Bulletin of the *Transilvania* University of Braşov • Vol 11(60), No. 2 - 2018
Series III: Mathematics, Informatics, Physics, 57-76

UNIQUENESS OF MEROMORPHIC FUNCTION SHARING THREE SETS WITH ITS DIFFERENCE OPERATOR

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Abstract

In this paper, we investigate the uniqueness problem of meromorphic function that shares three sets with its difference operator and obtain some results which extend some earlier results.

2010 *Mathematics Subject Classification:* Primary 30D35.
Key words and phrases: meromorphic function, weighted sharing, uniqueness, difference operator.

1 Introduction, Definitions and Results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane. We assume that the reader is familiar with the standard notations of Nevanlinna value distribution theory as explained in [7], [18]. For a meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic function of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ for all r outside a possible exceptional set of finite logarithmic measure. The meromorphic function α is said to be a small function of f if $T(r, \alpha) = S(r, f)$. We define difference operators by

$$\Delta_c f(z) = f(z + c) - f(z), \;\; \Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z)),$$

where c is a nonzero complex number and $n \geq 2$ is a positive integer. If $c = 1$, we denote $\Delta_c f(z) = \Delta f(z)$.

If $f - a$ and $g - a$ have the same set of zeros with same multiplicities then we say that f and g share the value a CM and if $f - a$ and $g - a$ have the same set of zeros ignoring multiplicities we say that f and g share the value a IM. In

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addition, if f and g share ∞ CM, then we say that $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM. For $a \in \mathbb{C} \cup \{\infty\}$ and $S \subset \mathbb{C} \cup \{\infty\}$, we denote by

$$E_f(S) = \cup_{a \in S}\{z : f(z) - a = 0, \quad \textit{counting multiplicity}\},$$

$$\overline{E}_f(S) = \cup_{a \in S}\{z : f(z) - a = 0, \quad \textit{ignoring multiplicity}\}.$$

If $E_f(S) = E_g(S)$, we say that f and g share the set S CM and if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM. Especially, if $S = \{a\}$ then we get the definition of usual value sharing. In this direction we need the following definitions.

Definition 1. *[10] Let l be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_l(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq l$ and $l + 1$ times if $m > l$. If $E_l(a; f) = E_l(a; g)$, we say that f, g share the value a with weight l.*

The definition implies that if f, g share a value a with weight l, then z_0 is an a-point of f with multiplicity $m(\leq l)$ if and only if it is an a-point of g with multiplicity $m(\leq l)$ and z_0 is an a-point of f with multiplicity $m(> l)$ if and only if it is an a-point of g with multiplicity $n(> l)$, where m is not necessarily equal to n.

We write f, g share (a, l) to mean that f, g share the value a with weight l. Clearly if f, g share (a, l) then f, g share (a, p) for any integer p, $0 \leq p < l$. Also we note that f, g share a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 2. *[10] Let l be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ and $S \subset \mathbb{C} \cup \{\infty\}$ we denote by $E_f(S, l) = \cup_{a \in S} E_l(a; f)$. Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.*

Definition 3. *[9] Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points of f. For a positive integer k we denote by $N(r, a; f \mid\leq k)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than k. By $\overline{N}(r, a; f \mid\leq k)$ we denote the corresponding reduced counting function. Analogously, we can define $N(r, a; f \mid\geq k)$ and $\overline{N}(r, a; f \mid\geq k)$. Clearly, $\overline{N}(r, a; f) = N(r, a; f \mid= 1) + \overline{N}(r, a; f \mid\geq 2)$.*

Definition 4. *[16, 17] Let f and g be two nonconstant meromorphic functions such that f and g share the value a IM and z_0 be an a-point of f with multiplicity p and an a-point of g with multiplicity q. We denote by $\overline{N}_L(r, a; f)$ the reduced counting function of those a-points of f for which $p > q$. In the same way, we can define $\overline{N}_L(r, a; g)$. Also we denote by $N_E^{1)}(r, a; f)$ the counting function of those a-points of f where $p = q = 1$.*

Definition 5. *[5, 7] Let f and g share the value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from multiplicities of the corresponding a-points of g. Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.*

Henceforth we denote by S_0, S_1, S_2 and S_3 the sets $S_0 = \{w : w^n + aw^{n-m} + b = 0$, where n and $n-m$ have no common factor and $m \geq 2$ $\}$, $S_1 = \{1,\ \omega,\ \omega^2,\ \ldots,\ \omega^{n-1}\}$, where $\omega^n = 1$, $S_2 = \{\infty\}$ and $S_3 = \{0\}$.

The investigation on the uniqueness of meromorphic function sharing sets with its shift and difference operator is an important subfield of uniqueness theory. In this field a lot of research work has been done by many researchers (see [8], [12], [13]).

In 2010, Zhang [19] investigated the relationship between $f(z)$ and its shift $f(z+c)$ sharing two sets and obtained the following results.

Theorem A. *Let $f(z)$ be a nonconstant meromorphic function of finite order and $n \geq 4$ be an integer. If $E_{f(z)}(S_j, \infty) = E_{f(z+c)}(S_j, \infty)$ $(j = 1,\ 2)$, then $f(z) = tf(z+c)$, where $t^n = 1$.*

Theorem B. *Let $f(z)$ be a nonconstant meromorphic function of finite order and n, m be two positive integers such that $n \geq 2m+4$. If $E_{f(z)}(S_j, \infty) = E_{f(z+c)}(S_j, \infty)$ $(j = 0,\ 2)$, then $f(z) = f(z+c)$.*

In 2012, Chen and Chen [4] replaced $f(z+c)$ with difference operator $\Delta_c f(z)$ and obtained the following results.

Theorem C. *Let $f(z)$ be a nonconstant meromorphic function of finite order and $n \geq 7$ be an integer. If $E_{f(z)}(S_1, 2) = E_{\Delta_c f(z)}(S_1, 2)$ and $E_{f(z)}(S_2, \infty) = E_{\Delta_c f(z)}(S_2, \infty)$, then $\Delta_c f(z) = tf(z)$, where $t^n = 1$ and $t \neq -1$.*

Theorem D. *Let $f(z)$ be a nonconstant meromorphic function of finite order and $n \geq 3$ be an integer. If $E_{f(z)}(S_1, 2) = E_{\Delta_c f(z)}(S_1, 2)$, $E_{f(z)}(S_2, \infty) = E_{\Delta_c f(z)}(S_2, \infty)$ and $\overline{N}(r, \infty; f) + \overline{N}(r, 0; f) = S(r, f)$, then $\Delta_c f(z) = tf(z)$, where $t^n = 1$ and $t \neq -1$.*

Theorem E. *Let $f(z)$ be a nonconstant meromorphic function of finite order and $n \geq 7$ be an integer. If $E_{f(z)}(S_1, 2) = E_{\Delta_c f(z)}(S_1, 2)$, $E_{f(z)}(S_2, 0) = E_{\Delta_c f(z)}(S_2, 0)$ and $\limsup\limits_{r\to\infty} \dfrac{N(r, 0; f)}{T(r, f)} < 1$, then $\Delta_c f(z) = tf(z)$, where $t^n = 1$ and $t \neq -1$.*

Theorem F. *Let $f(z)$ be a nonconstant meromorphic function of finite order and n be an integer such that $n \geq \frac{15\alpha}{2} + 4$ for $0 < \alpha \leq 2$. If $E_{f(z)}(S_1, 0) = E_{\Delta_c f(z)}(S_1, 0)$, $E_{f(z)}(S_2, \infty) = E_{\Delta_c f(z)}(S_2, \infty)$ and $\overline{N}(r, \infty; f) + \overline{N}(r, 0; f) \leq \alpha T(r, f)$, then $\Delta_c f(z) = tf(z)$, where $t^n = 1$ and $t \neq -1$.*

Theorem G. *Let $f(z)$ be a nonconstant meromorphic function of finite order and n, m be two positive integers such that $n \geq 2m+4$. If $E_{f(z)}(S_j, \infty) = E_{\Delta_c f(z)}(S_j, \infty)$ $(j = 0,\ 2)$ and $N(r, 0; \Delta_c f) = T(r, f) + S(r, f)$, then $f(z) = \Delta_c f(z)$.*

In 2016, Banerjee and Bhattacharyya [2] obtained the following results.

