Bulletin of the *Transilvania* University of Braşov • Vol 11(60), No. 2 - 2018
Series III: Mathematics, Informatics, Physics, 1-18

SOME RESULTS FOR A NEW THREE STEPS ITERATION SCHEME IN BANACH SPACES

H. A. ABASS[1], **A. A. MEBAWONDU***[2] and **O. T. MEWOMO**[3]

Abstract

In this paper, we introduce a three step iteration scheme and establish that this iterative method can be used to approximate fixed point of weak contraction mappings in the frame work of Banach spaces. We also establish that our newly proposed iterative scheme is faster than some existing iterative processes in literature and stability of this iterative scheme is established. Furthermore, we prove that this iterative method is equivalent to M iterative scheme introduced by Ullah et al. in [19], the iterative scheme introduced by Karakaya et al. in [11] and the Mann iterative iteration process. Finally, we established that the rate of convergence of our newly proposed iterative scheme is the same as that of M iteration scheme introduced by Ullah et al. in [19], the iterative scheme introduced by Karakaya et al. in [11] and we present an analytic proof and also a numerical example to support our claim.

2000 *Mathematics Subject Classification:* 47H09; 47H10; 49J20; 49J40 .
Key words: new iterative, weak contraction, normed space, stability.

1 Introduction

Let $(X, \|\cdot\|)$ denote a real Banach space and C be a nonempty closed and convex subset of X. A fixed point problem for a mapping $T : C \to C$ is; find $x \in C$ such that

$$Tx = x. \tag{1}$$

The set of all fixed points of T is denoted by $F(T)$. The theory of fixed point has overtime become an invaluable area of study as many problems in mathematics, engineering, physics, economics, game theory, etc can be transformed into a fixed

[1]School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa, e-mail: hammedabass548@gmail.com
[2]**Corresponding author*, School of Mathematics, Statistics and Computer Science,, University of KwaZulu-Natal, Durban, South Africa, e-mail: dele@aims.ac.za, mebawondua@stu.ukzn.ac.za
[3]School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa, e-mail: mewomo@ukzn.ac.za

point problem. In general to solve fixed point problems analytically is almost impossible and thus the need to consider iterative solution for fixed point problems arises. Over the years researchers have developed several iterative schemes for solving fixed point problems for different operators but the research is still on going in order to develop a faster and more efficient iterative algorithms. Fixed point iteration is valuable for application if it satisfies the following needed requirements:

1. it should converge to a fixed point of an operator;
2. it should be T-stable;
3. it should be fast compare to other existing iteration in literature;
4. it should show data dependence result.

The Picard iterative process

$$x_{n+1} = Tx_n, \ \forall n \in \mathbb{N}, \tag{2}$$

is one of the earliest iterative process used to approximate a solution of (1), where T is a contraction mapping. If T is nonexpansive, the Picard iterative process fails to approximate a solution of (1) even when the existence of the fixed point is guaranteed. To overcome this limitation, researchers in this area developed different iterative processes to approximate fixed points of nonexpansive mappings and other mappings more general than nonexpansive mappings. Some notable iteration schemes that have been developed in this direction over time includes; Mann iteration [10], Ishikawa iteration [6], Krasnosel'skii iteration [13] and the iteration scheme of Noor iteration [14].
In 2007, Agrawal et al., [3] introduced the following iteration process: For any chosen $x_0 \in C$ construct a sequence $\{x_n\}$ by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n, \quad n \geq 1, \end{cases} \tag{3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. They were able to established that the rate of convergence of their iterative process is better than that of the Mann iteration for contractive mappings and that the rate of convergence of the process is the same as Picard iteration.
In 2014, Gursoy and Karakay in [8] introduce new iteration process called Picard-S iteration process, as follows; Let C be a convex subset of a normed space X and $T : C \to C$ be any nonlinear mapping. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n Tx_n, \\ y_n = (1-\beta_n)Tx_n + \beta_n Tz_n \\ x_{n+1} = Ty_n, \quad n \geq 1, \end{cases} \tag{4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. They proved that the Picard-S iteration process can be used to approximate the fixed point of contraction mappings. They gave a numerical example to justify the fact that the Picard-S iteration converges faster than the iterations of Picard, Mann, Noor, Abbas et al.,[2] and a host of others in the literature.

Recently, Thakur et al., [18] introduced the following iterative scheme in the frame work of Banach space. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ y_n = T((1-\alpha_n)x_n + \alpha_n z_n), \\ x_{n+1} = T y_n, \quad n \geq 1, \end{cases} \tag{5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. They established that their iterative scheme is faster than Picard, Mann, Ishikawa, Agrawal [3], Noor and Abbas et al.,[2] iteration process. They gave a numerical example to justify their claim.

