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A NONLINEAR SECOND-ORDER PARTIAL DIFFERENTIAL EQUATION-BASED ALGORITHM FOR ADDITIVE NOISE REDUCTION

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Abstract

An effective nonlinear anisotropic diffusion-based algorithm for image restoration is proposed in this work. The technique considered here employs a novel second-order partial differential equation (PDE) model, composed of a hyperbolic equation and several boundary conditions. Our method provides satisfactory feature-preserving filtering results and overcomes successfully the blurring and staircase effects. It also produces sharper edges because of its hyperbolic equation that is based on a second time derivative. The proposed PDE model is well-posed, admitting a unique and weak solution under certain assumptions, which is computed by using an iterative finite difference-based explicit numerical approximation scheme. Some successful restoration experiments and method comparisons are also described in this paper.

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Key words: image restoration, additive noise, nonlinear hyperbolic partial differential equation, well-posed anisotropic diffusion model, finite differences, numerical approximation scheme.

1 Introduction

The partial differential equations have been increasingly used for solving various image processing and analysis tasks in the last three decades. Image denoising and restoration domain has been successfully approached using PDE-based algorithms [1].

The second-order nonlinear PDE-based denoising schemes represent considerably better smoothing solutions than the conventional two-dimension image filters [2], since those

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classic filters may generate the unintended blurring effect that affects the edges and other essential details. These second-order diffusion-based techniques that provide satisfactory feature-preserving filtering results and overcome successfully the image blurring, have been developed in PDE form, such as the influential Perona-Malik anisotropic diffusion scheme and other models derived from it [3, 4], or variational form, such as the total-variation based models inspired by TV-ROF Denoising [5, 6, 7, 8].

Unfortunately, they could also generate the so-called staircasing, or blocky, effect [9]. The nonlinear fourth-order diffusion-based algorithms, such as those inspired by the influential isotropic diffusion scheme introduced by You and Kaveh [10], overcome successfully the staircase effect, but may produce over-filtering and speckle noise.

Improving the second-order PDE models such that they become able to deal with all the undesirable effects represents another solution. We have proposed many improved nonlinear PDE-based restoration techniques in our past works [11, 12, 13]. In this paper we consider a novel hyperbolic diffusion-based detail-preserving technique for additive noise removal. The proposed nonlinear second-order hyperbolic PDE denoising model is detailed in the following section. The differential model is solved in the third section using an iterative finite difference method-based numerical approximation algorithm that is constructed by us.

The image restoration experiments and the method comparisons performed by us are discussed in the fourth section. The conclusions are drawn in the fifth section and the paper ends with a list of references.

2 Nonlinear Second-order Hyperbolic PDE Model

A novel second-order nonlinear PDE-based model for additive denoising is proposed in this section. It is based on the next hyperbolic differential equation with several boundary conditions, having its origin in the P-M anisotropic diffusion [3] and the mean curvature motion (MCM)[14] and improving both of them:

$$\begin{cases} \lambda \dfrac{\partial^2 u}{\partial t^2} + \alpha \dfrac{\partial u}{\partial t} - \xi(\|\nabla u\|)\mathrm{div}(\psi_u(\|\nabla u\|)\nabla u) + E \cdot \nabla u = 0 \\ u(x,y,0) = u_0(x,y),\ \forall (x,y) \in \Omega \\ \dfrac{\partial u}{\partial t}(0,x,y) = u_1(x,y) \\ u(t,x,y) = 0,\ \forall (x,y) \in \partial\Omega \end{cases} \tag{1}$$

where $\alpha, \lambda \in (0,1]$ and the drift term is modeled as:

$$E(x,y) = \left(e^{-\frac{x^2+y^2}{\nu}}, e^{-\frac{x+y}{r}} \right),\ \nu, r \geq 1 \tag{2}$$

It contains a component that controls the speed of the diffusion process and enhances the image boundaries, which is based on the next function:

$$\begin{cases} \xi : (0,\infty) \to (0,\infty) \\ \xi(s) = \gamma(\beta s^k + \eta)^{\frac{1}{k+1}} \end{cases} \tag{3}$$

where $\gamma, \eta, \beta \in (0,4]$ and $k \in [0,2]$.

The edge-stopping function of this PDE-based denoising model depends on the evolving image and has the following form:

$$\psi_u : [0,\infty) \to [0,\infty),\ \psi_u(s) = \frac{\varepsilon}{\varphi(u)\left|\ln\left(\frac{s}{\varphi(u)}\right)\right|^k + \rho\left(\frac{s}{\varphi(u)}\right)^{k+1}} \tag{4}$$

where $\rho, \varepsilon \in (1,5]$.