Theorem H. *Let $f(z)$ be a nonconstant meromorphic function of finite order and $n \geq 6$ be an integer. If $E_{f(z)}(S_1, 2) = E_{\Delta_c f(z)}(S_1, 2)$ and $E_{f(z)}(S_2, 0) = E_{\Delta_c f(z)}(S_2, 0)$, then $\Delta_c f(z) = t f(z)$, where $t^n = 1$ and $t \neq -1$.*

Theorem I. *Let $f(z)$ be a nonconstant meromorphic function of finite order and $n \geq 7$ be an integer. If $E_{f(z)}(S_1, 1) = E_{\Delta_c f(z)}(S_1, 1)$ and $E_{f(z)}(S_2, 0) = E_{\Delta_c f(z)}(S_2, 0)$, then $\Delta_c f(z) = t f(z)$, where $t^n = 1$ and $t \neq -1$.*

In 2017, Deng, Liu and Yang [6] proved the following result.

Theorem J. *Let $f(z)$ be a nonconstant meromorphic function of finite order and $E_{f(z)}(S_1, l) = E_{\Delta_c f(z)}(S_1, l)$, $E_{f(z)}(S_2, \infty) = E_{\Delta_c f(z)}(S_2, \infty)$. If $l \geq 2$, $n \geq 5$ or $l = 1$, $n \geq 7$, then $\Delta_c f(z) = t f(z)$, where $t^n = 1$ and $t \neq -1$.*

In this paper, with the notion of weighted sharing we investigate the relationship between $f(z)$ and its difference operator $\Delta_c f(z)$ sharing three sets and obtained the following results.

Theorem 1. *Let $f(z)$ be a nonconstant meromorphic function of finite order and $E_{f(z)}(S_0, l) = E_{\Delta_c f(z)}(S_0, l)$, $E_{f(z)}(S_j, \infty) = E_{\Delta_c f(z)}(S_j, \infty)$ $(j = 2,\ 3)$. If one of the conditions (i) $l \geq 2$, $n \geq 2m+3$; (ii) $l = 1$, $n \geq \frac{5m}{2}+4$; (iii) $l = 0$, $n \geq 5m+8$ holds, then $\Delta_c f(z) = f(z)$.*

Theorem 2. *Let $f(z)$ be a nonconstant meromorphic function of finite order and $E_{f(z)}(S_0, l) = E_{\Delta_c f(z)}(S_0, l)$, $E_{f(z)}(S_2, \infty) = E_{\Delta_c f(z)}(S_2, \infty)$, $E_{f(z)}(S_3, 0) = E_{\Delta_c f(z)}(S_3, 0)$. If one of the conditions (i) $l \geq 2$, $n \geq 3m + 5$; (ii) $l = 1$, $n \geq \frac{7m}{2} + \frac{11}{2}$; (iii) $l = 0$, $n \geq 6m + 9$ holds, then $\Delta_c f(z) = f(z)$.*

Theorem 3. *Let $f(z)$ be a nonconstant meromorphic function of finite order and $E_{f(z)}(S_0, l) = E_{\Delta_c f(z)}(S_0, l)$, $E_{f(z)}(S_2, 0) = E_{\Delta_c f(z)}(S_2, 0)$, $E_{f(z)}(S_3, \infty) = E_{\Delta_c f(z)}(S_3, \infty)$. If one of the conditions (i) $l \geq 2$, $n \geq 2m + 3$; (ii) $l = 1$, $n \geq \frac{5m}{2} + \frac{9}{2}$; (iii) $l = 0$, $n \geq 5m + 8$ holds, then $\Delta_c f(z) = f(z)$.*

Theorem 4. *Let $f(z)$ be a nonconstant meromorphic function of finite order and $E_{f(z)}(S_0, l) = E_{\Delta_c f(z)}(S_0, l)$, $E_{f(z)}(S_j, 0) = E_{\Delta_c f(z)}(S_j, 0)$ $(j = 2,\ 3)$. If one of the conditions (i) $l \geq 2$, $n \geq 3m + 5$; (ii) $l = 1$, $n \geq \frac{7m}{2} + 6$; (iii) $l = 0$, $n \geq 6m + 10$ holds, then $\Delta_c f(z) = f(z)$.*

2 Lemmas

In this section we present some lemmas which will be needed in the sequel. We shall denote by H and V the following functions:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) \tag{1}$$

and

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}. \tag{2}$$

Lemma 1. *[16, 17] If two nonconstant meromorphic functions F and G share $(1,0)$ and $H \not\equiv 0$, then*

$$N_E^{1)}(r,1;F) \leq N(r,\infty;H) + S(r,F) + S(r,G).$$

Lemma 2. *[11] Suppose that F and G are two nonconstant meromorphic functions sharing $(1,1)$ and $H \not\equiv 0$, then*

$$N(r,1;F \mid= 1) \leq N(r,\infty;H) + S(r,F) + S(r,G).$$

Lemma 3. *[11] If two nonconstant meromorphic functions F and G share $(1,2)$, then*

$$\overline{N}_0(r,0;G') + \overline{N}(r,1;G \mid\geq 2) + \overline{N}_*(r,1;F,G) \leq \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + S(r,G),$$

where $\overline{N}_0(r,0;G)$ denotes the reduced counting function of those zeros of G' which are not the zeros of $G(G-1)$.

