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JANOWSKI SYMMETRICAL FUNCTIONS

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Abstract

In this paper, we introduce and study a new class of analytic functions obtained by merging the notions of Ruscheweyh derivatives [1, 10], Janowski functions [5], and (j,k)-symmetrical functions [3, 7] defined in the open unit disk. In particular, we investigate coefficient estimates, a sufficient condition, convolution condition, and neighborhood results related to the functions in the class.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

that are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \quad and \quad |z| < 1\}$, and suppose $\mathcal{S}$ denotes the subclass of $\mathcal{A}$ consisting of all functions that are univalent in $\mathcal{U}$. Also, let Ω be the family of functions w, analytic in $\mathcal{U}$ and satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathcal{U}$. If f and g are analytic in $\mathcal{U}$, we say that a function f is subordinate to a function g in $\mathcal{U}$, if there exists a function $w \in \Omega$ such that $f(z) = g(w(z))$, and we denote this by $f \prec g$. If g is univalent in $\mathcal{U}$ then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. The Hadamard product or convolution of two functions f and g in $\mathcal{A}$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

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is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathcal{U}.$$

Denote by $D^\sigma : \mathcal{A} \to \mathcal{A}$ the operator defined by

$$D^\sigma f(z) = \frac{z}{(1-z)^{\sigma+1}} * f(z), \ \sigma \geq -1, z \in \mathcal{U}. \tag{2}$$

Ruscheweyh [10] defined the symbol $D^n f$ for $n \in \{0, 1, 2, ...\}$, and the operator $D^\sigma f$ is called the Ruscheweyh derivative of order σ, see [1]. For f given by (1), we have

$$D^\sigma f(z) = z + \sum_{n=2}^{\infty} a_n R_n(\sigma) z^n, \tag{3}$$

where

$$R_n(\sigma) = \frac{(\sigma+1)(\sigma+2)...(\sigma+n-1)}{(n-1)!} = \frac{\Gamma(n+\sigma)}{(n-1)!.\Gamma(1+\sigma)}. \tag{4}$$

In [1], Ahuja defined and studied a generalized class $H_\sigma^b[A, B]$ of analytic function f of complex order b. A function f in $\mathcal{A}$ is said to be in class $H_\sigma^b[A, B]$ if it satisfies the condition

$$1 + \frac{(\sigma+1)}{b}\left(\frac{D^{\sigma+1} f}{D^\sigma f} - 1\right) \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, \tag{5}$$

where $\sigma > -1$ and $b \neq 0$ is an arbitrary fixed complex number. As observed in [1], the class $H_\sigma^b[A, B]$ is of special interest because for different choices of A, B $(-1 \leq B < A \leq 1)$ and non-zero complex number b, the class $H_\sigma^b[A, B]$ includes various classes of analytic functions that are starlike, convex, pre-starlike, Ruscheweyh class of order α $(0 \leq \alpha < 1)$, β-spiral-like $(\frac{-\pi}{2} < \beta < \frac{\pi}{2})$, starlike of complex order, convex of complex order , β-convex spiral-like, β-spiral-like functions of order α and others. In particular we observe that $H_0^1[A, B]$ and $H_1^1[A, B]$ were introduced in 1973 by Janowski [5]. For further details and references, one may refer to [1].

In order to define a new class of Janowski symmetrical functions associated with Ruschewyh derivatives defined in the open unit disk $\mathcal{U}$, we first recall the notion of k-fold symmetric functions defined in k-fold symmetric domain, where k is any positive integer. A domain $\mathcal{D}$ is said to be k-fold symmetric if a rotation of $\mathcal{D}$ about the origin through an angle $\frac{2\pi}{k}$ carries $\mathcal{D}$ onto itself. A function f is said to be k-fold symmetric in $\mathcal{D}$ if for every z in $\mathcal{D}$ we have

$$f\left(e^{\frac{2\pi i}{k}} z\right) = e^{\frac{2\pi i}{k}} f(z), \ z \in \mathcal{D}.$$

The family of all k-fold symmetric functions is denoted by $\mathcal{S}^k$, and for $k = 2$ we get the class of odd univalent functions. In 1995, Liczberski and Polubinski [7] constructed the theory of (j, k)-symmetrical functions for $(j = 0, 1, 2, \ldots, k-1)$

and ($k = 2, 3, \dots$). If $\mathcal{D}$ is k-fold symmetric domain and j any integer, then a function $f : \mathcal{D} \to \mathbb{C}$ is called (j, k)-symmetrical if for each $z \in \mathcal{D}$, $f(\varepsilon z) = \varepsilon^j f(z)$. We note that the (j, k)-symmetrical functions is a generalization of the notions of even, odd, and k-symmetrical functions

The theory of (j, k)-symmetrical functions has many interesting applications; for instance, in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan's uniqueness theorem for holomorphic mappings, see [7].