In 2017, Karakaya et al. [11] in introduce new iteration process, as follows; Let C be a convex subset of a normed space X and $T : C \to C$ be any nonlinear mapping. For each $x_0 \in C$, the sequence $\{c_n\}$ in C is defined by

$$\begin{cases} a_n = T c_n, \\ b_n = (1-\alpha_n)a_n + \alpha_n T a_n \\ c_{n+1} = T b_n, \quad n \geq 1, \end{cases} \tag{6}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. They proved that their iterative process converges faster than the iterative process (4) and a host of iterative schemes in literature.

In 2018, Ullah et al., in [19] introduce new iteration process called M iteration process, as follows; Let C be a convex subset of a normed space X and $T : C \to C$ be a nonlinear mapping. For each $u_0 \in C$, the sequence $\{u_n\}$ in C is defined by

$$\begin{cases} w_n = (1-\alpha_n)u_n + \alpha_n T u_n, \\ v_n = T w_n \\ u_{n+1} = T v_n, \quad n \geq 1, \end{cases} \tag{7}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. They proved that their iterative process converges faster than the iterative process (5) and a host of iterative schemes in literature.

Motivated by the above facts, we introduce a new modified iteration process in the frame work of Banach space. Let C be a convex subset of a normed space X and $T : C \to C$ be a nonlinear mapping. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = T x_n, \\ y_n = T z_n \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n, \quad n \geq 1, \end{cases} \tag{8}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$.

Definition 1. *Let C be a nonempty subset of a real Banach space X and T be a self mapping on C. Then T is said to be*

1. *an α-contraction mapping, if for each $x, y \in C$ and $\alpha \in (0, 1)$,* $\|Tx - Ty\| \leq \alpha \|x - y\|$.
2. *Kannan mapping, if there exists $b \in (0, \frac{1}{2})$ for each $x, y \in C$,* $\|Tx - Ty\| \leq b[\|x - Tx\| - \|y - Ty\|]$.
3. *Chatterjea mapping, if there exists $c \in (0, \frac{1}{2})$ for each $x, y \in C$,* $\|Tx - Ty\| \leq c[\|x - Ty\| - \|y - Tx\|]$.

Combining the above definitions Zamfirescu in [21], introduce another map called Zamfirescu mapping and proved the following result.

Theorem 1. *Let X be a complete metric space and $T : X \to X$ be a Zamfirescu mapping, that is there exist the real numbers a, b and c satisfying $0 \leq a < 1$ and $b, c \in (0, \frac{1}{2})$ such that for all $x, y \in X$ at least one of the following conditions holds:*

1. $\|Tx - Ty\| \leq a\|x - y\|$;
2. $\|Tx - Ty\| \leq b[\|x - Tx\| + \|y - Ty\|]$;
3. $\|Tx - Ty\| \leq c[\|x - Ty\| + \|y - Tx\|]$.

Then T has a unique fixed point say x^* and the Picard iterative process converges to x^*.

In [4], Berinde introduce a new class of mapping.

Definition 2. *Let X be a Banach space and C be a nonempty subset of X. A mapping $T : C \to C$ is said to be a Berinde mapping, if there exist $\delta \in [0, 1), L \geq 0$ and for each $x, y \in C$ such that*

$$||Tx - Ty|| \leq \delta||x - y|| + L\|x - Tx\|. \tag{9}$$

He was able to establish that the class of mapping satisfying (9) is wider than the class of mapping introduced and studied by Zamfirescu.
The purpose of this paper is to introduce a three step iteration scheme and establish that our newly proposed scheme is faster than some existing iterative processes in literature, stability and data dependency of this iterative scheme is established. Furthermore, we prove that this iterative method is equivalent to M iterative scheme introduced by Ullah et al. in [19], the iterative scheme introduced by Karakaya et al. in [11] and the Mann iterative iteration process. Finally, we established that the rate of convergence of our newly proposed iterative scheme is the same as that of M iteration scheme introduced by Ullah et al. in [19], the iterative scheme introduced by Karakaya et al. in [11] and the Mann iteration. Finally, we present an analytic proof and also a numerical example to support our claim.

2 Preliminaries

Let X be a Banach space and $S_X = \{x \in X : \|x\| \leq 1\}$ be a unit ball in X. For all $\alpha \in (0,1)$ and $x, y \in S_X$ such that $x \neq y$, if $\|(1-\alpha)x + \alpha y\| < 1$, then we say X is strictly convex. If X is a strictly convex Banach space and $\|x\| = \|y\| = \|(1-\lambda)y + \lambda x\| \ \ \forall x, y \in X$ and $\lambda \in (0,1)$, then $x = y$.