Lemma 4. *[1] If two nonconstant meromorphic functions F and G share $(1,s)$ where $0 \leq s < \infty$, then*

$$\begin{aligned} &\overline{N}(r,1;F) + \overline{N}(r,1;G) - N_E^{1)}(r,1;F) + \left(s - \frac{1}{2}\right)\overline{N}_*(r,1;F,G) \\ \leq\ & \frac{1}{2}[N(r,1;F) + N(r,1;G)]. \end{aligned}$$

Lemma 5. *[14] Let f be a nonconstant meromorphic function and $a_0(z)$, $a_1(z)$, . . . , $a_n(z)(\not\equiv 0)$ be meromorphic functions such that $T(r,a_i(z)) = S(r,f)$ for $i = 0, 1, 2, \ldots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ \ldots\ + a_1 f + a_0) = nT(r,f) + S(r,f).$$

Lemma 6. *[3, 5] Let $f(z)$ be a transcendental meromorphic function of finite order and $c \in \mathbb{C}\backslash\{0\}$. Then*

$$T(r, f(z+c)) = T(r,f) + S(r,f).$$

Lemma 7. *[3, 5] Let f be a nonconstant meromorphic function of finite order ρ and $c \in \mathbb{C}\backslash\{0\}$. Then for each $\epsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O(r^{\rho-1+\epsilon}).$$

Lemma 8. *[2] Let F, G share $(1,s)$ and $(\infty,0)$ where $0 \leq s < \infty$. Then*

$$\overline{N}_*(r,1;F,G) \leq \frac{1}{s+1}\{\overline{N}(r,0;F) + \overline{N}(r,0;G) + 2\overline{N}(r,\infty;F)\} + S(r,F) + S(r,G).$$

Lemma 9. *[2] Let F, G share $(1,s)$ and $(\infty,0)$ where $0 < s < \infty$. Then*

$$\overline{N}_*(r,1;F,G) \leq \frac{1}{s}\{\overline{N}(r,0;F) + \overline{N}(r,\infty;F)\} + S(r,F) + S(r,G).$$

Lemma 10. *[15, 16] If F, G share $(\infty, 0)$ and $V \equiv 0$, then $F \equiv G$.*

Lemma 11. *Let $F = \frac{f^n + af^{n-m}}{-b}$, $G = \frac{(\Delta_c f)^n + a(\Delta_c f)^{n-m}}{-b}$ and $V \not\equiv 0$. If f, $\Delta_c f$ share (∞, k), where $0 \leq k < \infty$, then*

$$(nk + n - 1)\overline{N}(r, \infty; f \mid\geq k+1) \leq N(r, \infty; V) + S(r, F) + S(r, G).$$

Proof. Since f, $\Delta_c f$ share (∞, k), it follows that F, G share (∞, nk) and so a pole of F with multiplicity r $(\geq nk + 1)$ is a pole of G with multiplicity s $(\geq nk + 1)$ and vice-versa. We note that F and G have no poles of multiplicity t where $nk < t < nk + n$, so from the definition of V we obtain

$$(nk + n - 1)\overline{N}(r, \infty; f \mid\geq k+1) \leq N(r, 0; V) \leq N(r, \infty; V) + S(r, F) + S(r, G).$$

This proves the lemma. □

Lemma 12. *Let F and G be defined as in Lemma 11 and $V \not\equiv 0$. If f, $\Delta_c f$ share (∞, ∞), then*

$$(n - 1)\overline{N}(r, \infty; f) \leq N(r, \infty; V) + S(r, F) + S(r, G).$$

Proof. Since f, $\Delta_c f$ share (∞, ∞), the poles of F and G are of equal multiplicities. But as F has no pole of multiplicity $< n$, from the definition of V we obtain

$$(n - 1)\overline{N}(r, \infty; f) \leq N(r, 0; V) \leq N(r, \infty; V) + S(r, F) + S(r, G).$$

This proves the lemma. □

3 Proof of the Theorems

Proof of the Theorem 1. Let F and G be the same as in Lemma 11 and $V \not\equiv 0$. Since $E_{f(z)}(S_0, l) = E_{\Delta_c f(z)}(S_0, l)$ and $E_{f(z)}(S_j, \infty) = E_{\Delta_c f(z)}(S_j, \infty)$ $(j = 2,\ 3)$, it follows that F and G share $(1, l)$, (∞, ∞) and $(0, \infty)$. We now discuss the following three cases separately.

Case 1.1. *First we suppose that $l \geq 2$ and $H \not\equiv 0$. By the second fundamental theorem of Nevanlinna we have*

$$T(r, F) \leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}(r, 1; F) - \overline{N}_0(r, 0; F') + S(r, F), \tag{3}$$

where $\overline{N}_0(r, 0; F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F - 1)$.

If F and G share $(1, s)$ where $0 \leq s < \infty$, (∞, ∞) and $(0, \infty)$, we see from the definition of V and H

$$N(r, \infty; V) \leq \overline{N}_*(r, 1; F, G) + S(r, F) + S(r, G) \tag{4}$$

and

$$N(r,\infty;H) \le \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G). \quad (5)$$

Since F and G share (∞,∞) and $(0,\infty)$, we have from Lemma 5

$$\overline{N}(r,0;G) = \overline{N}(r,0;F) = \overline{N}(r,0;f) + mT(r,f) + S(r,f) \quad (6)$$

$$\text{and} \qquad \overline{N}(r,\infty;G) = \overline{N}(r,\infty;F) = \overline{N}(r,\infty;f) + S(r,f). \quad (7)$$

Using (4), (7) and Lemma 9 for $s = 2$, we obtain from Lemma 12

$$(2n-3)\overline{N}(r,\infty;f) \le \overline{N}(r,0;F) + S(r,F). \quad (8)$$

As F and G share $(1,2)$, we obtain from (5), Lemma 2 and Lemma 3

$$\begin{aligned}
\overline{N}(r,1;F) &= N(r,1;F \mid= 1) + \overline{N}(r,1;F \mid\ge 2) \\
&= N(r,1;F \mid= 1) + \overline{N}(r,1;G \mid\ge 2) \\
&\le N(r,\infty;H) + \overline{N}(r,1;G \mid\ge 2) + S(r,F) + S(r,G) \\
&\le \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') \\
&\quad + \overline{N}(r,1;G \mid\ge 2) + S(r,F) + S(r,G) \\
&\le \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}_0(r,0;F') \\
&\quad + S(r,F) + S(r,G). \qquad (9)
\end{aligned}$$

Using (6)-(9) and Lemma 5, we obtain from (3)

$$\begin{aligned}
nT(r,f) &\le \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;G) \\
&\quad + \overline{N}(r,\infty;G) + S(r,F) + S(r,G) \\
&\le 2\overline{N}(r,0;F) + 2\overline{N}(r,\infty;F) + S(r,F) \\
&\le \left[2 + \frac{2}{2n-3}\right] \overline{N}(r,0;F) + S(r,F) \\
&\le \left[2 + \frac{2}{2n-3}\right] \{\overline{N}(r,0;f) + mT(r,f)\} + S(r,f) \\
&\le \left[2m + 2 + \frac{2m+2}{2n-3}\right] T(r,f) + S(r,f),
\end{aligned}$$

which contradicts the fact $n \ge 2m+3$. Hence $H \equiv 0$. Therefore from (1) we get

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating twice, we obtain

$$F = \frac{AG+B}{CG+D}, \quad (10)$$

where A, B, C, D are constants with $AD - BC \ne 0$ and A, C are not simultaneously zero as F is not constant. We now consider the following three subcases.

Subcase 1.1.1. *Let $AC \neq 0$. Then from (10) we have*

$$F - \frac{A}{C} = \frac{BC - AD}{C(CG + D)}. \tag{11}$$

From (11) we see that the zeros of $F - \frac{A}{C}$ correspond to the poles of G. Since F and G share (∞, ∞), we see from (11) that ∞ as well as $\frac{A}{C}$ are Picard's exceptional values of F. So by second fundamental theorem of Nevanlinna, Lemma 5 and (6), we get

$$\begin{aligned} nT(r,f) &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}\left(r, \frac{A}{C}; F\right) + S(r,F) \\ &\leq \overline{N}(r,0;f) + mT(r,f) + S(r,f), \end{aligned}$$

which is a contradiction as $n \geq 2m + 3$.