Denote the family of all (j, k)-symmetrical functions by $\mathcal{S}^{(j,k)}$. We observe that , $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ are the classes of even, odd and k-symmetric functions respectively. We have the following decomposition theorem:

Theorem 1 [7, Page 16] For every mapping $f : \mathcal{U} \mapsto \mathbb{C}$, and a k-fold symmetric set $\mathcal{U}$, there exists exactly one sequence of (j, k)-symmetrical functions $f_{j,k}$ such that

$$f(z) = \sum_{j=0}^{k-1} f_{j,k}(z),$$

where

$$f_{j,k}(z) = \frac{1}{k} \sum_{v=0}^{k-1} \varepsilon^{-vj} f\left(\varepsilon^v z\right), \ z \in \mathcal{U}. \tag{6}$$

Remark 1. *Equivalently,* (6) *may be written as*

$$f_{j,k}(z) = \sum_{n=1}^{\infty} \delta_{n,j} a_n z^n, \quad a_1 = 1, \tag{7}$$

where

$$\delta_{n,j} = \frac{1}{k} \sum_{v=0}^{k-1} \varepsilon^{(n-j)v} = \begin{cases} 1, & n = lk + j; \\ 0, & n \neq lk + j; \end{cases}, \tag{8}$$

$$(l \in \mathbb{N}, \ k = 1, 2, \dots, \ j = 0, 1, 2, \dots, k - 1)$$

We amalgamate the notion of Janowski functions, Ruscheweyh derivative and (j, k)-symmetrical functions to originate a new class $\mathcal{S}^{(j,k)}[A, B, \alpha, b, \sigma]$.

Definition 1. *For arbitrary fixed numbers* A, B *and* $\alpha(-1 \le B < A \le 1, 0 \le \alpha < 1)$, *a function* f *in* $\mathcal{A}$ *is said to belong to the class* $\mathcal{S}^{(j,k)}[A, B, \alpha, b, \sigma]$ *if*

$$1 - \frac{2}{b} + \frac{2}{b} \frac{D^{\sigma+1} f(z)}{D^\sigma f_{j,k}(z)} \prec \frac{1 + [(1 - \alpha)A + \alpha B]z}{1 + Bz}, \qquad z \in \mathcal{U},$$

where $f_{j,k}$ *is defined by* (7), $b \neq 0, \sigma \geq -1$, *and* $D^\sigma f$ *is the generalised Rushceweyh derivative of* f *introduced by* (2).

For special values of j, k, A, B, α, b and σ, Definition 1 yields several known and new subclasses of $\mathcal{A}$, for instance, we have the following known subclasses of $\mathcal{A}$.

$$\mathcal{S}^{(j,k)}[A, B, 0, 2, 0] = \mathcal{S}^{(j,k)}[A, B], [2].$$

$$\mathcal{S}^{(1,1)}[A,B,0,b,\sigma] = H^b_\lambda[A,B], [1].$$
$$\mathcal{S}^{(1,1)}[A,B,\alpha,2,0] = \mathcal{S}[A,B,\alpha],\ [9].$$
$$\mathcal{S}^{(1,k)}[A,B,0,2,0] = \mathcal{S}^{(k)}[A,B],\ [6].$$
$$\mathcal{S}^{(1,k)}[1,-1,0,2,0] = \mathcal{S}^*_k = \mathcal{S}^*_k[1,-1],\ [12].$$

We need to recall the following neighborhood concept introduced by Goodman [4] and generalized by Ruscheweyh [11]