Definition 3. *A Banach space X is said to be smooth if*

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{10}$$

exists for all $x, y \in S_X$.

Berinde [4] proposed a method to compare the fastness of two sequences.

Lemma 1. *[4] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers converging to a and b respectively. If $\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = 0$, then a_n converges faster than $\{b_n\}$.*

Definition 4. *[9] Let $\{x_n\}$ be any arbitrary sequence in C. Then, an iteration procedure $x_{n+1} = f(T, x_n)$, converging to fixed point p, is said to be T-stable or stable with respect to T, if for $\epsilon_n = \|x_{n+1} - f(T, x_n)\|$, $\forall n \in \mathbb{N}$, we have $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} x_n = p$.*

Lemma 2. *[20] Let $\{\Psi_n\}$ and $\{\Phi_n\}$ be nonnegative real sequence satisfying the following inequality:*

$$\Psi_{n+1} \leq (1 - \phi_n)\Psi_n + \Phi_n,$$

where $\phi_n \in (0,1)$ for all $n \in \mathbb{N}, \sum_{n=0}^{\infty} \phi_n = \infty$ and $\lim_{n\to\infty} \frac{\Phi_n}{\phi_n} = 0$, then $\lim_{n\to\infty} \Psi_n = 0$.

3 Main Result

In this section, we show that the iteration (8) converges faster than a host of other iterations in literature and we established that the rate of convergence of our newly proposed iterative scheme is the same as that of M iterative scheme introduced by Ullah et al. in [19] and the iterative scheme introduced by Karakaya et al. in [11].

Theorem 2. *Let C be a nonempty closed and convex subset of a normed space X. Let T be a mapping satisfying (9) with $F(T) \neq \emptyset$. Let $\{x_n\}$ be the iterative process defined in (8) with sequence $\{\alpha_n\} \subset (0,1)$ satisfying $\sum_{n=0=}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a unique fixed point of T.*

Proof. Let $x^* \in F(T)$ and from (8), we have

$$\begin{aligned} \|z_n - x^*\| &= \|Tx_n - x^*\| = \|Tx^* - Tx_n\| \\ &\leq \delta\|x - x_n\| + L\|x^* - Tx^*\| \\ &= \delta\|x_n - x^*\|. \end{aligned} \tag{11}$$

Using (8) and (11), we have

$$\begin{aligned}\|y_n - x^*\| &= \|Tz_n - x^*\| \\ &\leq \delta\|z_n - x^*\| \\ &\leq \delta^2\|x_n - x^*\|.\end{aligned} \tag{12}$$

Finally, using (8) and (12), we have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|(1-\alpha_n)y_n + \alpha_n Ty_n - x^*\| \\ &\leq (1-\alpha_n)\|y_n - x^*\| + \alpha_n\|Ty_n - Tx^*\| \\ &\leq (1-\alpha_n)\|y_n - x^*\| + \alpha_n\delta\|y_n - x^*\| \\ &\leq \delta^2(1-(1-\delta)\alpha_n)\|x_n - x^*\|.\end{aligned} \tag{13}$$

From (13), we have the following inequalities:

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \delta^2(1-(1-\delta)\alpha_n)\|x_n - x^*\| \\ \|x_n - x^*\| &\leq \delta^2(1-(1-\delta)\alpha_{n-1})\|x_{n-1} - x^*\| \\ &\vdots \\ \|x_1 - x^*\| &\leq \delta^2(1-(1-\delta)\alpha_0)\|x_0 - x^*\|.\end{aligned} \tag{14}$$

From (14), we have that

$$\|x_{n+1} - x^*\| \leq \|x_0 - x^*\|\delta^{2(n+1)}\prod_{m=0}^{n}(1-(1-\delta)\alpha_m). \tag{15}$$

Since $\{\alpha_m\} \subset (0,1)$ and $\delta \in (0,1)$ for all $m \in \mathbb{N}$, we have $1-(1-\delta)\alpha_m \in (0,1)$. We recall the inequality $1 - x \leq e^{-x}$ for all $x \in [0,1]$, thus from (15), we have

$$\|x_{n+1} - x^*\| \leq \frac{\|x_0 - x^*\|\delta^{2(n+1)}}{e^{(1-\delta)\sum_{m=0}^{n}\alpha_m}}.$$

Taking the limit of both sides of the above inequalities, we have $\lim_{n\to\infty}\|x_n - x^*\| = 0$. Hence, $\{x_n\}$ converges strongly to the fixed point of T. □

Remark 1. *Supposing all the conditions in Theorem 2 holds, using similar approach as in Theorem 2, it is easy to show that the iterative processes* (7)*,* (6)*,* (5) *and* (4) *converges to a unique fixed point.*

Theorem 3. *Let C be a nonempty closed and convex subset of a normed space X and $T : C \to C$ be a mapping satisfying* (9) *with $F(T) \neq \emptyset$. Let $\{x_n\}, \{u_n\}$ and $\{c_n\}$ be defined by the iteration process* (8)*,* (7) *and* (6) *respectively with $\alpha_n \in (a,1)$, where $a \in (0,1)$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then the following are equivalent:*

1. *the iteration process (8) converges to the fixed point x^* of T;*
2. *the iteration process (7) converges to the fixed point x^* of T;*

3. the iteration process (6) converges to the fixed point x^* of T.