Subcase 1.1.2. *Let $A \neq 0$ and $C = 0$. Then from (10), we get*

$$F = \frac{A}{D}G + \frac{B}{D}. \tag{12}$$

If F has no 1-point, then using (6), (7) and Lemma 5 we obtain by the second fundamental theorem

$$\begin{aligned} nT(r,f) &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + S(r,F) \\ &\leq \overline{N}(r,0;f) + mT(r,f) + \overline{N}(r,\infty;f) + S(r,f), \end{aligned}$$

which is a contradiction as $n \geq 2m + 3$.

If F and G have some 1-points then from (12) we get $\frac{A}{D} + \frac{B}{D} = 1$.

Suppose $\frac{B}{D} \neq 0$. Hence 0 and $\frac{B}{D}$ are the Picard's exceptional value of F as well as G since F and G share $(0, \infty)$. So using (7) and Lemma 5, we get by second fundamental theorem

$$\begin{aligned} nT(r,f) &\leq \overline{N}(r,\infty;F) + S(r,F) \\ &\leq \overline{N}(r,\infty;f) + S(r,f), \end{aligned}$$

which contradicts $n \geq 2m + 3$.

Next suppose $\frac{B}{D} = 0$. Then $\frac{A}{D} = 1$ and so $F = G$. Therefore

$$f^n + af^{n-m} = (\Delta_c f)^n + a(\Delta_c f)^{n-m}.$$

Next suppose that $\Delta_c f = hf$. If h is not constant, then

$$f^m(h^n - 1) = -a(h^{n-m} - 1), \tag{13}$$

$$i.e., \quad f^m = \frac{-a(h^{n-m} - 1)}{h^n - 1} = \frac{-a(h - u_1)(h - u_2)\ldots(h - u_{n-m-1})}{(h - v_1)(h - v_2)\ldots(h - v_{n-1})}, \tag{14}$$

where $u_j = e^{\frac{2\pi}{j}i}$ *for* $j = 1,\ 2,\ \ldots,\ n-m-1$ *and* $v_k = e^{\frac{2\pi}{k}i}$ *for* $k = 1,\ 2,\ \ldots,\ n-1$. *Since* n *and* $n-m$ *have no common factors, all* u_j *and* v_k *are different. Suppose* z_0 *is the zero of* $h - v_k$ *with multiplicity* s_k *for* $k = 1,\ 2,\ \ldots,\ n-1$. *Hence* z_0 *is the pole of* f *with multiplicity at least* m. *Thus*

$$m\overline{N}(r, v_k; h) \leq N(r, v_k; h) \leq T(r, h) + S(r, h). \tag{15}$$

Then by (15), we have

$$2 \geq \sum_{k=1}^{n-1} \Theta(v_k; h) = \sum_{k=1}^{n-1}\left(1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, v_k; h)}{T(r, h)}\right) \geq \sum_{k=1}^{n-1}\left(1 - \frac{1}{m}\right) = (n-1)\left(1 - \frac{1}{m}\right),$$

which is impossible as $n \geq 2m + 3$ *and* $m \geq 2$. *Therefore* h *is constant. Since* f *is a nonconstant meromorphic function, we deduce from (13) that* $h = 1$. *Hence* $\Delta_c f = f$.

Subcase 1.1.3. *Let* $A = 0$ *and* $C \neq 0$. *Then from (10), we have*

$$F = \frac{1}{\frac{C}{B}G + \frac{D}{B}}. \tag{16}$$

If F *has no* 1*-point, then as in Subcase* 1.1.2 *we obtain a contradiction.*

If F *and* G *have some* 1*-points then from (16) we get* $\frac{C}{B} + \frac{D}{B} = 1$.

Suppose $\frac{D}{B} \neq 0$. *We see that* ∞ *is the Picard's exceptional value of* F *as* F, G *share* (∞, ∞). *So by second fundamental theorem of Nevanlinna, Lemma 5 and (6), we get*

$$\begin{aligned} nT(r, f) &\leq \overline{N}(r, 0; F) + \overline{N}\left(r, \frac{B}{D}; F\right) + S(r, F) \\ &= \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + S(r, F) \\ &\leq 2\overline{N}(r, 0; f) + 2mT(r, f) + S(r, f), \end{aligned}$$

which is contradiction as $n \geq 2m + 3$.

Next suppose $\frac{D}{B} = 0$. *Then* $\frac{C}{B} = 1$ *and so* $FG = 1$. *Noting that* f *and* $\Delta_c f$ *share* $(0, \infty)$ *and* (∞, ∞) *we have* $N\left(r, \infty; \frac{\Delta_c f}{f}\right) = S(r, f)$ *and hence* $T\left(r, \frac{\Delta_c f}{f}\right) = S(r, f)$, *by Lemma 7. Therefore using Lemmas 5 and 6, we get*

$$\begin{aligned} 2nT(r, f) &= 2T(r, F) + O(1) \leq T\left(r, \frac{1}{F^2}\right) + S(r, F) \leq T\left(r, \frac{G}{F}\right) + S(r, F) \\ &\leq (n-m)T\left(r, \frac{\Delta_c f}{f}\right) + T(r, (\Delta_c f)^m + a) + T(r, f^m + a) + S(r, f) \\ &\leq 3mT(r, f) + S(r, f), \end{aligned}$$

which is a contradiction as $n \geq 2m + 3$.

Case 1.2. *Let $l=1$ and $H\not\equiv 0$. By second fundamental theorem of Nevanlinna, we have*

$$\begin{aligned} T(r,F)+T(r,G) &\leq \overline{N}(r,1;F)+\overline{N}(r,0;F)+\overline{N}(r,\infty;F) \\ &+\overline{N}(r,1;G)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G) \\ &-\overline{N}_0(r,0;F')-\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \quad (17) \end{aligned}$$

As F and G share $(1,1)$, we have $N_E^{1)}(r,1;F)=N(r,1;F\mid=1)$ and hence using (5), Lemma 2, Lemmas 4 and 8 for $s=1$, we get

$$\begin{aligned} &\overline{N}(r,1;F)+\overline{N}(r,1;G) \\ \leq\ & \frac{1}{2}[T(r,F)+T(r,G)+\overline{N}_*(r,1;F,G)] \\ &+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G) \\ \leq\ & \frac{1}{4}[2T(r,F)+2T(r,G)+\overline{N}(r,0;F)+\overline{N}(r,0;G)+2\overline{N}(r,\infty;F)] \\ &+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \quad (18) \end{aligned}$$

Using (4), (7) and Lemma 8 for $s=1$, we obtain from Lemma 12

$$(2n-4)\overline{N}(r,\infty;f)\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F)+S(r,G). \quad (19)$$

From (17)-(19) and Lemma 5, we get

$$\begin{aligned} &n(T(r,f)+T(r,\Delta_c f)) \\ \leq\ & \frac{5}{2}\overline{N}(r,0;F)+\frac{5}{2}\overline{N}(r,0;G)+5\overline{N}(r,\infty;F)+S(r,F)+S(r,G) \\ \leq\ & \left[\frac{5}{2}+\frac{5}{2n-4}\right][\overline{N}(r,0;F)+\overline{N}(r,0;G)]+S(r,F)+S(r,G) \\ \leq\ & \left[\frac{5m}{2}+\frac{5}{2}+\frac{5m+5}{2n-4}\right]\{T(r,f)+T(r,\Delta_c f)\}+S(r,f)+S(r,\Delta_c f), \end{aligned}$$

which is a contradiction as $n\geq \frac{5m}{2}+4$. Hence $H\equiv 0$. The rest of the theorem follows from the proof of Case 1.1.

