\varepsilon_m=\frac{m+1}{m+2}\varepsilon$. If $h\in L^1(I_1,\mathbf{R})$ and $m\geq 0$ we put

$$q_0^1(h)(t)=|h(t)|+q(t)+L(t)(|\tilde{h}(t)|+\int_0^t L(s)|\tilde{h}(s)|ds),\ \ t\in I_1$$

and

$$q_{m+1}^1(h)=(I^{\alpha,\rho}M)^m(\frac{1}{\rho^\alpha\Gamma(\alpha+1)}|q_0^1(h)|_{1,1}+\varepsilon_{m+1}).$$

Since the map $q_0^1(.) = q_0(.)$ is continuous, we find that $q_m^1 : L^1(I_1, \mathbf{R}) \to L^1(I_1, \mathbf{R})$ is also continuous.

Set $g_0^1(h) = h$. In what follows, we show that for any $m \geq 1$ there exists $g_m^1 : L^1(I_1, \mathbf{R}) \to L^1(I_1, \mathbf{R})$ continuous with the properties

$$g_m^1(h) = h, \quad \forall h \in \mathcal{T}_{I_1}(x_0), \tag{a_1}$$

$$g_m^1(h)(t) \in F(t, \widetilde{g_{m-1}^1(h)}(t), V(\widetilde{g_{m-1}^1(h)})(t)) \quad a.e.\ (I_1), \tag{b_1}$$

$$|g_1^1(h)(t) - g_0^1(h)(t)| \leq q_0^1(h)(t) + \varepsilon_0 \quad a.e.\ (I_1), \tag{c_1}$$

$$|g_m^1(h)(t) - g_{m-1}^1(h)(t)| \leq M(t) q_{m-1}^1(h) \quad a.e.\ (I_1), \quad m \geq 2. \tag{d_1}$$

If $h \in L^1(I_1, \mathbf{R})$, we define

$$\Psi_1^1(h) = \{f \in L^1(I_1, \mathbf{R});\ \ f(t) \in F(t, \widetilde{h}(t), V(\widetilde{h}(t))(t))\ \ a.e.(I_1)\},$$

$$\Phi_1^1(h) = \begin{cases} \{h\} & \text{if } h \in \mathcal{T}_{I_1}(x_0), \\ \Psi_1^1(h) & \text{otherwise.} \end{cases}$$

We apply Lemma 4 (with $\phi(h) = h$) and we deduce that $\Phi_1^1 : L^1(I_1, \mathbf{R}) \to \mathcal{D}(I_1, \mathbf{R})$ is l. s. c. Using (5) we obtain that the set

$$H_1^1(h) = cl\{f \in \Phi_1^1(u); \quad |f(t) - h(t)| < q_0^1(h)(t) + \varepsilon_0 \quad a.e.\ (I_1)\}$$

is not empty for any $h \in L^1(I_1, \mathbf{R})$. We apply Lemma 2 to obtain a selection g_1^1 of H_1^1 which is continuous and verifies (a_1)-(c_1).

Assume that $g_i^1(.),\ i = 1, \ldots m$ satisfying (a_1)-(d_1) are already constructed. Therefore, from Hypothesis 1 and (b_1), (d_1) we infer

$$\begin{gathered} d(g_m^1(h)(t), F(t, \widetilde{g_m^1(h)}(t), V(\widetilde{g_m^1(h)})(t)) \leq L(t)(|\widetilde{g_{m-1}^1(h)}(t) - \widetilde{g_m^1(h)}(t)| + \\ \int_0^t L(s)|\widetilde{g_{m-1}^1(h)}(s) - \widetilde{g_m^1(h)}(s)|ds) \leq M(t)(I^{\alpha,\rho}M) q_m^1(h) = M(t)(q_{m+1}^1(h) - \\ s_m) < M(t) q_{m+1}^1(h), \end{gathered} \tag{6}$$

where $s_m := (I^{\alpha,\rho}M)^m(\varepsilon_{m+1} - \varepsilon_m) > 0$.

