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# EXISTENCE OF SOLUTIONS FOR $p(x)$-LAPLACIAN DIRICHLET PROBLEM BY TOPOLOGICAL DEGREE

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**Abstract**

In this paper, we prove the existence of at least one solution for the Dirichlet problem of $p(x)$-Laplacian

$$-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x, u, \nabla u),$$

by using the topological degree theory for a class of demicontinuous operators of generalized $(S_+)$ type. The right hand side $f$ is a Carathéodory function satisfying some non-standard growth conditions.

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## 1 Introduction

The $p(x)$-Laplacian has been used in the modelling of electrorheological fluids ([10]) and in image processing ([1, 4]). Up to these days, a great deal of results have been obtained for solutions to equations related to this operator.

We consider the following nonlinear degenerated elliptic problem

$$\begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u) = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $p(\cdot)$ is log-Hölder continuous with values in $(1, \infty)$. By using the degree theory for $p(\cdot) \equiv p$ with values in $(2, N)$, Kim and Hong studied in ([7]) the problem

$$\begin{cases} -\triangle_p u = u + f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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In [5] Fan and Zhang presents several sufficient conditions for the existence of solutions for the problem (1) with $f$ independent of $\nabla u$.

The aim of this paper is to prove an existence of at least weak solution for (1) extending and refining the results in [5, 7] by using the topological degree theory for a class of demicontinuous operators of generalized $(S_+)$ type.

This paper is divided into four sections. In the second section, we introduce some classes of operators of generalized $(S_+)$ type and the topological degree. In the third section, we present some basic properties of generalized Lebesgue-Sobolev spaces $W_0^{1,p(x)}$ and several important properties of $p(x)$-Laplacian operator. Finaly, in the fourth section, we give some existence results of weak solutions of problem (1).

## 2 Some classes of operators and topological degree

Let $X$ and $Y$ be two real Banach spaces and $\Omega$ a nonempty subset of $X$. The symbol $\rightarrow$ ($\rightharpoonup$) stands for strong (weak) convergence. We recall that a mapping $F : \Omega \subset X \rightarrow Y$ is

- *bounded,* if it takes any bounded set into a bounded set;
- *demicontinuous,* if for any $(u_n) \subset \Omega$, $u_n \rightarrow u$ implies $F(u_n) \rightharpoonup F(u)$;
- *compact* if it is continuous and the image of any bounded set is relatively compact.

Let $X$ be a real reflexive Banach space with dual $X^*$. A mapping $F : \Omega \subset X \rightarrow X^*$ is said to be

- *of class* $(S_+)$, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $limsup\langle Fu_n, u_n - u\rangle \leq 0$, it follows that $u_n \rightarrow u$;
- *quasimonotone* , if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, it follows that $limsup\langle Fu_n, u_n - u\rangle \geq 0$.

For any operator $F : \Omega \subset X \rightarrow X$ and any bounded operator $T : \Omega_1 \subset X \rightarrow X^*$ such that $\Omega \subset \Omega_1$, we say that $F$

- satisfies condition $(S_+)_T$, if for any $(u_n)\Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $limsup\langle Fu_n, y_n - y\rangle \leq 0$, we have $u_n \rightarrow u$;
- has the property $(QM)_T$, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$, we have $limsup\langle Fu_n, y - y_n\rangle \geq 0$.

For any $\Omega \subset X$, we consider the following classes of operators:

$$\mathcal{F}_1(\Omega) := \{F : \Omega \rightarrow X^* \mid F \text{ is bounded, demicontinuous and satifies condition } (S_+)\},$$
$$\mathcal{F}_{\mathrm{T}}(\Omega) := \{F : \Omega \rightarrow X \mid F \text{ is demicontinuous and satifies condition} (S_+)_T\}.$$

For any $\Omega \subset D_F$, where $D_F$ denotes the domain of $F$, and any $T \in \mathcal{F}_1(\Omega)$.
Let $\mathcal{O}$ be the collection of all bounded open set in $X$. Define

$$\mathcal{F}(\mathrm{X}) := \{\mathrm{F} \in \mathcal{F}_{\mathrm{T}}(\bar{\mathrm{G}}) \mid \mathrm{G} \in \mathcal{O}, \mathrm{T} \in \mathcal{F}_1(\bar{\mathrm{G}})\},$$

Here, $T \in F_1(\bar{\mathrm{G}})$ is called an *essential inner map* to $F$.