Proof. We start by showing that (1) ⇒ (2). Suppose that the iteration (8) converges to the fixed point x^*, of T that is $\lim_{n\to\infty} \|x_n - x^*\| = 0$. Using (8), (7) and the fact that $x^* = Tx^*$, we have

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &= \|(1-\alpha_n)y_n + \alpha_n Ty_n - Tv_n\| \\
&\leq \alpha_n\|Ty_n - Tv_n\| + (1-\alpha_n)\|y_n - Tv_n\| \\
&\leq \alpha_n\delta\|y_n - v_n\| + \alpha_n L\|y_n - Ty_n\| \\
&\quad + (1-\alpha_n)\|y_n - Ty_n\| + (1-\alpha_n)\|Ty_n - Tv_n\| \\
&\leq \alpha_n\delta\|y_n - v_n\| + (\alpha_n L + 1 - \alpha_n)\|y_n - Ty_n\| \\
&\quad + (1-\alpha_n)\delta\|y_n - v_n\| + (1-\alpha_n)L\|y_n - Ty_n\| \\
&= \delta\|y_n - v_n\| + (1-\alpha_n + L)\|y_n - Ty_n\|.
\end{aligned} \tag{16}$$

Now, observe that

$$\|y_n - v_n\| \leq \delta\|z_n - w_n\| + L\|z_n - Tz_n\| \tag{17}$$

and

$$\begin{aligned}
\|z_n - w_n\| &\leq (1-\alpha_n)\|u_n - Tx_n\| + \alpha_n\|Tx_n - Tu_n\| \\
&\leq (1-\alpha_n)\|u_n - x_n\| + (1-\alpha_n)\|x_n - Tx_n\| + \alpha\delta\|x_n - u_n\| \\
&\quad + \alpha_n L\|x_n - Tx_n\| \\
&= (1-(1-\delta)\alpha_n)\|x_n - u_n\| + (1-\alpha_n+\alpha_n L)\|x_n - Tx_n\| \\
&\leq (1-(1-\delta)\alpha_n)\|x_n - u_n\| + (1-\alpha_n+\alpha_n L)[\|x_n - x^*\| \\
&\quad + \|Tx^* - Tx_n\|] \\
&\leq (1-(1-\delta)\alpha_n)\|x_n - u_n\| + (1-\alpha_n+\alpha_n L)(1+\delta)\|x_n - x^*\|.
\end{aligned}$$

Also, we have that

$$\begin{aligned}
\|z_n - Tz_n\| &\leq \|z_n - x^*\| + \|Tx^* - Tz_n\| \\
&\leq (1+\delta)\|z_n - x^*\| \\
&\leq (1+\delta)\delta\|x_n - x^*\|
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - Ty_n\| &\leq (1+\delta)\|y_n - x^*\| \\
&\leq (1+\delta)\delta^2\|x_n - x^*\|.
\end{aligned}$$

Therefore, (17) becomes

$$\begin{aligned}
&\|y_n - v_n\| \leq \delta(1-(1-\alpha_n)\delta)\|x_n - u_n\| + \delta(1+\delta)(1-\alpha_n+\alpha L)\|x_n - x^*\| \\
&\quad + L\delta\|x_n - x^*\|
\end{aligned}$$

and so (16) becomes

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\leq \delta^2(1-(1-\delta)\alpha_n)\|x_n-u_n\|\\ &+\delta^2(1+\delta)(2+2L-2\alpha_n+\alpha_nL)\|x_n-x^*\|\\ &\leq (1-(1-\delta)\alpha_n)\|x_n-u_n\|+\\ &\delta^2(1+\delta)(2+2L-2\alpha_n+\alpha_nL)\|x_n-x^*\|.\end{aligned}$$

