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EXPONENTIAL GROWTH FOR A SEMI-LINEAR VISCOELASTIC HEAT EQUATION WITH $L^p_\rho(\mathbb{R}^n)$-NORM IN BI-LAPLACIAN TYPE

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Abstract

The problem considered here is a class of semi-linear visco-elastic heat equations in bi-Laplacian type. We introduce a weighted space to overcome the difficulties in the non-compactness of some operators and some useful Sobolev embedding inequalities. Under certain conditions on the parameters p, ρ, η, we prove that the local solutions grow as an exponential function in the L^p_ρ-norm, i.e. $\|u\|^p_{L^p_\rho(\mathbb{R}^n)} \to +\infty$ as t tends to $+\infty$.

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1 Introduction and related results

In this paper, we are interested in the exponential growth as $t \longrightarrow +\infty$ for the following problem

$$\begin{cases} u' + \Phi\Delta_x^2 \left(u - \int_0^t \eta\,(t-s)\, u\,(x,s)\, ds \right) & = |u|^{p-2}\, u, \\ \\ u\,(x,0) = u_0\,(x)\,, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n, t \in (0,T), p, n \geq 2, u\,(x,t) \equiv u, \quad \rho\,(x) \equiv \rho, \quad \Phi\,(x) \equiv \Phi$ and u_0 is the initial data which is chosen in suitable spaces. We assume that the functions ρ, Φ and η satisfy conditions:

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(H1) $\Phi(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^*_+$, and $(\Phi(x))^{-1} = \rho(x), \quad \forall x \in \mathbb{R}^n$. the coefficient $\Phi(x)$ represents the speed of sound at the point $x \in \mathbb{R}^n$.

(H2) $\rho : \mathbb{R}^n \longrightarrow \mathbb{R}^*_+,\ \rho(x) \in C^{0,\sigma}(\mathbb{R}^n)$ with $\sigma \in (0,1)$ and $\rho \in L^{n/2}(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$.

(H3) It is assumed for function η that $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, bounded of C^1, and non-increasing functions, and for all $s \geq 0$,

$$\eta(s) \geq 0, \quad \eta'(s) \leq 0 \quad \text{and } 1 - \int_0^{+\infty} \eta(s)ds = k > 0. \tag{2}$$

The exponent p is a real constant and satisfies,

$$\begin{cases} p > 1 & \text{if} \quad n = 1,2,3,4 \\ 1 < p < \frac{n}{n-4} & \text{if} \quad n > 4 \end{cases} \tag{3}$$

Further on, we use the following short notations:

$$\|u(x,t)\|_{L^r(\mathbb{R}^n)} \equiv \|u\|_r\,,\ r > 1.$$

For the case when $\Phi \equiv 1$, many researchers studied the initial boundary value problem of the type of (1) in a bounded domain. The main results are mainly concerned with the existence/nonexistence, stabilities and long-time dynamics, and many results may be found in the literature ([3], [6], [9], [13], [14], [18], [22]...)

In [35], the authors considered the following p-Laplacian evolution equation with a nonlocal source term

$$u' - div\Big(|\nabla u|^{p-2}\nabla u\Big) = u^m \int_\Omega u^n(y,t)dy \tag{4}$$

subject to initial data

$$u(x,0) = u_0(x), \qquad x \in \Omega$$

and weighted non-linear non-local boundary and conditions

$$u(x,t) = \int_\Omega \phi(x,y)u^l(y,t)dy, x \in \Omega, t \in (0,T),$$

where $p, l, m > 0, m \geq 0$ and Ω is an open bounded domain of $\mathbb{R}^N, N \geq 1$. The author proved the global existence and blow-up in time properties of non-negative solutions by using the upper and lower method under certain conditions on different parameters.

Levine and al. [15] got the global existence and non-existence of solution for the equation:

$$|u|^{l-2}\, u' - div(|\nabla u|^{m-2}\, \nabla u) = f(u). \tag{5}$$

Pucci and Serrin [25] discussed the stability of solution for (5). Pang and al. [22], [21] and Berrimi and Messaoudi [3] gave the sufficient and optimal conditions for the blow-up results to a class of solutions of (5) with positive/negative initial energy.