The conductance parameter in (4) is constructed as the following statistics-based function:

$$\varphi(u) = \|\zeta\mu(\|\nabla u\|) + \delta\ median(\|\nabla u\|)\| \tag{5}$$

where $\zeta, \delta \in (1,2)$, $\mu(\ )$ returns the average value of the argument and $median(\ )$ computes the median value. This diffusivity (edge-stopping) function is properly modeled for an effective diffusion-based denoising process, being positive, monotonically decreasing and converging to zero [1, 3].

The proposed nonlinear second-order hyperbolic diffusion-based model provides an effective additive noise removal and deblurring. Since it is also based on a second time derivative, it removes successfully the diffusion effect in the vicinity of the image boundaries, thus producing much sharper edges and better details.

Also, the considered PDE restoration model is well-posed, since a unique weak solution exists under certain conditions. That solution, which represents the recovered image, is determined by solving numerically this nonlinear PDE-based scheme. The numerical algorithm that solves the hyperbolic model is described in the next section.

3 Numerical Approximation Algorithm

We construct a numerical approximation scheme using the finite-difference method to solve the proposed differential model [15]. First, the time and space coordinates are quantized as follows:

$$x = ih,\ y = jh,\ t = n\Delta t,\ i \in \{1,\ldots,I\},\ j \in \{1,...,J\},\ n \in \{1,...,N\} \tag{6}$$

Then, the hyperbolic PDE in (1) is expressed as:

$$\lambda\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial u}{\partial t} = \xi(\|\Delta u\|)\left(\frac{\partial}{\partial x}(\psi_u(|\nabla u|)u_x) + \frac{\partial}{\partial y}(\psi_u(|\nabla u|)u_y)\right) - E\cdot\nabla u \tag{7}$$

Its left term is discretized by using finite differences [15], as follows:

$$\begin{aligned}\lambda\frac{u_{i,j}^{n+\Delta t} + 2u_{i,j}^{n} - u_{i,j}^{n-\Delta t}}{(\Delta t)^2} + \alpha\frac{u_{i,j}^{n+\Delta t} - u_{i,j}^{n}}{\Delta t}\\ = u_{i,j}^{n+\Delta t}\frac{(\lambda + \alpha\Delta t)}{(\Delta t)^2} + u_{i,j}^{n}\frac{2\lambda - \alpha\Delta t}{(\Delta t)^2} - u_{i,j}^{n-\Delta t}\frac{\lambda}{(\Delta t)^2}\end{aligned} \tag{8}$$

Next, the right term of (7) is approximated by using the central differences [14]. So, one computes $\xi_{i,j} = \xi(\|u_{i,j}\|)$ and $\psi_{i,j} = \psi_u(\|u_{i,j}\|)$, where

$$\|u_{i,j}\| \approx \sqrt{\left(\frac{u_{i+h,j} - u_{i-h,j}}{2h}\right)^2 + \left(\frac{u_{i,j+h} - u_{i,j-h}}{2h}\right)^2}.$$

Then, $\frac{\partial}{\partial x}\left(\psi_u(\|\nabla u\|)u_x\right)$ is discretized as $\psi_{i+\frac{h}{2},j}(u_{i+h,j} - u_{i,j}) - \psi_{i-\frac{h}{2},j}(u_{i,j} - u_{i-h,j})$, while $\frac{\partial}{\partial y}\left(\psi_u(\|\nabla u\|)u_y\right)$ is approximated as $\psi_{i,j+\frac{h}{2}}(u_{i,j+h} - u_{i,j}) - \psi_{i,j-\frac{h}{2}}(u_{i,j} - u_{i,j-h})$, where

$$\psi_{i\pm\frac{h}{2},j} = \frac{\psi_{i\pm h,j} + \psi_{i,j}}{2}, \quad \psi_{i,j\pm\frac{h}{2}} = \frac{\psi_{i,j\pm h} + \psi_{i,j}}{2} \tag{9}$$