We denote by

$$\begin{aligned}\Psi_n &= \|x_n-u_n\|\\ \phi_n &= (1-\delta)\alpha_n \in (0,1)\\ \Phi_n &= \delta^2(1+\delta)(2+2L-2\alpha_n+\alpha_nL)\|x_n-x^*\|.\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n-x^*\|=0$ and $\lim_{n\to\infty}\frac{\Phi_n}{\phi_n}=0$. Clearly, all the assumption in Lemma 2 are satisfied, we then have that $\|x_n-u_n\|\to 0$ as $n\to\infty$. Therefore, we have

$$\|u_n-x^*\|\leq\|x_n-u_n\|+\|x_n-x^*\|\to 0 \quad\text{as}\quad n\to\infty.$$

Hence $\lim_{n\to\infty}\|u_n-x^*\|=0$. That is the iteration (7) converges to the fixed point x^* of T.
Secondly, we show that (2) ⇒ (3). Suppose that the iteration (7) converges to x^* of T that is $\lim_{n\to\infty}\|u_n-x^*\|=0$. Using (7), (6) and the fact that $x^*=Tx^*$, we have

$$\begin{aligned}\|u_{n+1}-c_{n+1}\| &= \|Tv_n-Tb_n\|\\ &\leq \delta\|v_n-b_n\|+L\|v_n-Tv_n\|.\end{aligned}\tag{18}$$

Now, observe that

$$\begin{aligned}\|v_n-b_n\| &\leq (1-\alpha_n)\|a_n-Tw_n\|+\alpha_n\|Ta_n-Tw_n\|\\ &\leq (1-\alpha_n)[\|Tw_n-w_n\|+\|w_n-a_n\|]+\alpha_n\delta\|w_n-a_n\|\\ &+\alpha_nL\|w_n-Tw_n\|\\ &= (1-\alpha_n+\alpha_nL)\|Tw_n-w_n\|+(1-(1-\delta)\alpha_n)\|w_n-a_n\|.\end{aligned}\tag{19}$$

Also,

$$\begin{aligned}\|w_n-a_n\| &\leq (1-\alpha_n)\|u_n-Tc_n\|+\alpha_n\|Tu_n-Tc_n\|\\ &\leq (1-\alpha_n)[\|u_n-Tu_n\|+\|Tu_n-Tc_n\|]+\alpha_n\|Tu_n-Tc_n\|\\ &= (1-\alpha_n)\|u_n-Tu_n\|+\|Tu_n-Tc_n\|\\ &\leq (1-\alpha_n)\|u_n-Tu_n\|+\delta\|u_n-c_n\|+L\|u_n-Tu_n\|\\ &= (1-\alpha_n+L)\|Tu_n-u_n\|+\delta\|u_n-c_n\|\\ &\leq (1-\alpha_n+L)[\|Tu_n-Tx^*\|+\|x^*-u_n\|]+\delta\|u_n-c_n\|\\ &\leq (1-\alpha+L)(1+\delta)\|u_n-x^*\|++\delta\|u_n-c_n\|\end{aligned}$$

and

$$\begin{aligned}\|Tw_n - w_n\| &\leq \|Tw_n - Tx^*\| + \|x^* - w_n\| \\ &= \|Tx^* - Tw_n\| + \|x^* - w_n\| \\ &\leq (1+\delta)\|w_n - x^*\| \\ &\leq (1+\delta)(1-(1-\delta)\alpha_n)\|u_n - x^*\|.\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}\|v_n - Tv_n\| &\leq \|v_n - x^*\| + \|Tx^* - Tv_n\| \\ &\leq (1+\delta)\|v_n - x^*\| \\ &\leq (1+\delta)\delta(1-(1-\delta)\alpha_n)\|u_n - x^*\|.\end{aligned}$$

Therefore, we have that (19) becomes

$$\begin{aligned}\|v_n - b_n\| \leq \delta(1-(1-\delta)\alpha_n)\|u_n - c_n\| &+ [(1-\alpha_n+\alpha_n L)(1+\delta)(1-(1-\delta)\alpha_n) \\ &+ (1-(1-\delta)\alpha_n)[(1-\alpha_n+l)(1+\delta)]\|u_n - x^*\|.\end{aligned}$$

Finally, we have that

$$\begin{aligned}\|u_{n+1} - c_{n+1}\| &\leq \delta^2(1-(1-\delta)\alpha_n)\|u_n - c_n\| \\ &\quad + \delta[(1-\alpha_n+\alpha_n L)(1+\delta)(1-(1-\delta)\alpha_n) \\ &\quad + (1-(1-\delta)\alpha_n)(1-\alpha_n+L)(1+\delta) \\ &\quad + L(1+\delta)(1-(1-\delta)\alpha_n)]\|u_n - x^*\| \\ &\leq (1-(1-\delta)\alpha_n)\|u_n - c_n\| \\ &\quad + \delta[(1-\alpha_n+\alpha_n L)(1+\delta)(1-(1-\delta)\alpha_n) \\ &\quad + (1-(1-\delta)\alpha_n)(1-\alpha_n+L)(1+\delta) \\ &\quad + L(1+\delta)(1-(1-\delta)\alpha_n)]\|u_n - x^*\|.\end{aligned}$$