When $\Phi \neq 1$, in the pioneer paper [10], the author considered a semi-linear hyperbolic problem

$$u_{tt} - \Phi(x)\Delta_x u + \delta u' + \lambda f(u) = \eta(x), x \in \mathbb{R}^n, t > 0,$$

for $\delta > 0, n \geq 3$ and $\rho(x) = (\Phi(x))^{-1}$ related with $L^{\frac{n}{2}}(\mathbb{R}^n)$. The author introduced an energy space $D^{1,2}(\mathbb{R}^n) \times L^2_g(\mathbb{R}^n)$ and proved a local existence of solutions and global attractor.

Papadopoulos and Stavarakakis [23] studied a degenerate nonlocal quasi-linear wave equation of Kirchhoff type with a weak dissipative term and established the existence blow up results of

$$u_{tt} - \Phi(x)\left\|\nabla u(t)\right\|^2 \Delta_x u + \delta u' = |u|^a u, x \in \mathbb{R}^n, t \geq 0,$$

where the weighted function related with $L^{\frac{n}{2}}(R^n) \cap L^\infty(\mathbb{R}^n)$.

For the viscoelastic problem

$$u_{tt} - \Delta_x u + \int_0^t g(t-s)\Delta_x u(x,s)ds = 0, x \in \mathbb{R}^n, t > 0$$

Kafini and Messaoudi [7], looked into the equation, for compactly supported initial data u_0, u_1

$$u(x,0) = u_0(x), u'(x,0) = u_1(x), \qquad x \in \mathbb{R}^n$$

For an exponentially decaying relaxation function g, they obtained a polynomial decay for the first energy of solution.

Recently, Zennir [31] considered the following problem

$$\rho(x)\left(|u'|^{q-2}u'\right)' - M(\|\nabla_x u\|_2^2)\Delta_x u + \int_0^t g(t-s)\Delta_x u(s)ds = 0, x \in \mathbb{R}^n, t > 0 \quad (6)$$

where $q, n \geq 2$ and M is a positive C^1 function satisfying for $s \geq 0, m_0 > 0, m_1 \geq 0, \gamma \geq 1$, $M(s) = m_0 + m_1 s^\gamma$. The author proved a very general decay result of solutions for a wider class of relaxation functions.

In the present work, we consider problem (1) with appropriate conditions on p, η, and then we show that the local solutions grow exponentially, when the initial energy is negative/positive. We will see that the influence of the memory term is unable to stabilize the problem.

We organize our article as follows: First, we give the preliminary results, then we give the proof of our main result, in two cases:

1) With negative initial energy.
2) With positive initial energy.

2 Preliminaries and technical Lemmas

We need to define the weighted spaces in the following definition

Definition 2.1. *[27] We define the function spaces of our problem and its norm as follows:*

$$\mathcal{D}^{2,2}(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-4)}(\mathbb{R}^n) : \Delta_x f \in L^2(\mathbb{R}^n) \right\} \tag{7}$$

and the spaces $L^2_g(\mathbb{R}^n)$ to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(f,h)_{L^2_g(\mathbb{R}^n)} = \int_{\mathbb{R}^n} gfhdx.$$

For $1 < q < \infty$, if f is a measurable function on $\mathbb{R}^n$, we define

$$\|f\|_{L^q_g(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} g|f|^q dx \right)^{1/q}. \tag{8}$$

and that $\mathcal{D}^{2,2}(\mathbb{R}^n)$ can be embedded continuously in $L^{2n/(n-4)}(\mathbb{R}^n)$,i.e there exists $k > 0$ such that

$$\|u\|_{L^{2n/(n-4)}} \leq k\|u\|_{\mathcal{D}^{2,2}}. \tag{9}$$

The separable Hilbert space $L^2_g(\mathbb{R}^n)$ with

$$(f,f)_{L^2_g(\mathbb{R}^n)} = \|f\|^2_{L^2_g(\mathbb{R}^n)}.$$

consist of all f for which $\|f\|_{L^q_g(\mathbb{R}^n)} < \infty, 1 < q < +\infty$.

The following Lemma generalized version of Poincaré's inequality is frequently used.

Lemma 2.2. *(K. J. Brown [4], Lemma 2.1) Let $\rho \in L^{\frac{n}{2}}(\mathbb{R}^n)$. Then there exists $\delta = k^{-2} \|\rho\|^{-1}_{L^{\frac{n}{2}}} > 0$ s.t.*

$$\int_{\mathbb{R}^n} |\Delta_x u|^2 \, dx \geq \delta \int_{\mathbb{R}^n} \rho \, |u|^2 \, dx, \qquad \forall u \in C_0^\infty(\mathbb{R}^n)$$

Lemma 2.3. *(K. J. Brown [4], Lemma 2.1) Suppose that $\rho \in L^{\frac{2n}{2n-pn+2p}}(\mathbb{R}^n)$. Then, the continuous embedding*

$$D^{2,2}(\mathbb{R}^n) \subset L^p_\rho(\mathbb{R}^n), \qquad \forall 1 \leq p \leq \frac{2n}{n-2}$$

holds.