Definition 2. *For any $f \in \mathcal{A}$, ρ-neighborhood of function f can be defined as:*

$$\mathcal{N}_\rho(f) = \left\{ g \in \mathcal{A} : g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad \sum_{n=2}^{\infty} n|a_n - b_n| \leq \rho \right\}. \tag{9}$$

For $e(z) = z$, we can see that

$$\mathcal{N}_\rho(e) = \left\{ g \in \mathcal{A} : g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad \sum_{n=2}^{\infty} n|b_n| \leq \rho \right\}. \tag{10}$$

Ruschewegh [11] proved, among other results that for all $\eta \in \mathbb{C}$, with $|\mu| < \rho$,

$$\frac{f(z) + \eta z}{1 + \eta} \in \mathcal{S}^* \Rightarrow \mathcal{N}_\rho(f) \subset \mathcal{S}^*.$$

In this paper, we investigate coefficient estimates, a sufficient condition and convolution property. Finally motivated by Definition 2, we give an analogous definition of neighborhood for the class $\mathcal{S}^{(j,k)}[A,B,\alpha,b,\sigma]$ and then investigate a related neighborhood result for this new class.

2 Main results

We need the following lemma to prove our main results.

Lemma 1. *[8, Lemma 2.1] Let $p \in \mathcal{P}[A,B,\alpha]$ and suppose $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, then for $n \geq 1$ we have*

$$|c_n| \leq (1-\alpha)(A-B).$$

Theorem 1. *Let the function f defined by $f(z) = z + a_2 z^2 + \ldots$ be in the class $\mathcal{S}^{(j,k)}[A,B,\alpha,b,\sigma]$. Then for $n \geq 2$, $-1 \leq B < A \leq 1, b \neq 0, \sigma > -1$, $0 \leq \alpha < 1$, we have*

$$|a_n| \leq \prod_{r=1}^{n-1} \frac{r\{[|b|(A-B)(1-\alpha) - 2(\sigma+1)]\delta_{r,j} + 2(\sigma+r)\}}{2(\sigma+r)[(\sigma+r+1) - (\sigma+1)\delta_{r+1,j}]}, \tag{11}$$

where $\delta_{n,j}$ are given by (8). The inequality is sharp.

Proof. By Definition 1, we have

$$1-\frac{2}{b}+\frac{2}{b}\frac{D^{\sigma+1}f(z)}{D^{\sigma}f_{j,k}(z)}=p(z),$$

where $p(z)=1+\sum_{n=1}^{\infty}c_nz^n\in\mathcal{P}[A,B,\alpha]$. This gives

$$2D^{\sigma+1}f(z)+(b-2)D^{\sigma}f_{j,k}(z)=bD^{\sigma}f_{j,k}(z)\left(1+\sum_{n=1}^{\infty}c_nz^n\right),$$

or

$$2[D^{\sigma+1}f(z)-D^{\sigma}f_{j,k}(z)]=bD^{\sigma}f_{j,k}(z)\left(\sum_{n=1}^{\infty}c_nz^n\right).$$

Using (1), (3), (7), and simplifying the resulting expression we obtain

$$\frac{2}{(\sigma+1)}\sum_{n=1}^{\infty}[(\sigma+n)-(\sigma+1)\delta_{n,j}]R_n(\sigma)a_nz^n=\left(b\sum_{n=1}^{\infty}R_n(\sigma)\delta_{n,j}a_nz^n\right)\left(\sum_{n=1}^{\infty}c_nz^n\right).$$

By using Cauchy product formula in this inequality above and equating coefficients of z^n on both sides, we get

$$a_n=\frac{b(\sigma+1)}{2[(\sigma+n)-(\sigma+1)\delta_{n,j}]R_n(\sigma)}\sum_{r=1}^{n-1}R_{n-r}(\sigma)\delta_{n-r,j}a_{n-r}c_r.$$

By Lemma 1, we get

$$|a_n|\le\frac{(A-B)(1-\alpha)|b|(\sigma+1)}{2[(\sigma+n)-(\sigma+1)\delta_{n,j}]R_n(\sigma)}\sum_{r=1}^{n-1}R_r(\sigma)\delta_{r,j}|a_r|. \tag{12}$$