For $h \in L^1(I_1, \mathbf{R})$, we put

$$\Psi_{m+1}^1(h) = \{f \in L^1(I_1, \mathbf{R});\ \ f(t) \in F(t, \widetilde{g_m^1(h)}(t), V(\widetilde{g_m^1(h)})(t)) \quad a.e.\ (I_1)\},$$

$$\Phi_{m+1}^1(h) = \begin{cases} \{h\} & \text{if } h \in \mathcal{T}_{I_1}(x_0), \\ \Psi_{m+1}^1(h) & \text{otherwise.} \end{cases}$$

Again, Lemma 4 (applied for $\phi(h) = g_m^1(h)$) allows to conclude that $\Phi_{m+1}^1(.)$ is l.s.c. with nonempty closed decomposable values. At the same time, from (6), if $h \in L^1(I_1, \mathbf{R})$, the set

$$H_{m+1}^1(h) = cl\{f \in \Phi_{m+1}^1(h); \quad |f(t) - g_{m+1}^1(h)(t)| < M(t) q_{m+1}^1(h) \quad a.e.\ (I_1)\}$$

is nonempty. As above, via Lemma 2, we obtain a selection g^1_{m+1} of H^1_{m+1} continuous with (a_1)-(d_1).

We conclude that

$$|g^1_{m+1}(h)-g^1_m(h)|_{1,1}\leq (I^{\alpha,\rho}M)^m(\frac{1}{\rho^\alpha\Gamma(\alpha+1)}|q^1_0(h)|_{1,1}+\varepsilon)$$

which means that the sequence $\{g^1_m(h)\}_{m\in\mathbf{N}}$ is a Cauchy sequence in the Banach space $L^1(I_1,\mathbf{R})$. Take $g^1(h)\in L^1(I_1,\mathbf{R})$ its limit. Since the mapping $s\to|q^1_0(h)|_{1,1}$ is continuous, thus it is locally bounded and the Cauchy condition is satisfied by $\{g^1_m(h)\}_{m\in\mathbf{N}}$ locally uniformly with respect to h. Therefore, $g^1(.):L^1(I_1,\mathbf{R})\to L^1(I_1,\mathbf{R})$ is continuous.

Taking into account (a_1) we find that $g^1(h)=h,\quad\forall h\in\mathcal{T}_{I_1}(x_0)$ and from the hypotheses that the values of F are closed and (b_1) we find that

$$g^1(h)(t)\in F(t,\widetilde{g^1(h)}(t),V(\widetilde{g^1(h)})(t)),\quad a.e.\,(I_1)\quad\forall h\in L^1(I_1,\mathbf{R}).$$

At the final step of the induction procedure we assume that $g^i(.):L^1(I_i,\mathbf{R})\to L^1(I_i,\mathbf{R})$, $i=2,...,k-1$ are constructed and satisfying (I)-(III) and we construct $g^k(.):L^1(I_k,\mathbf{R})\to L^1(I_k,\mathbf{R})$ continuous with (I)-(III).

We introduce the map $g^k_0:L^1(I_k,\mathbf{R})\to L^1(I_k,\mathbf{R})$

$$g^k_0(h)(t)=g^{k-1}(h|_{I_{k-1}})(t)\chi_{I_{k-1}}+h(t)\chi_{I_k\setminus I_{k-1}}(t)\tag{7}$$

Since $g^{k-1}(.)$ is continuous and for $h_0,h\in L^1(I_k,\mathbf{R})$ we have

$$|g^k_0(h)-g^k_0(h_0)|_{1,k}\leq|g^{k-1}(h|_{I_{k-1}})-g^{k-1}(h_0|_{I_{k-1}})|_{1,k-1}+\int_{k-1}^k|h(t)-h_0(t)|dt,$$

and we deduce that $g^k_0(.)$ is continuous.

At the same time, the equality $g^{k-1}(h)=h$, $\forall h\in\mathcal{T}_{I_{k-1}}(x_0)$ and (7) allows to obtain

$$g^k_0(h)=h,\quad\forall h\in\mathcal{T}_{I_k}(x_0).$$

For $h\in L^1(I_k,\mathbf{R})$, we define

$$\Psi^k_1(h)=\{l\in L^1(I_k,\mathbf{R});\ \widetilde{l(t)}=g^{k-1}(h|_{I_{k-1}})(t)\chi_{I_{k-1}}(t)+n(t)\chi_{I_k\setminus I_{k-1}}(t),$$
$$n(t)\in F(t,\widetilde{g^k_0(h)}(t),V(\widetilde{g^k_0(h)})(t))\ a.e.\,([k-1,k])\}$$

$$\Phi^k_1(h)=\begin{cases}\{h\} & \text{if } h\in\mathcal{T}_{I_k}(x_0),\\ \Psi^k_1(h) & \text{otherwise.}\end{cases}$$