Lemma 2.4. *([11], Lemma 2.4) Let $\rho \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then, the continuous embedding*

$$L^p_\rho(\mathbb{R}^n) \subset L^q_\rho(\mathbb{R}^n)$$

holds for any $1 \leq q \leq p < \infty$.

Lemma 2.5. *For any* $u \in \mathcal{D}^{2,2}(\mathbb{R}^n)$

$$\|u\|_{L^2_g(\mathbb{R}^n)} \leq \|g\|_{L^{n/2}(\mathbb{R}^n)} \|\Delta_x u\|_{L^2(\mathbb{R}^n)}. \tag{10}$$

We also set

$$E_1 = \left(\frac{1}{2} - \frac{1}{p}\right)\lambda^2 \quad where \quad \lambda = C^{\frac{-1}{p-2}} B^{\frac{-p}{p-2}}. \tag{11}$$

3 Main results and proofs

We are now ready to present the main exponential growth result. We begin by stating the local existence in time for (1) (see [11], [25]) .

We will use without mention the evolution triple for the spaces, which is

$$D^{2,2}(\mathbb{R}^n) \subset L^2_\rho(\mathbb{R}^n) \subset D^{-1,2}(\mathbb{R}^n), \tag{12}$$

Definition 3.1. *The weak solutions of* (1) *are given by the function* u *s.t.*

$$u \in L^2\left(0, T; D^{2,2}(\mathbb{R}^n)\right), \quad u' \in L^2\left(0, T; L^2_\rho(\mathbb{R}^n)\right),$$

where

$$u(x, 0) = u_0(x) \in D^{2,2}(\mathbb{R}^n).$$

And for all $\Psi \in C_0^\infty([0,T] \times \mathbb{R}^n)$, u *satisfies the generalized formula*

$$\begin{aligned} &\textstyle\int_0^t \int_{\mathbb{R}^n} \rho u'(\tau)\Psi(\tau)\, dx d\tau - \int_0^t \int_{\mathbb{R}^n} \rho\, |u(\tau)|^{p-2} u(\tau)\Psi(\tau)\, dx d\tau \\ &\textstyle - \int_0^t \int_{\mathbb{R}^n} \Delta_x \left(\int_0^s \eta(t-\nu)\, u(\nu)\, ds - u(\tau)\right)\Delta_x \Psi(\tau)\, dx d\tau = 0, \quad \forall t \geq 0. \end{aligned} \tag{13}$$

The energy functional $E(t)$ associated with our problem is given as follows

$$E(t) = \tfrac{1}{2}\left(1 - \textstyle\int_0^t \eta(s)\, ds\right)\|\Delta_x u\|_2^2 + \tfrac{1}{2}(\eta \diamond \Delta_x u)(t) - \tfrac{1}{p}\|u\|^p_{L^p_\rho}. \tag{14}$$

where

$$(\eta \diamond \Delta_x w)(t) = \int_0^t \eta(t-s)\, \|\Delta_x w(x,t) - \Delta_x w(x,s)\|_2^2\, ds.$$

Lemma 3.2. *The energy functional introduced in* (14) *is a non-increasing function along the solution of* (1) *and satisfies*

$$E'(t) = -\|u'\|^2_{L^2_\rho} + \tfrac{1}{2}\left((\eta' \diamond \Delta_x u)(t) - \eta(t)\|\Delta_x u\|_2^2\right) \leq 0, \forall t \geq 0 \tag{15}$$

and then by ***(H3)***

$$E(t) \leq E(0), \quad \text{for all } \ t \in [0, T). \tag{16}$$

Proof. We multiply the equation (1) by $\rho(x)u'$, and we integrate by parts over $\mathbb{R}^n$, we have

$$E(t) - E(0) = -\textstyle\int_0^t \left(\|u_\tau\|^2_{L^{2\rho}} - \tfrac{1}{2}(\eta' \diamond \Delta_x u)(\tau) + \tfrac{1}{2}\eta(\tau)\|\Delta_x u\|_2^2\right) d\tau,$$

This gives (16) for all $t \geq 0$. □

3.1 Growth with negative initial energy

Theorem 3.3. *Suppose that the (3) holds, for $u_0(x) \in D^{2,2}(\mathbb{R}^n)$ satisfying*

$$E(0) < 0,$$

and

$$\int_0^t \eta(s)\,ds < \frac{p-2}{p-1}. \tag{17}$$

Then the solution of the problem (1) grows exponentially with the L^p_ρ-norm.