It now suffices to prove that

$$\frac{(A-B)(1-\alpha)|b|(\sigma+1)}{2[(\sigma+n)-(\sigma+1)\delta_{n,j}]R_n(\sigma)}\sum_{r=1}^{n-1}R_r(\sigma)\delta_{r,j}|a_r| \tag{13}$$
$$\le\prod_{r=1}^{n-1}\frac{r\{[|b|(A-B)(1-\alpha)-2(\sigma+1)]\delta_{r,j}+2(\sigma+r)\}}{2(\sigma+r)[(\sigma+r+1)-(\sigma+1)\delta_{r+1,j}]}.$$

In order to prove (13), we use the induction method. This is a routine to verify that the result is true for $n=2$ and 3. Let the hypothesis be true for $n=m$. Then

$$\begin{aligned}|a_m| &\le \frac{(A-B)(1-\alpha)|b|(\sigma+1)}{2[(\sigma+m)-(\sigma+1)\delta_{m,j}]R_m(\sigma)}\sum_{r=1}^{m-1}R_r(\sigma)\delta_{r,j}|a_r|\\ &\le\prod_{r=1}^{m-1}\frac{r\{[|b|(A-B)(1-\alpha)-2](\sigma+1)\delta_{r,j}+2(\sigma+r)\}}{2(\sigma+r)[(\sigma+r+1)-(\sigma+1)\delta_{r+1,j}]}\end{aligned}$$

Multiplying both sides by the expression

$$\frac{m\{[|b|(A-B)(1-\alpha)-2(\sigma+1)]\delta_{m,j}+2(\sigma+m)\}}{2(\sigma+m)[(\sigma+m+1)-(\sigma+1)\delta_{m+1,j}]}$$

we get

$$\begin{aligned}
&\prod_{r=1}^{m}\frac{r\{[|b|(A-B)(1-\alpha)-2](\sigma+1)\delta_{r,j}+2(\sigma+r)\}}{2(\sigma+r)[(\sigma+r+1)-(\sigma+1)\delta_{r+1,j}]}\\
&\geq \begin{cases}\frac{m\{[|b|(A-B)(1-\alpha)-2(\sigma+1)]\delta_{m,j}+2(\sigma+m)\}}{2(\sigma+m)[(\sigma+m+1)-(\sigma+1)\delta_{m+1,j}]}\\ \times\frac{(A-B)(1-\alpha)|b|(\sigma+1)}{2[(\sigma+m)-(\sigma+1)\delta_{m,j}]R_m(\sigma)}\sum_{r=1}^{m-1}R_r(\sigma)\delta_{r,j}|a_r|,\end{cases}\\
&=\frac{(A-B)(1-\alpha)|b|(\sigma+1)}{2R_{m+1}(\sigma)[(\sigma+m+1)-(\sigma+1)\delta_{m+1,j}]}\times\\
&\quad\times\left\{1+\frac{R_m(\sigma)(A-B)(1-\alpha)|b|(\sigma+1)\delta_{m,j}}{2R_m(\sigma)[(\sigma+m+1)-(\sigma)\delta_{m,j}]}\right\}\sum_{r=1}^{m-1}R_r(\sigma)\delta_{r,j}|a_r|\\
&\geq\frac{(A-B)(1-\alpha)|b|(\sigma+1)}{2R_{m+1}(\sigma)[(\sigma+m+1)-(\sigma+1)\delta_{m+1,j}]}\left\{\sum_{r=1}^{m}R_r(\sigma)\delta_{r,j}|a_r|\right\}.
\end{aligned}$$

That is

$$\begin{aligned}
|a_{m+1}|&\leq\frac{(A-B)(1-\alpha)|b|(\sigma+1)}{2[(\sigma+m+1)-(\sigma+1)\delta_{m+1,j}]R_{m+1}(\sigma)}\sum_{r=1}^{m}R_r(\sigma)\delta_{r,j}|a_r| \quad (14)\\
&\leq\prod_{r=1}^{m}\frac{r\{[|b|(A-B)(1-\alpha)-2](\sigma+1)\delta_{r,j}+2(\sigma+r)\}}{2(\sigma+r)[(\sigma+r+1)-(\sigma+1)\delta_{r+1,j}]}
\end{aligned}$$