**Lemma 1.** *[7, Lemma 2.3] Suppose that $T \in \mathcal{F}_1(\bar{\mathrm{G}})$ is continuous and $S : D_S \subset X^* \to X$ is demicontinuous such that $T(\bar{G}) \subset D_s$, where $G$ is a bounded open set in a real reflexive Banach space $X$. Then the following statement are true:*

**(i)** *If $S$ is quasimonotone, then $I + SoT \in \mathcal{F}_{\mathrm{T}}(\bar{\mathrm{G}})$, where $I$ denotes the identity operator.*

**(ii)** *If $S$ is of class $(S_+)$, then $SoT \in \mathcal{F}_{\mathrm{T}}(\bar{\mathrm{G}})$*

**Definition 1.** *Let $G$ be a bounded open subset of a real reflexive Banach space $X$, $T \in \mathcal{F}_1(\bar{\mathrm{G}})$ be continuous and let $F, S \in \mathcal{F}_{\mathrm{T}}(\bar{\mathrm{G}})$. The affine homotopy $H : [0, 1] \times \bar{G} \to X$ defined by*

$$H(t, u) := (1 - t)Fu + tSu \text{ for } (t, u) \in [0, 1] \times \bar{G}$$

*is called an admissible affine homotopy with the common continuous essential inner map $T$.*

**Remark 1** (Lemma 2.5 [7])**.** *The above affine homotopy satisfies condition $(S_+)_T$.*

As in[7], we introduce a suitable topological degree for the class $\mathcal{F}(\mathrm{X})$:

**Theorem 1.** *Let*

$$\mathcal{M} = \{(\mathrm{F}, \mathrm{G}, \mathrm{h}) | \mathrm{G} \in \mathcal{O}, \mathrm{T} \in \mathcal{F}_1(\bar{\mathrm{G}}), \mathrm{F} \in \mathcal{F}_{\mathrm{T}}(\bar{\mathrm{G}}), \mathrm{h} \notin \mathrm{F}(\partial \mathrm{G})\}.$$

*There exists a unique degree function $d : \mathcal{M} \to \mathbb{Z}$ that satisfies the following properties:*

1. *(Existence) if $d(F, G, h) \neq 0$ , then the equation $Fu = h$ has a solution in $G$,*
2. *(Additivity) Let $F \in \mathcal{F}_{\mathrm{T}}(\bar{\mathrm{G}})$. If $G_1$ and $G_2$ are two disjoint open subset of $G$ such that $h \notin F(\bar{G} \setminus (G_1 \cup G_2))$, then we have*

$$d(F, G, h) = d(F, G_1, h) + d(F, G_2, h),$$

3. *(Homotopy invariance) Suppose that $H : [0, 1] \times \bar{G} \to X$ is an admissible affine homotopy with a common continuous essential inner map and $h : [0, 1] \to X$ is a continuous path in $X$ such that $h(t) \notin H(t, \partial G)$ for all $t \in [0, 1]$ ,then the value of $d(H(t, .), G, h(t))$ is constant for all $t \in [0, 1]$,*

4. *(Normalization) For any* $h \in G$*, we have*

$$d(I, G, h) = 1,$$

5. *(Boundary dependence) If* $F, S \in \mathcal{F}_{\mathrm{T}}(\bar{\mathrm{G}})$ *coincide on* $\partial G$ *and* $h \notin F(\partial G)$*, then*

$$d(F, G, h) = d(S, G, h).$$

**Remark 2.** *[7, Definition 3.3] The above degree is defined as follows:*

$$d(F, G, h) := d_B(F|_{\bar{G_0}}, G_0, h),$$

*where* $d_B$ *is the Berkovits degree [2] and* $G_0$ *is any open subset of* $G$ *with* $F^{-1}(h) \subset G_0$ *and* $F$ *is bounded on* $\bar{G}_0$*.*

# 3 The spaces $W_0^{1,p(x)}(\Omega)$ and properties of $p(x)-$Laplacian operator

## 3.1 The spaces $W_0^{1,p(x)}(\Omega)$

We introduce the setting of our problem with some auxiliary results of the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. For convenience, we only recall some basic facts with will be used later, we refer to [6, 9, 13] for more details.

Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, with a Lipschitz boundary denoted by $\partial\Omega$. Denote

$$C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) | \inf_{x \in \bar{\Omega}} h(x) > 1\}.$$

For any $h \in C_+(\bar{\Omega})$, we define

$$h^+ := max\{h(x), x \in \bar{\Omega}\}, h^- := min\{h(x), x \in \bar{\Omega}\}.$$

For any $p \in C_+(\bar{\Omega})$ we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u;\ u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < +\infty\}$$

endowed with *Luxemburg norm*

$$|u|_{p(x)} = \inf\{\lambda > 0/\rho_{p(x)}(\frac{u}{\lambda}) \leq 1\}.$$

where

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx, \quad \forall u \in L^{p(x)}(\Omega).$$

$(L^{p(x)}(\Omega), |\ .\ |_{p(x)})$ is a Banach space [9, Theorem 2.5], separable and reflexive [9, Corollary 2.7]. Its conjugate space is $L^{p'(x)}(\Omega)$ where $1/p(x) + 1/p'(x) = 1$ for all $x \in \Omega$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, Hölder inequality holds [9, Theorem 2.1]

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(x)}|v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)}. \tag{2}$$

Notice that if $(u_n)$ and $u \in L^{p(.)}(\Omega)$ then the following relations hold true (see [6])

$$|u|_{p(x)} < 1 (= 1; > 1) \quad \Leftrightarrow \quad \rho_{p(x)}(u) < 1 (= 1; > 1),$$

$$|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|^{p^-}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^+}_{p(x)}, \tag{3}$$

$$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|^{p^+}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^-}_{p(x)}, \tag{4}$$

$$\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0. \tag{5}$$

From (3) and (4), we can deduce the inequalities

$$|u|_{p(x)} \leq \rho_{p(x)}(u) + 1, \tag{6}$$

$$\rho_{p(x)}(u) \leq |u|^{p^-}_{p(x)} + |u|^{p^+}_{p(x)}. \tag{7}$$

If $p_1, p_2 \in C_+(\bar\Omega), p_1(x) \leq p_2(x)$ for any $x \in \bar\Omega$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.
Next, we define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ as

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega)/|\nabla u| \in L^{p(x)}(\Omega)\}.$$

It is a Banach space under the norm

$$||u|| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

We also define $W_0^{1,p(.)}(\Omega)$ as the subspace of $W^{1,p(.)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $||\ .\ ||$. If the exponent $p(.)$ satisfies the log-Hölder continuity condition, i.e. there is a constant $\alpha > 0$ such that for every $x, y \in \Omega, x \neq y$ with $|x - y| \leq \frac{1}{2}$ one has

$$|p(x) - p(y)| \leq \frac{\alpha}{-\log|x-y|}\ , \tag{8}$$

then we have the Poincaré inequality (see [8, 11]), i.e. the exists a constant $C > 0$ depending only on $\Omega$ and the function $p$ such that

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)}, \forall u \in W_0^{1,p(.)}(\Omega). \tag{9}$$

In particular, the space $W_0^{1,p(.)}(\Omega)$ has a norm $|\ .\ |$ given by

$$|u|_{1,p(x)} = |\nabla u|_{p(.)} \text{ for all } u \in W_0^{1,p(x)}(\Omega),$$

which is equivalent to $||\ .\ ||$. In addition, we have the compact embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$(see [9]). The space $(W_0^{1,p(x)}(\Omega), |\ .\ |_{1,p(x)})$ is a Banach space, separable and reflexive (see [6, 9]). The dual space of $W_0^{1,p(x)}(\Omega)$, denoted $W^{-1,p'(x)}(\Omega)$, is equipped with the norm

$$|v|_{-1,p'(x)} = \inf\{|v_0|_{p'(x)} + \sum_{i=1}^{N} |v_i|_{p'(x)}\},$$

where the infinimum is taken on all possible decompositions $v = v_0 - divF$ with $v_0 \in L^{p'(x)}(\Omega)$ and $F = (v_1, ..., v_N) \in (L^{p'(x)}(\Omega))^N$.

### 3.2 Properties of $p(x)-$Laplacian operator

We discuss the $p(x)-$Laplacian operator

$$-\Delta_{p(x)} u := -div(|\nabla u|^{p(x)-2} \nabla u).$$

Consider the following functional:

$$J(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx, \quad u \in W_0^{1,p(x)}(\Omega).$$

We know that (see [3]), $J \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$, and the $p(x)-$Laplacian operator is the derivative operator of $J$ in the weak sense.
We denote $L = J' : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$, then

$$\langle Lu, v\rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx, \text{ for all } u, v \in W_0^{1,p(x)}(\Omega).$$

**Theorem 2.** *[3, Theorem 3.1]*

**(i)** $L : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ *is a continuous, bounded and strictly monotone operator;*

**(ii)** $L$ *is a mapping of class* $(S_+)$*;*

**(iii)** $L$ *is a homeomorphism.*

## 4 Existence of solutions

In this section, we study the Dirichlet boundary value problem (1) based on the degree theory in Section 2, where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with a Lipschitz boundary $\partial\Omega$, $p \in C_+(\bar{\Omega})$ satisfy the log-Hölder continuity condition (8), $1 < p^- \leq p(x) \leq p^+ < \infty$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a real-valued function such that:

$(f_1)$ $f$ satisfies the Carathéodory condition, that is, $f(., \eta, \zeta)$ is measurable on $\Omega$ for all $(\eta, \zeta) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x, ., .)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for a.e. $x \in \Omega$.