Proof. As in [29] We set

$$H(t) := -E(t). \tag{18}$$

By (15) we have

$$\frac{d}{dt}H(t) := -\frac{d}{dt}E(t).$$

Consequently, we have

$$H(0) > 0.$$

This imply that

$$H(t) \geq H(0) > 0. \tag{19}$$

Then,

$$H(t) - \frac{1}{p}\|u\|^p_{L^p_\rho} \leq 0.$$

Thus

$$\frac{1}{p}\|u\|^p_{L^p_\rho} \geq H(t) \geq H(0) > 0. \tag{20}$$

for every t in $[0;T)$.
Let us now define another functional

$$L(t) := H(t) + \frac{\epsilon}{2}\|u\|^2_{L^2_\rho}, \tag{21}$$

for ϵ small positive constant. By differentiating the functional $L(t)$ and using (1), we get

$$\begin{aligned}
L'(t) &= H'(t) + \epsilon \textstyle\int_{\mathbb{R}^n} \rho u u' dx \\
&= H'(t) + \epsilon \textstyle\int_{\mathbb{R}^n} \rho u \left[-\Phi\Delta_x^2\left(u - \int_0^t \eta(t-s)\,u(s)\,ds\right) + |u|^{p-2}u\right] dx \\
&= H'(t) + \epsilon\left[-\|\Delta_x u\|_2^2 + \textstyle\int_{\mathbb{R}^n}\int_0^t \eta(t-s)\,\Delta_x u(s)\,\Delta_x u ds dx + \|u\|^p_{L^p_\rho}\right] \\
&= H'(t) + \epsilon\left[-\|\Delta_x u\|_2^2 - \textstyle\int_{\mathbb{R}^n}\int_0^t \eta(t-s)\,\Delta_x u\,(\Delta_x u(t) - \Delta_x u(s))\,ds dx \right. \\
&\quad \left. + \textstyle\int_{\mathbb{R}^n}\int_0^t \eta(t-s)\,|\Delta_x u|^2\,ds dx + \|u\|^p_{L^p_\rho}\right]
\end{aligned} \tag{22}$$

By Cauchy-Schwartz and Young's inequalities, we estimate $I(t)$ where

$$I(t) = \int_{\mathbb{R}^n} \int_0^t \eta(t-s)\, \Delta_x u \left(\Delta_x u(t) - \Delta_x u(s)\right) ds dx$$

$$\begin{aligned} I(t) &\leq {}^t_0 \left(\int_{\mathbb{R}^n} \eta(t-s) \left|\Delta_x u\right|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \eta(t-s) \left|\left(\Delta_x u(t) - \Delta_x u(s)\right)\right|^2 dx\right)^{\frac{1}{2}} ds \\ &\leq \frac{1}{2}\left(\int_0^t \eta(s)\, ds \left\|\Delta_x u\right\|_2^2 + \left(\eta \diamond \Delta_x u\right)(t)\right). \end{aligned} \tag{23}$$

By (23), (22), we get

$$L'(t) = H'(t) - \epsilon\left(1 - \frac{1}{2}\int_0^t \eta(s)\, ds\right) \left\|\Delta_x u\right\|_2^2 - \frac{\epsilon}{2}\left(\eta \diamond \Delta_x u\right)(t) + \epsilon \left\|u\right\|_{L_\rho^p}^p, \tag{24}$$

we then substitute for $\|u\|_{L_\rho^p}^p$ from (14) hence, (24) becomes

$$\begin{aligned} L'(t) &= H'(t) - \epsilon\left(1 - \frac{1}{2}\int_0^t \eta(s)\, ds\right) \left\|\Delta_x u\right\|_2^2 - \frac{\epsilon}{2}\eta \diamond \Delta_x u(t) \\ &+ \epsilon\left[pH(t) + \frac{p}{2}\left(1 - \int_0^t \eta(s)\, ds\right)\left\|\Delta_x u\right\|_2^2 + \frac{p}{2}\eta \diamond \Delta_x u(t)\right] \\ &\geq H'(t) + \epsilon p H(t) + \epsilon\left(\frac{p}{2} - 1 - \left(\frac{p}{2} - \frac{1}{2}\right)\int_0^t \eta(s)\, ds\right)\left\|\Delta_x u\right\|_2^2 \\ &+ \epsilon\left(\frac{p}{2} - \frac{1}{2}\right)\left(\eta \diamond \Delta_x u\right)(t), \end{aligned} \tag{25}$$