The last inequality shows that inequality (13) is true for $n=m+1$. This completes the proof. The coefficient inequality (11) is sharp for the analytic function

$$g(z)=z+\sum_{n=2}^{\infty}\prod_{r=1}^{n-1}\frac{r\{[|b|(A-B)(1-\alpha)-2(\sigma+1)]\delta_{r,j}+2(\sigma+r)\}}{2(\sigma+r)[(\sigma+r+1)-(\sigma+1)\delta_{r+1,j}]}x_n z^n,$$

where $-1\leq B<A\leq 1, 0\leq\alpha<1, b\neq 0, \sigma>-1$ and $\delta_{n,j}$ is defined by (8) and $\sum_{n=2}^{\infty}|x_n|=1$.

We next find a sufficient condition for a function f to belong to the class $\mathcal{S}^{(j,k)}[A,B,\alpha,b,\sigma]$.

Theorem 2. *If $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ is analytic function in $\mathcal{U}$, then $f\in\mathcal{S}^{(j,k)}[A,B,\alpha,b,\sigma]$ provided the condition*

$$\sum_{n=2}^{\infty}\{2[(\sigma+n)-(\sigma+1)\delta_{n,j}]+|\chi_n|\}R_n(\sigma)|a_n|\leq(1-\alpha)(A-B)(\sigma+1)|b| \quad (15)$$

holds, where $R_n(\sigma)$ is given by (4) and

$$\chi_n=2B(\sigma+n)-[(1-\alpha)(A-B)b+2B](\sigma+1)\delta_{n,j} \quad (16)$$

for $-1\leq B<A\leq 1, 0\leq\alpha<1, b\neq 0, \sigma>-1$. The result is sharp.

Proof. Letting $F(z) = 1 - \frac{2}{b} + \frac{2}{b}\frac{D^{\sigma+1}f(z)}{D^{\sigma}f_{j,k}(z)}$, assume the inequality (15) holds and let $|z| = 1$. Then it suffices to show that the function F suffices the condition

$$\left|\frac{1-F}{BF - [(1-\alpha)A + \alpha B]}\right| \leq 1. \tag{17}$$

We have

$$\left|\frac{2[D^{\sigma+1}f(z) - D^{\sigma}f_{j,k}(z)]}{[(1-\alpha)(A-B)b + 2B]D^{\sigma}f_{j,k}(z) - 2BD^{\sigma+1}f(z)}\right|$$

$$= \left|\frac{2\sum_{n=2}^{\infty}[(\sigma+n) - (\sigma+1)\delta_{n,j}]R_n(\sigma)a_n z^{n-1}}{(1-\alpha)(B-A)(\sigma+1)b + \sum_{n=2}^{\infty}\{\chi_n\}R_n(\sigma)a_n z^{n-1}}\right|$$

$$\leq \frac{2\sum_{n=2}^{\infty}[(\sigma+n) - (\sigma+1)\delta_{n,j}]R_n(\sigma)|a_n||z|^{n-1}}{(1-\alpha)(A-B)(\sigma+1)|b| - \sum_{n=2}^{\infty}|\chi_n|R_n(\sigma)|a_n||z|^{n-1}}$$

$$\leq \frac{2\sum_{n=2}^{\infty}[(\sigma+n) - (\sigma+1)\delta_{n,j}]R_n(\sigma)|a_n|}{(1-\alpha)(A-B)(\sigma+1)|b| - \sum_{n=2}^{\infty}|\chi_n|R_n(\sigma)|a_n|}.$$

This last expression is bounded above by 1, which implies that $f \in \mathcal{S}^{(j,k)}[A, B, \alpha, b, \sigma]$. The coefficient inequality (15) is sharp for the analytic function

$$g(z) = z + \frac{(1-\alpha)(A-B)(\sigma+1)|b|}{\{2[(\sigma+n) - (\sigma+1)\delta_{n,j}] + |\chi_n|\}\, R_n(\sigma)} z^n,$$

where $-1 \leq B < A \leq 1, 0 \leq \alpha < 1, b \neq 0, \sigma > -1$ and $\delta_{n,j}$ is defined by (8).