$(f_2)$ $f$ has the growth condition

$$|f(x,\eta,\zeta)| \leq c(k(x) + |\eta|^{q(x)-1} + |\zeta|^{q(x)-1})$$

for a.e. $x \in \Omega$ and all $(\eta,\zeta) \in \mathbb{R} \times \mathbb{R}^N$, where $c$ is a positive constant, $k \in L^{p'(x)}(\Omega)$ and $1 < q^- \leq q(x) \leq q^+ < p^-$.

**Definition 2.** *We call that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1) if*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx = \int_\Omega f(x,u,\nabla u) v dx, \quad \forall v \in W_0^{1,p(x)}(\Omega).$$

**Lemma 2.** *Under assumptions $(f_1)$ and $(f_2)$, the operator $S : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ setting by*

$$\langle Su, v\rangle = -\int_\Omega f(x,u,\nabla u) v dx, \quad \forall u,v \in W_0^{1,p(x)}(\Omega)$$

*is compact.*

*Proof.* Let $\phi : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be an operator defined by

$$\phi u(x) := -f(x,u,\nabla u) \text{ for } u \in W_0^{1,p(x)}(\Omega) \text{ and } x \in \Omega.$$

We first show that $\phi$ is bounded and continuous.
For each $u \in W_0^{1,p(x)}(\Omega)$, we have the growth condition $(f_2)$, the inequalities (6) and (7) that

$$\begin{aligned} |\phi u|_{p'(x)} &\leq \rho_{p'(x)}(\phi u) + 1 \\ &= \int_\Omega |f(x,u(x),\nabla u(x)|^{p'(x)} + 1 \\ &\leq const(\rho_{p'(x)}(k) + \rho_{r(x)}(u) + \rho_{r(x)}(\nabla u)) + 1 \\ &\leq const(|k|_{p'(x)}^{p'^+} + |u|_{r(x)}^{r^+} + |u|_{r(x)}^{r^-} + |\nabla u|_{r(x)}^{r^+} + |\nabla u|_{r(x)}^{r^-}) + 1, \end{aligned}$$

where $r(x) = (q(x)-1)p'(x) < p(x)$. By the continuous embedding $L^{p(x)} \hookrightarrow L^{r(x)}$ and the Poincaré inequality (9), we have

$$|\phi u|_{p'(x)} \leq const(|k|_{p'(x)}^{p'^+} + |u|_{1,p(x)}^{r^+} + |u|_{1,p(x)}^{r^-}) + 1$$

This implies that $\phi$ is bounded on $W_0^{1,p(x)}(\Omega)$.
To show that $\phi$ is continuous, let $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. Then $u_n \to u$ in $L^{p(x)}(\Omega)$ and $\nabla u_n \to \nabla u$ in $(L^{p(x)}(\Omega))^N$. Hence there exist a subsequence $(u_k)$ of $(u_n)$ and measurable functions $h$ in $L^{p(x)}(\Omega)$ and $g$ in $(L^{p(x)}(\Omega))^N$ such that

$$u_k(x) \to u(x) \text{ and } \nabla u_k(x) \to \nabla u(x),$$

$$|u_k(x)| \leq h(x) \text{ and } |\nabla u_k(x)| \leq |g(x)|$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$. Since $f$ satisfies the Carathodory condition, we obtain that

$$f(x, u_k(x), \nabla u_k(x)) \to f(x, u(x), \nabla u(x)) \text{ a.e. } x \in \Omega.$$

it follows from $(f_2)$ that

$$|f(x, u_k(x), \nabla u_k(x))| \leq c(k(x) + |h(x)|^{q(x)-1} + |g(x)|^{q(x)-1})$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
Since

$$k + |h|^{q(x)-1} + |g(x)|^{q(x)-1} \in L^{p'(x)}(\Omega),$$

and taking into account the equality

$$\rho_{p'(x)}(\phi u_k - \phi u) = \int_\Omega |f(x, u_k(x), \nabla u_k(x)) - f(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

the dominated convergence theorem and the equivalence (5) implies that

$$\phi u_k \to \phi u \text{ in } L^{p'(x)}(\Omega).$$

Thus the entire sequence $(\phi u_n)$ converges to $\phi u$ in $L^{p'(x)}(\Omega)$.
Since the embedding $I : W_0^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$ is compact, it is known that the adjoint operator $I^* : L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is also compact. Therefore, the composition $I^* o\phi : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is compact. This completes the proof. □

**Theorem 3.** *Under assumptions* $(f_1)$ *and* $(f_2)$*, problem (1) has a weak solution* $u$ *in* $W_0^{1,p(x)}(\Omega)$.