Taking $\delta_1 = \frac{p}{2} - 1 - \left(\frac{p}{2} - \frac{1}{2}\right)\int_0^t \eta(s)\, ds$, then by (14), we get

$$\begin{aligned} L'(t) &\geq H'(t) + \epsilon\left(p - \delta_1\right) H(t) + \epsilon\delta_1 \left\|\Delta_x u\right\|_2^2 + \epsilon\left(\frac{p}{2} - \frac{1}{2}\right)\eta \diamond \Delta_x u(t) \\ &- \epsilon\delta_1 \frac{1}{2}\left(1 - \int_0^t \eta(s)\, ds\right)\left\|\Delta_x u\right\|_2^2 - \frac{1}{2}\epsilon\delta_1 \eta \diamond \Delta_x u(t) + \frac{1}{p}\epsilon\delta_1 \left\|u\right\|_{L_\rho^p}^p \\ &\geq H'(t) + \epsilon\left(p - \delta_1\right) H(t) + \epsilon\delta_1\left(\frac{1}{2} - \int_0^t \eta(s)\, ds\right)\left\|\Delta_x u\right\|_2^2 \\ &+ \frac{\epsilon}{2}\left(p - 1 - \delta_1\right)\eta \diamond \Delta_x u(t) + \frac{\epsilon\delta_1}{p}\left\|u\right\|_{L_\rho^p}^p \\ &\geq H'(t) + C_1 H(t) + C_2 \eta \diamond \Delta_x u(t) + C_3 \left\|\Delta_x u\right\|_2^2 + C_4 \left\|u\right\|_{L_\rho^p}^p, \end{aligned} \tag{26}$$

by inequality (17), we can get $\delta_1 > 0$ and $C_i > 0, \forall\, i = 1, 2, 3, 4$.
This implies

$$L'(t) \geq C\left(H(t) + H'(t) + \left(\eta \diamond \Delta_x u\right)(t) + \left\|\Delta_x u\right\|_2^2 + \left\|u\right\|_{L_\rho^p}^p\right) \tag{27}$$

where

$$C = min\left\{1, C_1, C_2, C_3, C_4\right\}.$$

Therefore by (21), we have

$$\begin{aligned} L\left(t\right) &= H\left(t\right) + \tfrac{\epsilon}{2}\left(\left\|\Delta_x u\right\|_2^2 + \left\|u\right\|_{L_\rho^2}^2\right) \\ &\leq H\left(t\right) + \tfrac{\epsilon}{2}\left(\left\|\Delta_x u\right\|_2^2 + c_0\left(\left\|u\right\|_{L_\rho^p}^p\right)^{\frac{2}{p}}\right) \end{aligned} \tag{28}$$

Since $\frac{2}{p} < 1$, now applying the following inequality

$$a^r \leq \left(1 + \frac{1}{b}\right)(a+b), \quad \textit{for all } \ a, b \in \mathbb{R}^+ \quad \textit{and} \quad r \in [0,1]. \tag{29}$$

Then from (20), we have

$$\begin{aligned} \left(\left\|u\right\|_{L_\rho^p}^p\right)^{\frac{2}{p}} &\leq \left(1 + \tfrac{1}{H(0)}\right)\left(\left\|u\right\|_{L_\rho^p}^p + H\left(0\right)\right) \\ &\leq c_1\left\|u\right\|_{L_\rho^p}^p, \end{aligned} \tag{30}$$

where $c_1 = c_0(1 + (H\left(0\right))^{-1})(p+1)p^{-1}$.

by substituting (30) in (28), we get

$$\begin{aligned} L\left(t\right) &\leq H\left(t\right) + \tfrac{\epsilon}{2}\left\|\Delta_x u\right\|_2^2 + \tfrac{\epsilon c_1}{2}\left\|u\right\|_{L_\rho^p}^p \\ &\leq H\left(t\right) + H'\left(t\right) + \tfrac{\epsilon}{2}\left\|\Delta_x u\right\|_2^2 + \tfrac{\epsilon c_1}{2}\left\|u\right\|_{L_\rho^p}^p + \eta \diamond \Delta_x u\left(t\right) \\ &\leq d_0\left(H\left(t\right) + H'\left(t\right) + \left(\eta \diamond \Delta_x u\right)\left(t\right) + \left\|\Delta_x u\right\|_2^2 + \left\|u\right\|_{L_\rho^p}^p\right), \end{aligned} \tag{31}$$

this implies that, for some positive constant Γ s. t.