Once again Lemma 4 (applied for $\phi(h)=g^k_0(h)$) implies that $\Phi^k_1(.):L^1(I_k,\mathbf{R})\to\mathcal{D}(I_k,\mathbf{R})$ is l.s.c.. In addition, if $h\in L^1(I_k,\mathbf{R})$ one may write

$$\begin{aligned}&d(\widetilde{g^k_0(t)},F(t,\widetilde{g^k_0(h)}(t),V(\widetilde{g^k_0(h)})(t))=d(h(t),F(t,\widetilde{g^k_0(h)}(t),\\&V(\widetilde{g^k_0(h)}(t))\chi_{I_k\setminus I_{k-1}}\leq q^k_0(h)(t)\quad a.e.\,(I_k),\end{aligned}\tag{8}$$

where

$$q_0^k(h)(t)=|h(t)|+q(t)+L(t)(|\widetilde{g_0^k(h)}(t)|+\int_0^t L(s)|\widetilde{g_0^k(h)}(s)|ds).$$

Obviously, $q_0^k: L^1(I_k,\mathbf{R})\to L^1(I_k,\mathbf{R})$ is continuous. If $m\geq 0$ we define

$$q_{m+1}^k(h)=(I^{\alpha,\rho}M)^m(\frac{k^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}|q_0^k(h)|_{1,k}+\varepsilon_{m+1})$$

and from the continuity of $q_0^k(.)$ we deduce the continuity of $q_m^k: L^1(I_k,\mathbf{R})\to L^1(I_k,\mathbf{R})$.

Finally, we provide the existence of $g_m^k: L^1(I_k,\mathbf{R})\to L^1(I_k,\mathbf{R})$ continuous such that

$$g_m^k(h)(t)=g^{k-1}(h|_{I_{k-1}})(t)\quad \forall t\in I_{k-1}, \tag{a_k}$$

$$g_m^k(h)=h\quad \forall h\in \mathcal{T}_{I_k}(x_0), \tag{b_k}$$

$$g_m^k(h)(t)\in F(t,\widetilde{g_{m-1}^k(h)}(t),V(\widetilde{g_{m-1}^k(h)})(t))\quad a.e.\ (I_k), \tag{c_k}$$

$$|g_1^k(h)(t)-g_0^k(h)(t)|\leq q_0^k(h)(t)+\varepsilon_0\quad a.e.\ (I_k), \tag{d_k}$$

$$|g_m^k(h)(t)-g_{m-1}^k(h)(t)|\leq M(t)q_{m-1}^k(h)\quad a.e.\ (I_k),\quad m\geq 2. \tag{e_k}$$

Set

$$H_1^k(h)=cl\{f\in\Phi_1^k(h);\quad |f(t)-g_0^k(h)(t)|<q_0^k(h)(t)+\varepsilon_0\quad a.e.\ (I_k)\}.$$

Using (8), $H_1^k(h)\neq\emptyset$ for any $h\in L^1(I_1,\mathbf{R})$. Taking into account Lemma 2 and the fact that the maps g_0^k, q_0^k are continuous we find a continuous selection g_1^k of H_1^k with (a_k)-(d_k).

If $g_i^k(.)$, $i=1,\dots m$ with (a_k)-(e_k) are already constructed, from (e_k) one may write

$$\begin{aligned}&d(g_m^k(h)(t),F(t,\widetilde{g_m^k(h)}(t),V(\widetilde{g_m^k(h)})(t))\leq L(t)(|\widetilde{g_{m-1}^k(h)}(t)-\widetilde{g_m^k(h)}(t)|+\\&\textstyle\int_0^t L(s)|\widetilde{g_{m-1}^k(h)}(s)-\widetilde{g_m^k(h)}(s)|ds)\leq M(t)(q_{m+1}^k(h)-s_m)<M(t)q_{m+1}^k(h),\end{aligned} \tag{9}$$

where $s_m:=(I^{\alpha,\rho}M)^m(\varepsilon_{m+1}-\varepsilon_m)>0$.