Case 1.3. *Let $l=0$ and $H\not\equiv 0$. By second fundamental theorem of Nevanlinna, we have*

$$\begin{aligned} T(r,F)+T(r,G) &\leq \overline{N}(r,1;F)+\overline{N}(r,0;F)+\overline{N}(r,\infty;F) \\ &+\overline{N}(r,1;G)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G) \\ &-\overline{N}_0(r,0;F')-\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \quad (20) \end{aligned}$$

Since F and G share $(1,0)$, using (5), Lemma 1, Lemmas 4 and 8 for $s=0$, *we get*

$$\begin{aligned}
&\overline{N}(r,1;F)+\overline{N}(r,1;G)\\
\le\ &\frac{1}{2}[T(r,F)+T(r,G)+3\overline{N}_*(r,1;F,G)]\\
&+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\
\le\ &\frac{1}{2}[T(r,F)+T(r,G)+3\overline{N}(r,0;F)+3\overline{N}(r,0;G)+6\overline{N}(r,\infty;F)]\\
&+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \qquad (21)
\end{aligned}$$

Using (4), (7) and Lemma 8 for $s=0$, we obtain from Lemma 12

$$(n-3)\overline{N}(r,\infty;f)\le\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F)+S(r,G). \qquad (22)$$

From (20)-(22) and Lemma 5, we get

$$\begin{aligned}
&n(T(r,f)+T(r,\Delta_c f))\\
\le\ &5\overline{N}(r,0;F)+5\overline{N}(r,0;G)+10\overline{N}(r,\infty;F)+S(r,F)+S(r,G)\\
\le\ &\left[5+\frac{10}{n-3}\right](\overline{N}(r,0;F)+\overline{N}(r,0;G))+S(r,F)+S(r,G)\\
\le\ &\left[5m+5+\frac{10m+10}{n-3}\right]\{T(r,f)+T(r,\Delta_c f)\}+S(r,f)+S(r,\Delta_c f),
\end{aligned}$$

which is a contradiction as $n\ge 5m+8$. Hence $H\equiv 0$. The rest of the theorem follows from the proof of Case 1.1.

If $V\equiv 0$, then from Lemma 10 we have $F=G$ and so the theorem follows from Case 1.1. □

Proof of the Theorem 2. Let F and G be the same as in Lemma 11 and $V\not\equiv 0$. Since $E_{f(z)}(S_0,l)=E_{\Delta_c f(z)}(S_0,l)$, $E_{f(z)}(S_2,\infty)=E_{\Delta_c f(z)}(S_2,\infty)$ and $E_{f(z)}(S_3,0)=E_{\Delta_c f(z)}(S_3,0)$, it follows that F and G share $(1,l)$, (∞,∞) and $(0,0)$. We now discuss the following three cases separately.

Case 2.1. *First we suppose that $l\ge 2$ and $H\not\equiv 0$. By the second fundamental theorem of Nevanlinna we have*

$$T(r,F)\le\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)-\overline{N}_0(r,0;F')+S(r,F), \qquad (23)$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$.

If F and G share $(1,s)$ where $0\le s<\infty$, (∞,∞) and $(0,0)$, we see from the definition of V and H

$$N(r,\infty;V)\le\overline{N}_*(r,1;F,G)+\overline{N}_*(r,0;F,G)+S(r,F)+S(r,G) \qquad (24)$$

$$\begin{aligned}
\text{and}\quad N(r,\infty;H)\ \le\ &\overline{N}_*(r,0;F,G)+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;F')\\
&+\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \qquad (25)
\end{aligned}$$

Since F and G share (∞,∞) and $(0,0)$, we have from Lemma 5

$$\overline{N}(r,0;G)=\overline{N}(r,0;F)=\overline{N}(r,0;f)+mT(r,f)+S(r,f), \tag{26}$$

$$\overline{N}(r,\infty;G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f)+S(r,f), \tag{27}$$

and

$$\overline{N}_*(r,0;F,G)\le\overline{N}(r,0;F)=\frac{1}{2}[\overline{N}(r,0;F)+\overline{N}(r,0;G)]. \tag{28}$$

Using (24), (27) and Lemma 9 for $s=2$, we obtain from Lemma 12

$$(2n-3)\overline{N}(r,\infty;f)\le 3\overline{N}(r,0;F)+S(r,F). \tag{29}$$

As F and G share $(1,2)$, we obtain from (25), Lemma 2 and Lemma 3

$$\begin{aligned}\overline{N}(r,1;F) &= N(r,1;F\mid=1)+\overline{N}(r,1;F\mid\ge 2)\\ &= N(r,1;F\mid=1)+\overline{N}(r,1;G\mid\ge 2)\\ &\le N(r,\infty;H)+\overline{N}(r,1;G\mid\ge 2)+S(r,F)+S(r,G)\\ &\le \overline{N}_*(r,0;F,G)+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')\\ &\quad+\overline{N}(r,1;G\mid\ge 2)+S(r,F)+S(r,G)\\ &\le \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}_0(r,0;F')\\ &\quad+S(r,F)+S(r,G).\end{aligned} \tag{30}$$

Using (25)-(30) and Lemma 5, we obtain from (23)

$$\begin{aligned}nT(r,f) &\le 2\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)\\ &\quad+\overline{N}(r,\infty;G)+S(r,F)+S(r,G)\\ &\le 3\overline{N}(r,0;F)+2\overline{N}(r,\infty;f)+S(r,F)\\ &\le \left[3+\frac{6}{2n-3}\right]\overline{N}(r,0;F)+S(r,F)\\ &\le \left[3+\frac{6}{2n-3}\right]\{\overline{N}(r,0;f)+mT(r,f)\}+S(r,f)\\ &\le \left[3m+3+\frac{6m+6}{2n-3}\right]T(r,f)+S(r,f),\end{aligned}$$

which contradicts the fact $n\ge 3m+5$. Hence $H\equiv 0$. The rest of the theorem follows from Case 1.1 of Theorem 1.

Case 2.2. *Let $l=1$ and $H\not\equiv 0$. By the second fundamental theorem of Nevanlinna, we have*

$$\begin{aligned}T(r,F)+T(r,G) &\le \overline{N}(r,1;F)+\overline{N}(r,0;F)+\overline{N}(r,\infty;F)\\ &\quad+\overline{N}(r,1;G)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)\\ &\quad-\overline{N}_0(r,0;F')-\overline{N}_0(r,0;G')+S(r,F)+S(r,G).\end{aligned} \tag{31}$$

Since F *and* G *share* $(1,1)$, $N_E^{1)}(r,1;F) = N(r,1;F \mid= 1)$ *and so using (25), Lemma 2, Lemmas 4 and 8 for* $s = 1$, *we get*

$$\begin{aligned} & \overline{N}(r,1;F) + \overline{N}(r,1;G) \\ \leq\ & \frac{1}{2}[T(r,F) + T(r,G) + \overline{N}_*(r,1;F,G) + 2\overline{N}_*(r,0;F,G)] \\ & + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G) \\ \leq\ & \frac{1}{4}[2T(r,F) + 2T(r,G) + 3\overline{N}(r,0;F) + 3\overline{N}(r,0;G) + 2\overline{N}(r,\infty;F)] \\ & + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G). \end{aligned} \tag{32}$$