$$L'\left(t\right) \geq \Gamma L\left(t\right). \tag{32}$$

We integrate now (32) over $[0,t]$ to obtain

$$L\left(t\right) \geq L\left(0\right)e^{\gamma t}. \tag{33}$$

□

Lemma 3.4. *Let u be the solution of problem (1). There exists a positive constant β, such that*

$$\beta\left\|u\right\|_{L_\rho^p}^p \geq L\left(t\right). \tag{34}$$

Proof. Using (18), (21) and (29), to obtain

$$
\begin{aligned}
L(t) &= H(t) + \tfrac{\epsilon}{2}\left(\|\Delta_x u\|_2^2 + \|u\|_{L^2_\rho}^2\right) \\
&\leq \tfrac{1}{p}\|u\|_{L^p_\rho}^p - \tfrac{1}{2}\left(1-\epsilon-\int_0^t \eta(s)\,ds\right)\|\Delta_x u\|_2^2 - \tfrac{1}{2}(\eta \diamond \Delta_x u)(t) + \tfrac{\epsilon c_0}{2}\left(\|u\|_{L^p_\rho}^p\right)^{\frac{2}{p}} \\
&\leq\leq \tfrac{1}{p}\|u\|_{L^p_\rho}^p + \tfrac{\epsilon c_0}{2}\left(1+\tfrac{1}{H(0)}\right)\left(1+\tfrac{1}{p}\right)\|u\|_{L^p_\rho}^p \\
&\leq \left(\tfrac{1}{p} + \tfrac{\epsilon c_0}{2}\left(1+\tfrac{1}{H(0)}\right)\left(1+\tfrac{1}{p}\right)\right)\|u\|_{L^p_\rho}^p .
\end{aligned}
\tag{35}
$$

By (33) and (34), we deduce that the solution of (1) in the L^p_ρ-norm growths exponentially. □

3.2 Growth with positive initial energy

Lemma 3.5. *[28] Let u be a solution of* (1). *Suppose that (3) holds. Assume further that*

$$E(0) < E_1$$

and

$$\|u_0\|_{\mathcal{D}^{2,2}(\mathbb{R}^n)} > \lambda.$$

Then there exists a constant $\beta > \alpha$ such that

$$\|u\|_{\mathcal{D}^{2,2}(\mathbb{R}^n)} > \beta.$$

The following Theorem in our second main result.

Theorem 3.6. *Suppose that (3) holds. Let $u_0 \in \mathcal{D}^{2,2}(\mathbb{R}^n)$ satisfying $\|u_0\|_{\mathcal{D}^{2,2}(\mathbb{R}^n)} > \lambda$ and $E_1 > E(0) \geq 0$. Then the local solutions of the problem* (1) *grow up as an exponential function as $t \longrightarrow \infty$ with L^p_ρ-norm.*

Proof. Now, we set

$$H(t) = E_1 - E(t) \tag{36}$$

Then, by (15), we have

$$H'(t) = -E'(t) \geq 0$$

Consequently,

$$H(0) = E_1 - E(0) > 0$$

It is clear that,

$$0 < H(0) \leq H(t)$$

By (11), $E(t)$ and Lemma (3.5), we have

$$\begin{aligned} H(t) &= E_1 - \tfrac{1}{2}\left(1 - \int_0^t \eta(s)\,ds\right)\|\Delta_x u\|_2^2 - \tfrac{1}{2}(\eta \diamond \Delta_x u)(t) + \tfrac{1}{p}\|u\|_{L_\rho^p}^p \\ &\leq E_1 - \tfrac{1}{2}\lambda^2 + \tfrac{1}{p}\|u\|_{L_\rho^p}^p \\ &\leq -\tfrac{1}{p}\lambda^2 + \tfrac{1}{p}\|u\|_{L_\rho^p}^p \\ &< \tfrac{1}{p}\|u\|_{L_\rho^p}^p . \end{aligned} \tag{37}$$

This implies that

$$\frac{1}{p}\|u\|_{L_\rho^p}^p > H(t) \geq H(0) > 0. \tag{38}$$

Then, it is not hard to follow the steps of the proof for Theorem 3.3. □

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