We denote by

$$\begin{aligned}\Psi_n &= \|u_n - c_n\| \\ \phi_n &= (1-\delta)\alpha_n \in (0,1) \\ \Phi_n &= \delta[(1-\alpha_n+\alpha_n L)(1+\delta)(1-(1-\delta)\alpha_n) \\ &\quad + (1-(1-\delta)\alpha_n)(1-\alpha_n+L)(1+\delta) + L(1+\delta)\delta(1-(1-\delta)\alpha_n)]\|u_n - x^*\|.\end{aligned}$$

Since $\lim_{n\to\infty}\|u_n - x^*\| = 0$ and $\lim_{n\to\infty}\frac{\Phi_n}{\phi_n} = 0$. Clearly, all the assumption in Lemma 2 are satisfied, we then have that $\|u_n - c_n\| \to 0$ as $n \to \infty$. Therefore, we have

$$\|c_n - x^*\| \leq \|c_n - u_n\| + \|u_n - x^*\| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $\lim_{n\to\infty}\|c_n - x^*\| = 0$. That is the iteration (6) converges to the fixed point x^* of T.

Lastly, we show that (3) $\Rightarrow$ (1). Suppose that the iteration (6) converges to x^* of T that is $\lim_{n\to\infty}\|c_n - x^*\| = 0$. Using (8), (6) and the fact that $x^* = Tx^*$, we have

$$\begin{aligned}\|x_{n+1} - c_{n+1}\| &\leq (1-\alpha_n)\|y_n - Tb_n\| + \alpha_n\|Ty_n - Tb_n\| \\ &\leq (1-\alpha_n)\|y_n - b_n\| + (1-\alpha_n)\|b_n - Tb_n\| + \alpha_n\delta\|y_n - b_n\| \\ &+ L\alpha_n\|b_n - Tb_n\| \qquad (20) \\ &= (1-(1-\delta)\alpha_n)\|y_n - b_n\| + (1-\alpha_n+\alpha_n L)\|b_n - Tb_n\|.\end{aligned}$$

Now, observe that

$$\begin{aligned}\|b_n - y_n\| &\leq (1-\alpha_n)\|a_n - Tz_n\| + \alpha_n\|Ta_n - Tz_n\| \\ &\leq (1-\alpha_n)\|a_n - Ta_n\| + (1-a_n)\|Ta_n - Tz_n\| + \alpha_n\|Ta_n - Tz_n\| \\ &= (1-\alpha_n)\|a_n - Ta_n\| + \|Ta_n - Tz_n\| \qquad (21) \\ &\leq (1-\alpha_n+L)\|a_n - Ta_n\| + \delta\|a_n - z_n\|.\end{aligned}$$

Also, we have

$$\begin{aligned}\|a_n - z_n\| &= \|Tc_n - Tx_n\| \\ &\leq \delta\|c_n - x_n\| + L\|c_n - Tc_n\| \\ &\leq \delta\|c_n - x_n\| + L\|c_n - x^*\| + L\delta\|x^* - c_n\| \\ &= \delta\|c_n - x_n\| + (1+\delta)L\|c_n - x^*\|.\end{aligned}$$

More so, we have

$$\begin{aligned}\|a_n - Ta_n\| &\leq \|a_n - x^*\| + \|Tx^* - Ta_n\| \\ &\leq (1+\delta)\|a_n - x^*\| \\ &\leq \delta(1+\delta)\|c_n - x^*\|.\end{aligned}$$

Finally, we have

$$\begin{aligned}\|b_n - Tb_n\| &\leq (1+\delta)\|b_n - x^*\| \\ &\leq \delta(1+\delta)(1-(1-\delta)\alpha_n)\|c_n - x^*\|.\end{aligned}$$

Therefore, we have that (21) becomes

$$\|b_n - y_n\| \leq (1-\alpha_n+L)\delta(1+\delta)\|c_n - x^*\| + \delta^2\|c_n - x_n\| + \delta(1+\delta)L\|c_n - x^*\|$$

and (20) becomes

$$\begin{aligned}\|x_{n+1} - c_{n+1}\| &\leq \delta^2(1-(1-\delta)\alpha_n)\|c_n - x_n\| \\ &+ (1-(1-\delta)\alpha_n)\delta(1+\delta)(2-\alpha_n+L)\|c_n - x^*\| \\ &+ (1-\alpha_n+\alpha_n L)(1+\delta)^2\delta(1-(1-\delta)\alpha_n)\|c_n - x^*\| \\ &\leq (1-(1-\delta)\alpha_n)\|c_n - x_n\| + [(1-(1-\delta)\alpha_n)\delta(1+\delta)(2-\alpha_n+L) \\ &+ (1-\alpha_n+\alpha_n L)(1+\delta)^2\delta(1-(1-\delta)\alpha_n)]\|c_n - x^*\|\end{aligned}$$