Using (24), (27) and Lemma 8 for $s = 1$, *we obtain from Lemma 12*

$$(n-2)\overline{N}(r,\infty;f) \leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F) + S(r,G). \tag{33}$$

From (31)-(33) and Lemma 5, we get

$$\begin{aligned} & n(T(r,f) + T(r,\Delta_c f)) \\ \leq\ & \frac{7}{2}\overline{N}(r,0;F) + \frac{7}{2}\overline{N}(r,0;G) + 5\overline{N}(r,\infty;F) + S(r,F) + S(r,G) \\ \leq\ & \left[\frac{7}{2} + \frac{5}{n-2}\right][\overline{N}(r,0;F) + \overline{N}(r,0;G)] + S(r,F) + S(r,G) \\ \leq\ & \left[\frac{7m}{2} + \frac{7}{2} + \frac{5m+5}{n-2}\right](T(r,f) + T(r,\Delta_c f)) + S(r,f) + S(r,\Delta_c f), \end{aligned}$$

which is a contradiction as $n \geq \frac{7m}{2} + \frac{11}{2}$. *Hence* $H \equiv 0$. *The rest of the theorem follows from Case* 1.1 *of Theorem 1.*

Case 2.3. *Let* $l = 0$ *and* $H \not\equiv 0$. *By the second fundamental theorem of Nevanlinna, we have*

$$\begin{aligned} T(r,F) + T(r,G) \ \leq\ & \overline{N}(r,1;F) + \overline{N}(r,0;F) + \overline{N}(r,\infty;F) \\ & + \overline{N}(r,1;G) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) \\ & - \overline{N}_0(r,0;F') - \overline{N}_0(r,0;G') + S(r,F) + S(r,G). \end{aligned} \tag{34}$$

Since F *and* G *share* $(1,0)$, *using (25), Lemma 1, Lemmas 4 and 8 for* $s = 0$, *we get*

$$\begin{aligned} & \overline{N}(r,1;F) + \overline{N}(r,1;G) \\ \leq\ & \frac{1}{2}[T(r,F) + T(r,G) + 3\overline{N}_*(r,1;F,G) + 2\overline{N}_*(r,0;F,G)] \\ & + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G) \\ \leq\ & \frac{1}{2}[T(r,F) + T(r,G) + 4\overline{N}(r,0;F) + 4\overline{N}(r,0;G) + 6\overline{N}(r,\infty;F)] \\ & + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G). \end{aligned} \tag{35}$$

Using (24), (27), (28) and Lemma 8 for $s = 0$, we obtain from Lemma 12

$$(2n-6)\overline{N}(r,\infty;f) \leq 3(\overline{N}(r,0;F)+\overline{N}(r,0;G))+S(r,F)+S(r,G). \tag{36}$$

From (34)-(36) and Lemma 5, we get

$$\begin{aligned} & n(T(r,f)+T(r,\Delta_c f)) \\ \leq\ & 6\overline{N}(r,0;F)+6\overline{N}(r,0;G)+10\overline{N}(r,\infty;F)+S(r,F)+S(r,G) \\ \leq\ & \left[6+\frac{15}{n-3}\right](\overline{N}(r,0;F)+\overline{N}(r,0;G))+S(r,F)+S(r,G) \\ \leq\ & \left[6m+6+\frac{15m+15}{n-3}\right](T(r,f)+T(r,\Delta_c f))+S(r,f)+S(r,\Delta_c f), \end{aligned}$$

which is a contradiction as $n \geq 6m+9$. Hence $H \equiv 0$. The rest of the theorem follows from Case 1.1 *of Theorem 1.*

If $V \equiv 0$, then from Lemma 10 we have $F = G$ and so the theorem follows from Case 1.1 of Theorem 1. □

Proof of the Theorem 3. Let F and G be the same as in Lemma 11 and $V \not\equiv 0$. Since $E_{f(z)}(S_0,l) = E_{\Delta_c f(z)}(S_0,l)$, $E_{f(z)}(S_2,0) = E_{\Delta_c f(z)}(S_2,0)$ and $E_{f(z)}(S_3,\infty)$ $= E_{\Delta_c f(z)}(S_3,\infty)$, it follows that F and G share $(1,l)$, $(\infty,0)$ and $(0,\infty)$. We now discuss three cases as follows.

Case 3.1. *Let $l \geq 2$ and $H \not\equiv 0$. By the second fundamental theorem of Nevanlinna we have*

$$T(r,F) \leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)-\overline{N}_0(r,0;F')+S(r,F), \tag{37}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$.

If F and G share $(1,s)$ where $0 \leq s < \infty$, $(\infty,0)$ and $(0,\infty)$, we see from the definition of V and H

$$N(r,\infty;V) \leq \overline{N}_*(r,1;F,G)+S(r,F)+S(r,G) \tag{38}$$

$$\begin{aligned} \text{and} \quad N(r,\infty;H) \leq\ & \overline{N}_*(r,\infty;F,G)+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;F') \\ & +\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \end{aligned} \tag{39}$$

Since F and G share $(\infty,0)$ and $(0,\infty)$, we have from Lemma 5

$$\overline{N}(r,0;G)=\overline{N}(r,0;F)=\overline{N}(r,0;f)+mT(r,f)+S(r,f), \tag{40}$$

$$\overline{N}(r,\infty;G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f)+S(r,f), \tag{41}$$

$$\text{and} \quad \overline{N}_*(r,\infty;F,G) \leq \overline{N}(r,\infty;F). \tag{42}$$

Using (38), (41) and Lemma 9 for $s = 2$, *we obtain from Lemma 11 for* $k = 0$

$$(2n-3)\overline{N}(r,\infty;f) \leq \overline{N}(r,0;F) + S(r,F). \tag{43}$$

As F *and* G *share* $(1,2)$, *we obtain from (39), (42), Lemma 2 and Lemma 3*

$$\begin{aligned}
\overline{N}(r,1;F) &= N(r,1;F \mid= 1) + \overline{N}(r,1;F \mid\geq 2) \\
&= N(r,1;F \mid= 1) + \overline{N}(r,1;G \mid\geq 2) \\
&\leq N(r,\infty;H) + \overline{N}(r,1;G \mid\geq 2) + S(r,F) + S(r,G) \\
&\leq \overline{N}_*(r,\infty;F,G) + \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') \\
&\quad + \overline{N}(r,1;G \mid\geq 2) + S(r,F) + S(r,G) \\
&\leq \overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}_0(r,0;F') \\
&\quad + S(r,F) + S(r,G).
\end{aligned} \tag{44}$$

Using (40), (41), (43), (44) and Lemma 5, we obtain from (37)

$$\begin{aligned}
nT(r,f) &\leq \overline{N}(r,0;F) + 2\overline{N}(r,\infty;F) + \overline{N}(r,0;G) \\
&\quad + \overline{N}(r,\infty;G) + S(r,F) + S(r,G) \\
&\leq 2\overline{N}(r,0;F) + 3\overline{N}(r,\infty;F) + S(r,F) \\
&\leq \left[2 + \frac{3}{2n-3}\right] \overline{N}(r,0;F) + S(r,F) \\
&\leq \left[2 + \frac{3}{2n-3}\right] \{\overline{N}(r,0;f) + mT(r,f)\} + S(r,f) \\
&\leq \left[2m + 2 + \frac{3m+3}{2n-3}\right] T(r,f) + S(r,f),
\end{aligned}$$

which contradicts the fact $n \geq 2m+3$. *Hence* $H \equiv 0$. *The rest of the theorem follows from Case* 1.1 *of Theorem 1.*