*Proof.* Let $S : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be as in Lemma 2 and
$L : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$, as in subsection 3.2, setting by

$$\langle Lu, v\rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx, \text{ for all } u, v \in W_0^{1,p(x)}(\Omega).$$

Then $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1) if and only if

$$Lu = -Su \tag{10}$$

Thanks to the properties of the operator $L$ seen in Theorem 2 and in view of Minty-Browder Theorem (see [14], Theorem 26A), the inverse operator
$T := L^{-1} : W^{-1,p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega)$ is bounded, continuous and satisfies condition $(S_+)$. Moreover, note by Lemma 2 that the operator $S$ is bounded, continuous and quasimonotone.
Consequently, equation (10) is equivalent to

$$u = Tv \text{ and } v + SoTv = 0. \tag{11}$$

To solve equation (11), we will apply the degree theory introducing in section 2. To do this, we first claim that the set

$$B := \{v \in W^{-1,p'(x)}(\Omega) | v + tSoTv = 0 \text{ for some } t \in [0,1]\}$$

is bounded. Indeed, let $v \in B$. Set $u := Tv$, then $|Tv|_{1,p(x)} = |\nabla u|_{p(x)}$.
If $|\nabla u|_{p(x)} \leq 1$, then $|Tv|_{1,p(x)}$ is bounded.
If $|\nabla u|_{p(x)} > 1$, then we get by the implication (3), the growth condition $(f_2)$, the Hölder inequality (2), the inequality (7)and the Young inequality the estimate

$$\begin{aligned}
|Tv|_{1,p(x)}^{p^-} &= |\nabla u|_{p(x)}^{p-} \leq \rho_{p(x)}(\nabla u) \\
&= \langle Lu, u\rangle = \langle v, Tv\rangle \\
&= -t\langle SoTv, Tv\rangle \\
&= t\int_\Omega f(x,u,\nabla u)udx \\
&\leq const(\int_\Omega |k(x)u(x)|dx + \rho_{q(x)}(u) + \int_\Omega |\nabla u|^{q(x)-1}|u|dx) \\
&\leq const(2|k|_{p'(x)}|u|_{p(x)} + |u|_{q(x)}^{q^+} + |u|_{q(x)}^{q^-} + \frac{1}{q'^-}\rho_{q(x)}(\nabla u) + \frac{1}{q^-}\rho_{q(x)}(u)) \\
&\leq const(|u|_{p(x)} + |u|_{q(x)}^{q^+} + |u|_{q(x)}^{q^-} + |\nabla u|_{q(x)}^{q^+}).
\end{aligned}$$

From the Poincaré inequality (9) and the continuous embedding $L^{p(x)} \hookrightarrow L^{q(x)}$, we can deduct the estimate

$$|Tv|_{1,p(x)}^{p^-} \leq const(|Tv|_{1,p(x)} + |Tv|_{1,p(x)}^{q^+}).$$

It follows that $\{Tv | v \in B\}$ is bounded.
Since the operator $S$ is bounded, it is obvious from (11) that the set $B$ is bounded in $W^{-1,p'(x)}(\Omega)$. Consequently, there exists $R > 0$ such that

$$|v|_{-1,p'(x)} < R \text{ for all } v \in B.$$

This says that

$$v + tSoTv \neq 0 \text{ for all } v \in \partial B_R(0) \text{ and all } t \in [0,1].$$

From Lemma (1) it follows that

$$I + SoT \in \mathcal{F}_{\mathrm{T}}(\overline{\mathrm{B_R(0)}}) \text{ and } \mathrm{I = LoT} \in \mathcal{F}_{\mathrm{T}}(\overline{\mathrm{B_R(0)}}).$$

Consider a homotopy $H : [0,1] \times \overline{B_R(0)} \to W^{-1,p'(x)}(\Omega)$ given by

$$H(t,v) := v + tSoTv \text{ for } (t,v) \in [0,1] \times \overline{B_R(0)}.$$

Applying the homotopy invariance and normalization property of the degree $d$ stated in Theorem(1), we get

$$d(I + SoT, B_R(0), 0) = d(I, B_R(0), 0) = 1,$$

and hence there exists a point $v \in B_R(0)$ such that

$$v + SoTv = 0.$$

We conclude that $u = Tv$ is a weak solution of (1). This completes the proof. □

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