Theorem 3. *A function* $f \in \mathcal{S}^{(j,k)}[A, B, \alpha, b, \sigma]$ *if and only if*

$$\frac{1}{z}\left\{f(z) * \left[\varphi(z) * \left([(b-2)h(z) + 2k(z)](1 + Be^{i\phi}) - bh(z)(1 + [(1-\alpha)A + \alpha B]e^{i\phi})\right)\right]\right\} \neq 0, \tag{18}$$

for $-1 \leq B < A \leq 1, 0 \leq \alpha < 1, b \neq 0, \sigma > -1, 0 \leq \phi < 2\pi$ *and where* φ, k *and* h *are given by*

$$\varphi(z) = z + \sum_{n=2}^{\infty} R_n(\sigma) z^n, \tag{19}$$

$$k(z) = z + \sum_{n=2}^{\infty} \frac{(\sigma+n)}{(\sigma+1)} z^n, \tag{20}$$

$$h(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{(1-j)v} \frac{z}{1 - \varepsilon^v z} = z + \sum_{n=2}^{\infty} \delta_{n,j} z^n. \tag{21}$$

Proof. Suppose that $f \in \mathcal{S}^{(j,k)}[A, B, \alpha, b, \sigma]$. In view of the Definition 1, we have

$$1 - \frac{2}{b} + \frac{2}{b}\frac{D^{\sigma+1}f(z)}{D^{\sigma}f_{j,k}(z)} \prec \frac{1 + [(1-\alpha)A + \alpha B]z}{1 + Bz},$$

if and only if

$$\frac{(b-2)D^{\sigma}f_{j,k}(z) + 2D^{\sigma+1}f(z)}{bD^{\sigma}f_{j,k}(z)} \neq \frac{1 + [(1-\alpha)A + \alpha B]e^{i\phi}}{1 + Be^{i\phi}}, \forall z \in \mathcal{U}, 0 \leq \phi < 2\pi. \tag{22}$$

It is easy to verify that (22) is equivalent to

$$[(b-2)D^{\sigma}f_{j,k}(z) + 2D^{\sigma+1}f(z)](1 + Be^{i\phi}) - bD^{\sigma}f_{j,k}(z)(1 + [(1-\alpha)A + \alpha B]e^{i\phi}) \neq 0. \tag{23}$$

On the other hand, note that

$$D^{\sigma+1}f(z) = \frac{z}{(1-z)^{\sigma+2}} * f(z) = \varphi(z) * f(z) * k(z), \tag{24}$$

where the functions φ, k are given by (19) and (20). Therefore, we get

$$D^{\sigma}f_{j,k}(z) = \varphi(z) * f(z) * h(z), \tag{25}$$

where the function h is given by (21). Substituting (24) and (25) into (23) we get (18).

To find some neighborhood results for the class $f \in \mathcal{S}^{(j,k)}[A, B, \alpha, b, \sigma]$ analogous to those obtained by Ruschewegh [11], we introduce the following concept of neighborhood

Definition 3. *For $-1 \leq B < A \leq 1, 0 \leq \alpha < 1, b \neq 0, \sigma > -1, 0 \leq \phi < 2\pi$ and $\rho \geq 0$ we define $\mathcal{N}^{(j,k)}(A, B, \alpha, b, \sigma; f, \rho)$ the neighborhood of a function $f \in \mathcal{A}$ as*

$$\mathcal{N}^{(j,k)}(A, B, \alpha, b, \sigma; f, \rho) = \Big\{ g \in \mathcal{A} : g(z) = z + \sum_{n=2}^{\infty} b_n z^n, d(f, g) \tag{26}$$

$$= \sum_{n=2}^{\infty} \frac{\{2[(\sigma+n) - (\sigma+1)\delta_{n,j}] + |\chi_n|\} R_n(\sigma)}{(1-\alpha)(A-B)(\sigma+1)|b|} |b_n - a_n| \leq \rho \Big\},$$

where $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $\delta_{n,j}$ is defined by (8).

Remark 2. *For parametric values $j = k = A = -B = 1, \alpha = \sigma = 0$ and $b = 2$* (26) *reduces to* (9).