Since $\lim_{n\to\infty}\|c_n - x^*\| = 0$ and using similar approach as above. Clearly, all the assumption in Lemma 2 are satisfied, we then have that $\|c_n - x_n\| \to 0$ as $n \to \infty$. Therefore, we have

$$\|x_n - x^*\| \le \|c_n - x_n\| + \|c_n - x^*\| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $\lim_{n\to\infty}\|x_n - x^*\| = 0$. That is the iteration (8) converges to the fixed point x^* of T. □

We recall the Mann iteration [10]: For any chosen $u_0 \in C$ construct a sequence $\{u_n\}$ by

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n, \quad n \ge 1, \tag{22}$$

where $\{\alpha_n\}$ is a sequences in $(0, 1)$.

Theorem 4. *Let C be a nonempty closed and convex subset of a normed space X and $T : C \to C$ be a mapping satisfying (9) with $F(T) \neq \emptyset$. Let $\{x_n\}$ be defined by the iteration process (8) and $\{u_n\}$ be defined by the iteration process (22) with $\alpha_n \in (a, 1)$, where $a \in (0, 1)$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then the following are equivalent:*

1. *the iteration process (22) converges to the fixed point x^* of T;*
2. *the iteration process (8) converges to the fixed point x^* of T.*

Proof. Firstly, we show that (1) ⇒ (2). Suppose that the iteration (22) converges to x^* of T that is $\lim_{n\to\infty}\|u_n - x^*\| = 0$. Using (8), (22) and the fact that $x^* = Tx^*$, we have

$$\begin{aligned}\|u_{n+1} - x_{n+1}\| &\le (1-\alpha_n)\|u_n - y_n\| + \alpha_n\|Tu_n - Ty_n\| \\ &\le (1-(1-\delta)\alpha_n)\|u_n - y_n\| + \alpha_n L\|u_n - Tu_n\|.\end{aligned} \tag{23}$$

Now, observe that

$$\begin{aligned}\|u_n - y_n\| &= \|u_n - Tz_n\| \\ &\le \|u_n - Tu_n\| + \|Tu_n - Tz_n\| \\ &\le \delta\|u_n - z_n\| + (1+L)\|u_n - Tu_n\|,\end{aligned}$$

$$\begin{aligned}\|u_n - z_n\| &\le \|u_n - Tu_n\| + \|Tu_n - Tx_n\| \\ &\le (1+L)\|u_n - Tu_n\| + \delta\|u_n - x_n\| \\ &\le (1+L)(1+\delta)\|u_n - x^*\| + \delta\|u_n - x_n\|\end{aligned}$$

and

$$\|u_n - Tu_n\| \le (1+\delta)\|u_n - x^*\|.$$

We therefore have that

$$\|u_n - y_n\| \leq \delta(1+L)(1+\delta)\|u_n - x^*\| + \delta^2\|u_n - x_n\| + (1+L)(1+\delta)\|u_n - x^*\|,$$

so that (23) becomes

$$\begin{aligned}\|u_{n+1} - x_{n+1}\| &\leq (1-(1-\delta)\alpha_n)\delta(1+L)(1+\delta)\|u_n - x^*\| \\ &+ \delta^2(1-(1-\delta)\alpha_n)\|u_n - x_n\| \\ &+ (1-(1-\delta)\alpha_n)(1+L)(1+\delta)\|u_n - x^*\| + \alpha_n L(1+\delta)\|u_n - x^*\| \\ &\leq (1-(1-\delta)\alpha_n)\|u_n - x_n\| \\ &+ [(1-(1-\delta)\alpha_n)\delta(1+L)(1+\delta) + (1-(1-\delta)\alpha_n)(1+L)(1+\delta) \\ &+ \alpha_n L(1+\delta)]\|u_n - x^*\|.\end{aligned}$$

Since $\lim_{n\to\infty}\|u_n - x^*\| = 0$ and $\lim_{n\to\infty}\frac{\Phi_n}{\phi_n} = 0$. Clearly, all the assumption in Lemma 2 are satisfied, we then have that $\|u_n - x_n\| \to 0$ as $n \to \infty$. Therefore, we have

$$\|x_n - x^*\| \leq \|x_n - u_n\| + \|u_n - x^*\| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $\lim_{n\to\infty}\|x_n - x^*\| = 0$. That is the iteration (8) converges to the fixed point x^* of T.
Lastly, we show that (2) $\Rightarrow$ (1). Suppose that the iteration (8) converges to x^* of T that is $\lim_{n\to\infty}\|x_n - x^*\| = 0$. Using (8), (22) and the fact that $x^* = Tx^*$, we have