For $h\in L^1(I_k,\mathbf{R})$, we define

$$\begin{aligned}&\Psi_{m+1}^k(h)=\{l\in L^1(I_k,\mathbf{R});\ l(t)=g^{k-1}(h|_{I_{k-1}})(t)\chi_{I_{k-1}}(t)+\\&n(t)\chi_{I_k\setminus I_{k-1}}(t),\quad n(t)\in F(t,\widetilde{g_m^k(h)}(t),V(\widetilde{g_m^k(h)})(t))\quad a.e.\ ([k-1,k])\},\end{aligned}$$

$$\Phi_{m+1}^k(h)=\begin{cases}\{h\} & \text{if } h\in\mathcal{T}_{I_k}(x_0),\\ \Psi_{m+1}^k(h) & \text{otherwise.}\end{cases}$$

Applying Lemma 4 we obtain that $\Phi_{m+1}^k(.): L^1(I_k,\mathbf{R})\to\mathcal{P}(L^1(I_k,\mathbf{R}))$ has nonempty closed decomposable values and is l.s.c.. As above, the set

$$H_{m+1}^k(h)=cl\{f\in\Phi_{m+1}^k(h);\ |f(t)-g_{m+1}^k(h)(t)|<M(t)q_{m+1}^k(h)\quad a.e.\ (I_k)\}$$

is nonempty. Again, Lemma 2 allows to obtain a continuous selection g_{m+1}^k of H_{m+1}^k, verifying (a_k)-(e_k).

By (e_k) one has

$$|g^k_{m+1}(h) - g^k_m(h)|_{1,k} \leq (I^{\alpha,\rho}M)^m[\frac{k^{\rho\alpha}}{\Gamma(\alpha+1)}|q^k_0(h)|_{1,1} + \varepsilon].$$

Repeating the proof done in the first case we get the convergence of $\{g^k_m(h)\}_{m\in\mathbf{N}}$ to some $g^k(h) \in L^1(I_k, \mathbf{R})$. Moreover, $g^k(.) : L^1(I_k, \mathbf{R}) \to L^1(I_k, \mathbf{R})$ is continuous.

By (a_k) we have that

$$g^k(h)(t) = g^{k-1}(h|_{I_{k-1}})(t) \quad \forall t \in I_{k-1},$$

by (b_k) $g^k(h) = h$ $\forall h \in \mathcal{T}_{I_k}(x_0)$ and, finally, since the values of F are closed, from (c_k) we deduce that

$$g^k(h)(t) \in F(t, \widetilde{g^k(h)}(t), V(\widetilde{g^k(h)})(t)), \quad a.e.\ (I_k) \quad \forall h \in L^1(I_k, \mathbf{R}),$$

and the proof is complete. □

Remark 1. *By definition, a subspace X of a Hausdorff topological space Y is said to be a retract of Y if there exists a continuous function $h : Y \to X$ with $h(x) = x$, $\forall x \in X$.*

So, Theorem 1 states that for each $x_0 \in \mathbf{R}$, the set $\mathcal{T}(x_0)$ is a retract of the Banach space $L^1_{loc}([0,\infty), \mathbf{R})$.

References

[1] Almeida, R., Malinowski, A.B., Odzijewicz, T., *Fractional differential equations with dependence on the Caputo-Katugampola derivative*, J. Comput. Nonlin. Dyn. **11** (2016), ID 061017, 11 pp.

[2] Băleanu, D., Diethelm, K., Scalas E., Trujillo, J.J., *Fractional Calculus Models and Numerical Methods*, World Scientific, Singapore, 2012.

[3] Bressan A., Colombo, G., *Extensions and selections of maps with decomposable values*, Studia Math. **90** (1988), 69-86.

[4] Cernea, A., *A note on the solution set of a fractional differential inclusion*, Bul. Transilvania Univ. Braşov, Series III **5(54)** (2012), 83-92.

[5] Cernea, A., *On a fractional integrodifferential inclusion*, Electronic J. Qual. Theory Differ. Equ. **2014** (2014), no. 25, 1-11.

[6] Cernea, A., *A note on the solution set of a fractional integrodifferential inclusion*, Progress Fract. Differentiation Appl. **2** (2016), 13-18.

[7] Diethelm, K., *The Analysis of Fractional Differential Equations*, Springer, Berlin, 2010.

[8] Katugampola, U.N., *A new approach to generalized fractional derivative*, Bull. Math. Anal. Appl., **6** (2014), 1-15.

[9] Kilbas, A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[10] Miller, K., Ross, B., *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[11] Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego,1999.

[12] Zeng, S., Băleanu, D., Bai, Y., Wu, G., *Fractional differential equations of Caputo-Katugampola type and numerical solutions*, Appl. Math. Comput. **315** (2017), 549-554.