Theorem 4. *Let $f \in \mathcal{A}$, and for all complex numbers η, with $|\mu| < \rho$, if*

$$\frac{f(z) + \eta z}{1 + \eta} \in \mathcal{S}^{(j,k)}[A, B, \alpha, b, \sigma]. \tag{27}$$

Then

$$\mathcal{N}^{(j,k)}(A, B, \alpha, b, \sigma; f, \rho) \subset \mathcal{S}^{(j,k)}[A, B, \alpha, b, \sigma].$$

Proof. We assume that a function g defined by $g(z) = \sum_{n=2}^{\infty} b_n z^n$ is in the class $\mathcal{N}^{(j,k)}(A, B, \alpha, b, \sigma; f, \rho)$. In order to prove the theorem, we only need to prove that $g \in \mathcal{S}^{(j,k)}[A, B, \alpha, b, \sigma]$. We would prove this claim in next three steps.
We first note that Theorem 3 is equivalent to

$$f \in \mathcal{S}^{(j,k)}[A, B, \alpha, b, \sigma] \Leftrightarrow \frac{1}{z}[(f * t_\phi)(z)] \neq 0, \quad z \in \mathcal{U}, \tag{28}$$

where

$$t_\phi(z) = \frac{\varphi(z) * \left\{[(b-2)h(z) + 2k(z)](1 + Be^{i\phi}) - bh(z)(1 + [(1-\alpha)A + \alpha B]e^{i\phi})\right\}}{(1-\alpha)(B-A)(\sigma+1)|b|e^{i\phi}},$$

where $0 \leq \phi < 2\pi, -1 \leq B < A \leq 1, 0 \leq \alpha < 1, b \neq 0$ and φ, k and h are given by (19), (20) and (21) respectively. We can write $t_\phi(z) = z + \sum_{n=2}^{\infty} t_n z^n$,
where

$$t_n = \frac{\left\{2[(\sigma+n) - (\sigma+1)\delta_{n,j}] + \chi_n e^{i\phi}\right\} R_n(\sigma)}{(1-\alpha)(B-A)(\sigma+1)|b|e^{i\phi}}, \tag{29}$$

and where χ_n is defined by (16). Secondly we obtain that (27) is equivalent to

$$\left|\frac{f(z) * t_\phi(z)}{z}\right| \geq \rho; \tag{30}$$

because, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ and satisfy (27), then (28) is equivalent to

$$t_\phi \in \mathcal{S}^{(j,k)}[A, B, \alpha, b, \sigma] \Leftrightarrow \frac{1}{z}\left[\frac{f(z) * t_\phi(z)}{1+\eta}\right] \neq 0, \qquad |\eta| < \rho.$$

Thirdly letting $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ we notice that

$$\begin{aligned}
\left|\frac{g(z) * t_\phi(z)}{z}\right| &= \left|\frac{f(z) * t_\phi(z)}{z} + \frac{(g(z) - f(z)) * t_\phi(z)}{z}\right| \\
&\geq \rho - \left|\frac{(g(z) - f(z)) * t_\phi(z)}{z}\right|, \quad \text{(by using (30))} \\
&= \rho - \left|\sum_{n=2}^{\infty}(b_n - a_n)t_n z^n\right|, \\
&\geq |z|\sum_{n=2}^{\infty}\left[\frac{\left\{2[(\sigma+n) - (\sigma+1)\delta_{n,j}] + \chi_n e^{i\phi}\right\} R_n(\sigma)}{(1-\alpha)(B-A)(\sigma+1)|b|e^{i\phi}}\right]|b_n - a_n| \\
&\geq \rho - \rho = 0, \qquad \text{by applying (29).}
\end{aligned}$$

This proves that

$$\frac{g(z) * t_\phi(z)}{z} \neq 0, \qquad z \in \mathcal{U}.$$

In view of our observations (28), it follows that $g \in \mathcal{S}^{(j,k)}[A, B, \alpha, b, \sigma]$. This completes the proof of the theorem.
When $j = k = A = -B = 1, \alpha = \sigma = 0$ and $b = 2$ in the above theorem we get (10) proved by Ruschewyh in [11].

Corollary 1. *Let $\mathcal{S}^*$ be the class of starlike functions. Let $f \in \mathcal{A}$ and for all complex numbers η, with $|\mu| < \rho$, if*

$$\frac{f(z) + \eta z}{1 + \eta} \in \mathcal{S}^*. \tag{31}$$

Then $\mathcal{N}_\sigma(f) \subset \mathcal{S}^*$.

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