One may use $h = 1$ and $\Delta t = 1$, respectively. Thus, one obtains the following implicit approximation:

$$\begin{aligned} u_{i,j}^{n+1}(\lambda+\alpha) + u_{i,j}^{n}(2\lambda-\alpha) - u_{i,j}^{n-1}\lambda = \xi_{i,j}(\psi_{i+\frac{1}{2},j}(u_{i+1,j}^{n} - u_{i,j}^{n}) \\ -\psi_{i-\frac{1}{2},j}(u_{i,j}^{n} - u_{i-1,j}^{n}) + \psi_{i,j+\frac{1}{2}}(u_{i,j+1}^{n} - u_{i,j}^{n}) - \psi_{i,j-\frac{1}{2}}(u_{i,j}^{n} - u_{i,j-1}^{n})) - E\cdot\nabla u \end{aligned} \tag{10}$$

It leads to the next explicit iterative finite difference-based numerical approximation scheme:

$$\begin{aligned} u_{i,j}^{n+1} = \frac{1}{\alpha+\lambda}&\left(u_{i,j}^{n}\left(\alpha - 2\lambda - \xi_{i,j}\left(\psi_{i+\frac{1}{2},j} + \psi_{i-\frac{1}{2},j} + \psi_{i,j+\frac{1}{2}} + \psi_{i,j-\frac{1}{2}}\right)\right)\right. \\ &+ u_{i+1,j}^{n}\psi_{i+\frac{1}{2},j}\xi_{i,j} + u_{i-1,j}^{n}\psi_{i-\frac{1}{2},j}\xi_{i,j} + u_{i,j+1}^{n}\psi_{i,j+\frac{1}{2}}\xi_{i,j} \\ &\left.+ u_{i,j-1}^{n}\psi_{i,j-\frac{1}{2}}\xi_{i,j} + u_{i,j}^{n-1}\lambda\right) - \left(e^{\frac{x^2+y^2}{\nu}}, e^{\frac{x+y}{r}}\right)\left(\frac{u_{i+1,j}^{n} - u_{i-1,j}^{n}}{2}, \frac{u_{i,j+1}^{n} - u_{i,j-1}^{n}}{2}\right) \end{aligned} \tag{11}$$

The explicit numerical approximation scheme (11) is stable and consistent to the nonlinear hyperbolic PDE-based model. It is also converging fast to its weak solution representing the filtered image. It is then used in our successful image restoration experiments which are described in the next section.

4 Denoising Experiments and Method Comparison

The second order hyperbolic diffusion-based filtering technique proposed here has been applied to hundreds of images corrupted with various levels of white additive Gaussian noise (AWGN). Several well-known image collections, such as the volumes of the USC-SIPI database, have been used in our experiments.

The denoising performance of the proposed approach has been assessed using similarity metrics such as Peak Signal to Noise Ratio (PSNR), Mean Squared Error (MSE) and Structural Similarity Index (SSIM) [16]. Our hyperbolic PDE-based technique removes successfully the additive noise, while avoiding the multiplicative (speckle) noise. It overcomes properly the image blurring and also alleviates the staircasing effect.

Also, it preserves the image details very well, enhancing the edges and other features. And it has a low execution time, of around 1 second, because of its fast-converging numerical approximation scheme.

The described restoration framework outperforms the classic two-dimension filters, such as the Average, Gaussian and other conventional filters, by providing a much better smoothing and deblurring. It also performs better than nonlinear second-order diffusion-based models inspired by Perona-Malik scheme and the total variation approaches derived from TV-ROF Denoising, since it provides a better filtering, sharper details and reduces the unintended staircase effect. It may even represent a better denoising solution than fourth-order PDE-based schemes, like You-Kaveh model, because it avoids the speckle noise and the image over-filtering.

The average PSNR values achieved by some PDE and non-PDE filtering methods when applied on the chosen set of corrupted images are displayed in Table 1. One can see that our restoration techniques gets higher values than other models.

Table 1. Average PSNR values

Restoration Technique	**Average PSNR value**
This PDE-based model	28.3446 (dB)
2D Gaussian filter	23.4271(dB)
Average filter	25.0026 (dB)
Wiener filter	25.8983 (dB)
Perona-Malik	26.3244 (dB)
TV-ROF model	27.8782 (dB)

A method comparison example is described in Figure 1. The original *Lenna* image is displayed in (a) and its version corrupted by an additive Gaussian noise of $\mu = 0.1$ and $var = 0.08$ is depicted in (b).