Case 3.2. *Let* $l = 1$ *and* $H \not\equiv 0$. *By the second fundamental theorem of Nevanlinna, we have*

$$\begin{aligned}
T(r,F) + T(r,G) &\leq \overline{N}(r,1;F) + \overline{N}(r,0;F) + \overline{N}(r,\infty;F) \\
&\quad + \overline{N}(r,1;G) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) \\
&\quad - \overline{N}_0(r,0;F') - \overline{N}_0(r,0;G') + S(r,F) + S(r,G).
\end{aligned} \tag{45}$$

Since F *and* G *share* $(1,1)$, $N_E^{1)}(r,1;F) = N(r,1;F \mid= 1)$ *and so using (39), (42), Lemma 2, Lemmas 4 and 8 for* $s = 1$, *we get*

$$\begin{aligned}
&\overline{N}(r,1;F) + \overline{N}(r,1;G) \\
\leq\ &\frac{1}{2}[T(r,F) + T(r,G) + \overline{N}_*(r,1;F,G) + 2\overline{N}_*(r,\infty;F,G)] \\
&+ \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G) \\
\leq\ &\frac{1}{4}[2T(r,F) + 2T(r,G) + \overline{N}(r,0;F) + \overline{N}(r,0;G) + 6\overline{N}(r,\infty;F)] \\
&+ \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G).
\end{aligned} \tag{46}$$

Using (38), (41) and Lemma 8 for $s=1$, we obtain from Lemma 11 for $k=0$

$$(2n-4)\overline{N}(r,\infty;f) \leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F)+S(r,G). \tag{47}$$

Using (40), (41), (46), (47) and Lemma 5, we get from (45)

$$\begin{aligned} & n(T(r,f)+T(r,\Delta_c f)) \\ \leq\ & \frac{5}{2}\overline{N}(r,0;F)+\frac{5}{2}\overline{N}(r,0;G)+7\overline{N}(r,\infty;F)+S(r,F)+S(r,G) \\ \leq\ & \left[\frac{5}{2}+\frac{7}{2n-4}\right](\overline{N}(r,0;F)+\overline{N}(r,0;G))+S(r,F)+S(r,G) \\ \leq\ & \left[\frac{5m}{2}+\frac{5}{2}+\frac{7m+7}{2n-4}\right](T(r,f)+T(r,\Delta_c f))+S(r,f)+S(r,\Delta_c f), \end{aligned}$$

which is a contradiction as $n \geq \frac{5m}{2}+\frac{9}{2}$. Hence $H \equiv 0$. The rest of the theorem follows from Case 1.1 of Theorem 1.

Case 3.3. *Let $l=0$ and $H \not\equiv 0$. Using the second fundamental theorem of Nevanlinna, we have*

$$\begin{aligned} T(r,F)+T(r,G) \leq\ & \overline{N}(r,1;F)+\overline{N}(r,0;F)+\overline{N}(r,\infty;F) \\ & +\overline{N}(r,1;G)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G) \\ & -\overline{N}_0(r,0;F')-\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \end{aligned} \tag{48}$$

Since F and G share $(1,0)$, using (39), (42), Lemma 1, Lemmas 4 and 8 for $s=0$, we get

$$\begin{aligned} & \overline{N}(r,1;F)+\overline{N}(r,1;G) \\ \leq\ & \frac{1}{2}[T(r,F)+T(r,G)+3\overline{N}_*(r,1;F,G)+2\overline{N}_*(r,\infty;F,G)] \\ & +\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G) \\ \leq\ & \frac{1}{2}[T(r,F)+T(r,G)+3\overline{N}(r,0;F)+3\overline{N}(r,0;G)+8\overline{N}(r,\infty;F)] \\ & +\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \end{aligned} \tag{49}$$

Using (38), (41) and Lemma 8 for $s=0$, we obtain from Lemma 11 for $k=0$

$$(n-3)\overline{N}(r,\infty;f) \leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F)+S(r,G). \tag{50}$$

Using (40), (41), (49), (50) and Lemma 5, we get from (48)

$$\begin{aligned} & n(T(r,f)+T(r,\Delta_c f)) \\ \leq\ & 5\overline{N}(r,0;F)+5\overline{N}(r,0;G)+12\overline{N}(r,\infty;F)+S(r,F)+S(r,G) \\ \leq\ & \left[5+\frac{12}{n-3}\right](\overline{N}(r,0;F)+\overline{N}(r,0;G))+S(r,F)+S(r,G) \\ \leq\ & \left[5m+5+\frac{12m+12}{n-3}\right](T(r,f)+T(r,\Delta_c f))+S(r,f)+S(r,\Delta_c f), \end{aligned}$$

which is a contradiction as $n \geq 5m+8$. Hence $H \equiv 0$. The rest of the theorem follows from Case 1.1 of Theorem 1.

If $V \equiv 0$, then from Lemma 10 we have $F = G$ and so the theorem follows from Case 1.1 of Theorem 1. □

Proof of the Theorem 4. Let F and G be the same as in Lemma 11 and $V \not\equiv 0$. Since $E_{f(z)}(S_0, l) = E_{\Delta_c f(z)}(S_0, l)$ and $E_{f(z)}(S_j, 0) = E_{\Delta_c f(z)}(S_j, 0)$ $(j = 2,\ 3)$, it follows that F and G share $(1, l)$, $(\infty, 0)$ and $(0, 0)$. We now discuss the following cases.

Case 4.1. *We assume that $l \geq 2$ and $H \not\equiv 0$. By the second fundamental theorem of Nevanlinna we have*

$$T(r, F) \leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}(r, 1; F) - \overline{N}_0(r, 0; F') + S(r, F), \tag{51}$$

where $\overline{N}_0(r, 0; F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$.

If F and G share $(1, s)$ where $0 \leq s < \infty$, $(\infty, 0)$ and $(0, 0)$, from the definition of V and H it is obvious that

$$N(r, \infty; V) \leq \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, 0; F, G) + S(r, F) + S(r, G) \tag{52}$$

and

$$\begin{aligned} N(r, \infty; H) \quad \leq \quad & \overline{N}_*(r, 0; F, G) + \overline{N}_*(r, \infty; F, G) + \overline{N}_*(r, 1; F, G) \\ & + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') + S(r, F) + S(r, G). \end{aligned} \tag{53}$$

Since F and G share $(\infty, 0)$ and $(0, 0)$, we have from Lemma 5

$$\overline{N}(r, 0; G) = \overline{N}(r, 0; F) = \overline{N}(r, 0; f) + mT(r, f) + S(r, f), \tag{54}$$

$$\overline{N}(r, \infty; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f) + S(r, f), \tag{55}$$

$$\overline{N}_*(r, 0; F, G) \leq \overline{N}(r, 0; F) = \frac{1}{2}[\overline{N}(r, 0; F) + \overline{N}(r, 0; G)] \tag{56}$$

$$\textit{and} \qquad \overline{N}_*(r, \infty; F, G) \leq \overline{N}(r, \infty; F). \tag{57}$$

Using (52), (55), (56) and Lemma 9 for $s = 2$, we obtain from Lemma 11 for $k = 0$

$$(2n - 3)\overline{N}(r, \infty; f) \leq 3\overline{N}(r, 0; F) + S(r, F) + S(r, G). \tag{58}$$

As F and G share $(1, 2)$, we obtain from (53), (56), (57), Lemmas 2 and 3

$$\begin{aligned} \overline{N}(r, 1; F) \quad & = \quad N(r, 1; F \mid= 1) + \overline{N}(r, 1; F \mid\geq 2) \\ & = \quad N(r, 1; F \mid= 1) + \overline{N}(r, 1; G \mid\geq 2) \\ & \leq \quad N(r, \infty; H) + \overline{N}(r, 1; G \mid\geq 2) + S(r, F) + S(r, G) \\ & \leq \quad \overline{N}_*(r, 0; F, G) + \overline{N}_*(r, \infty; F, G) + \overline{N}_*(r, 1; F, G) + \overline{N}_0(r, 0; F') \\ & \qquad + \overline{N}_0(r, 0; G') + \overline{N}(r, 1; G \mid\geq 2) + S(r, F) + S(r, G) \\ & \leq \quad \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) \\ & \qquad + \overline{N}_0(r, 0; F') + S(r, F) + S(r, G). \end{aligned} \tag{59}$$

Using (54), (55), (58), (59) and Lemma 5, we obtain from (51)

$$\begin{aligned} nT(r,f) &\leq 3\overline{N}(r,0;F)+3\overline{N}(r,\infty;F)+S(r,F)\\ &\leq \left[3+\frac{9}{2n-3}\right]\overline{N}(r,0;F)+S(r,F)\\ &\leq \left[3+\frac{9}{2n-3}\right]\{\overline{N}(r,0;f)+mT(r,f)\}+S(r,f)\\ &\leq \left[3m+3+\frac{9m+9}{2n-3}\right]T(r,f)+S(r,f), \end{aligned}$$

which contradicts the fact that $n \geq 3m+5$. *Hence* $H \equiv 0$. *The rest of the theorem follows from Case* 1.1 *of Theorem 1.*

Case 4.2. *Let* $l = 1$ *and* $H \not\equiv 0$. *By the second fundamental theorem of Nevanlinna, we have*

$$\begin{aligned} T(r,F)+T(r,G) \leq\ & \overline{N}(r,1;F)+\overline{N}(r,0;F)+\overline{N}(r,\infty;F)\\ &+\overline{N}(r,1;G)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)\\ &-\overline{N}_0(r,0;F')-\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \end{aligned} \tag{60}$$

Since F *and* G *share* $(1,1)$, *we have* $N_E^{1)}(r,1;F) = N(r,1;F \mid= 1)$ *and so using (53), (56), (57), Lemma 2, Lemmas 4 and 8 for* $s = 1$, *we get*

$$\begin{aligned} &\overline{N}(r,1;F)+\overline{N}(r,1;G)\\ \leq\ & \frac{1}{2}[T(r,F)+T(r,G)+\overline{N}_*(r,1;F,G)]+\overline{N}_*(r,0;F,G)\\ &+\overline{N}_*(r,\infty;F,G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\ \leq\ & \frac{1}{4}[2T(r,F)+2T(r,G)+3\overline{N}(r,0;F)+3\overline{N}(r,0;G)+6\overline{N}(r,\infty;F)]\\ &+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \end{aligned} \tag{61}$$

Using (52), (55), (56) and Lemma 8 for $s = 1$, *we obtain from Lemma 11 for* $k = 0$

$$(n-2)\overline{N}(r,\infty;f) \leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F+S(r,G). \tag{62}$$

Using (54), (55), (61), (62) and Lemma 5, we get from (60)

$$\begin{aligned} &n(T(r,f)+T(r,\Delta_c f))\\ \leq\ & \frac{7}{2}\overline{N}(r,0;F)+\frac{7}{2}\overline{N}(r,0;G)+7\overline{N}(r,\infty;F)+S(r,F)+S(r,G)\\ \leq\ & \left[\frac{7}{2}+\frac{7}{n-2}\right](\overline{N}(r,0;F)+\overline{N}(r,0;G))+S(r,F)+S(r,G)\\ \leq\ & \left[\frac{7m}{2}+\frac{7}{2}+\frac{7m+7}{n-2}\right](T(r,f)+T(r,\Delta_c f))+S(r,f)+S(r,\Delta_c f), \end{aligned}$$

which is a contradiction as $n \geq \frac{7m}{2}+6$. *Hence* $H \equiv 0$. *The rest of the theorem follows from the proof of Case* 1.1 *of Theorem 1.*

Case 4.3. *Let* $l = 0$ *and* $H \not\equiv 0$. *By the second fundamental theorem of Nevanlinna, we have*

$$\begin{aligned} T(r,F)+T(r,G) &\leq \overline{N}(r,1;F)+\overline{N}(r,0;F)+\overline{N}(r,\infty;F) \\ &\quad +\overline{N}(r,1;G)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G) \\ &\quad -\overline{N}_0(r,0;F')-\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \quad (63) \end{aligned}$$

Since F *and* G *share* $(1,0)$, *using (53), (56), (57), Lemma 1, Lemmas 4 and 8 for* $s=0$, *we get*

$$\begin{aligned} &\overline{N}(r,1;F)+\overline{N}(r,1;G) \\ \leq\ & \frac{1}{2}[T(r,F)+T(r,G)+3\overline{N}_*(r,1;F,G)]+\overline{N}_*(r,0;F,G) \\ &+\overline{N}_*(r,\infty;F,G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G) \\ \leq\ & \frac{1}{2}[T(r,F)+T(r,G)+4\overline{N}(r,0;F)+4\overline{N}(r,0;G)+8\overline{N}(r,\infty;F)] \\ &+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \quad (64) \end{aligned}$$

Using (52), (55), (56) and Lemma 8 for $s=0$, *we obtain from Lemma 11 for* $k=0$

$$(2n-6)\overline{N}(r,\infty;f) \leq 3[\overline{N}(r,0;F)+\overline{N}(r,0;G)]+S(r,F)+S(r,G). \quad (65)$$

Using (54), (55), (64), (65) and Lemma 5, we get from (63)

$$\begin{aligned} &n(T(r,f)+T(r,\Delta_c f)) \\ \leq\ & 6\overline{N}(r,0;F)+6\overline{N}(r,0;G)+12\overline{N}(r,\infty;F)+S(r,F)+S(r,G) \\ \leq\ & \left[6+\frac{18}{n-3}\right](\overline{N}(r,0;F)+\overline{N}(r,0;G))+S(r,F)+S(r,G) \\ \leq\ & \left[6m+6+\frac{18m+18}{n-3}\right](T(r,f)+T(r,\Delta_c f))+S(r,f)+S(r,\Delta_c f), \end{aligned}$$

which is a contradiction as $n \geq 6m+10$. *Hence* $H \equiv 0$. *The rest of the theorem follows from Case* 1.1 *of Theorem 1.*

If $V \equiv 0$, then from Lemma 10 we have $F = G$ and so the theorem follows from Case 1.1 of Theorem 1. □

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