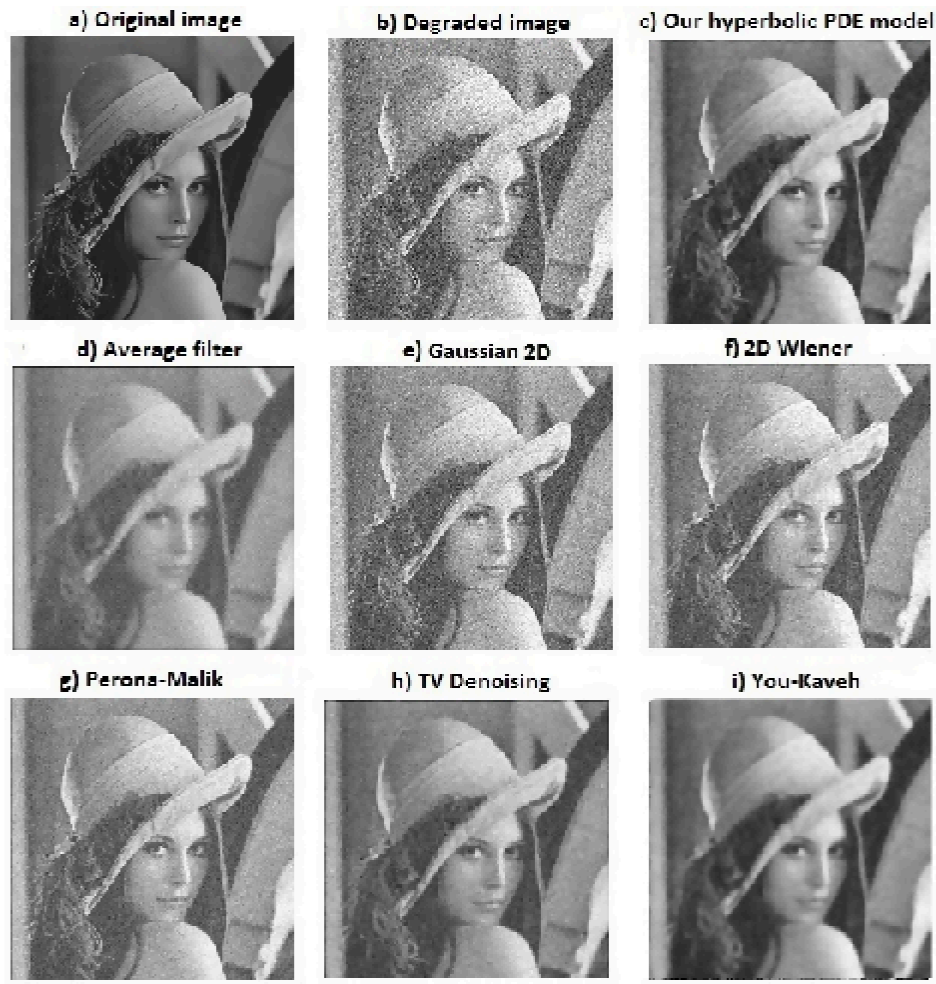

Figure 1: Method comparison example

The restoration achieved by our hyperbolic model is displayed in (c), the Average filtering is depicted in (d), Gaussian 2D filtering in (e), Wiener filtering in (f), output of Perona-Malik scheme in (g), TV-ROF Denoising in (h) and the denoising of You-Kaveh algorithm in (i).

5 Conclusions

We have proposed a nonlinear second order hyperbolic diffusion-based technique for additive white Gaussian noise removal in this paper. The considered PDE model is based on a nonlinear diffusion term that assures a strong feature-preserving image restoration, a component that has the role of controlling the speed of the diffusion process and a drift term introduced to enhance the image edges. Those edges are also defined by the second-order time derivative from the considered hyperbolic equation.

The unique weak solution of this differential model is computed by using a finite-difference method based iterative explicit numerical approximation algorithm that is stable, consistent to the PDE-based model and converges quite fast to it. It has been successfully applied in our restoration experiments which have proved the effectiveness of the proposed denoising approach. Our restoration framework has provided proper smoothing results while avoiding the undesirable effects. It outperforms not only the classic 2D filtering models, but also many nonlinear second-order and fourth-order PDE-based approaches. Also, it outperforms many of our past PDE-based denoising schemes, such as those based on parabolic anisotropic diffusion equations [11] or other diffusion-based models [17], which use equations of the porous materials [18], by providing not only a better noise removal but also better edges.

Given its edge-preserving and edge-defining character, this nonlinear diffusion-based filtering model could be further used for edge extraction and applied in the object detection domain.

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