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| &\leq (1-\alpha_n)\|y_n - u_n\| + \alpha_n\|Ty_n - Tu_n\| \\ &\leq (1-(1-\delta)\alpha_n)\|y_n - u_n\| + \alpha_n L\|y_n - Ty_n\|. \end{aligned} \tag{24}$$

Now, observe that

$$\|y_n - u_n\| \leq \|Tz_n - z_n\| + \|z_n - u_n\|,$$

$$\begin{aligned}\|Tz_n - z_n\| &\leq \|Tz_n - Tx^*\| + \|x^* - z_n\| \\ &\leq (1+\delta)\|z_n - x^*\| \\ &\leq \delta(1+\delta)\|x_n - x^*\|\end{aligned}$$

and

$$\begin{aligned}\|z_n - u_n\| &\leq \|Tx_n - x_n\| + \|x_n - u_n\| \\ &\leq \|Tx_n - Tx^*\| + \|x^* - x_n\| + \|x_n - u_n\| \\ &\leq (1+\delta)\|x_n - x^*\| + \|x_n - u_n\|.\end{aligned}$$

As such, we have

$$\|y_n - u_n\| \leq [\delta(1+\delta) + (1+\delta)]\|x_n - x^*\| + \|x_n - u_n\|.$$

More so, we have

$$\begin{aligned}\|y_n - Ty_n\| &\leq \|y_n - x^*\| + \|Tx^* - Ty_n\| \\ &\leq (1+\delta)\|y_n - x^*\| \\ &\leq \delta^2(1+\delta)\|x_n - x^*\|.\end{aligned}$$

We therefore have that (24) becomes

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| &\leq (1-(1-\delta)\alpha_n)\|x_n - u_n\| \\ &\quad + [(\delta(1+\delta) + (1+\delta))(1-(1-\delta)\alpha_n) + \alpha_n L\delta^2(1+\delta)]\|x_n - x^*\|.\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - x^*\| = 0$ and $\lim_{n\to\infty}\frac{\Phi_n}{\phi_n} = 0$. Clearly, all the assumption in Lemma 2 are satisfied, we then have that $\|x_n - u_n\| \to 0$ as $n \to \infty$. Therefore, we have

$$\|u_n - x^*\| \leq \|u_n - x_n\| + \|x_n - x^*\| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $\lim_{n\to\infty}\|u_n - x^*\| = 0$. That is the iteration (22) converges to the fixed point x^* of T. □

Corollary 1. *Let C be a nonempty closed and convex subset of a normed space X and $T : C \to C$ be a contraction mapping with $k \in (0,1), F(T) \neq \emptyset$ such that $\alpha_n \in (a,1)$, where $a \in (0,1)$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then the following are equivalent:*

1. *the iteration process (8) converges to the fixed point x^* of T;*
2. *the iteration process (7) converges to the fixed point x^* of T;*
3. *the iteration process (6) converges to the fixed point x^* of T;*
4. *the iteration process (22) converges to the fixed point x^* of T.*

Remark 2. *We claim that the iterative processes* (8), (7) *and* (6) *converges at the same rate and converges faster than iterative processes* (5) *and* (4)*, also, the iterative process* (5) *and* (4) *converges at the same rate, provided that the following upper bounds conditions hold $\alpha \leq \alpha_n < 1$ and $\beta\alpha \leq \beta_n\alpha_n < 1$ for some $\alpha, \beta > 0$ for all $n \in \mathbb{N}$.*

Proof of Claim: *From Theorem 2, we have the iteration process* (8) *takes the form*

$$\|x_{n+1} - x^*\| \leq \|x_0 - x^*\|\delta^{2(n+1)}\prod_{m=0}^{n}(1-(1-\delta)\alpha_m).$$

From our assumption that $\alpha_n \leq \alpha < 1$, we then have that

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \|x_0 - x^*\|\delta^{2(n+1)}\prod_{m=0}^{n}(1-(1-\delta)\alpha) \\ &= \|x_0 - x^*\|\delta^{2(n+1)}(1-(1-\delta)\alpha)^{n+1}.\end{aligned} \tag{25}$$

Using similar approach as above, it is easy to get that iterative process (5) *takes the form*

$$\begin{aligned}\|p_{n+1}-x^*\| &\leq \|p_0-x^*\|\delta^{2(n+1)}\prod_{m=0}^{n}(1-(1-\delta)\alpha_m\beta_m)\\ &= \|p_0-x^*\|\delta^{2(n+1)}(1-(1-\delta)\alpha\beta)^{n+1}\end{aligned}$$

and (4) *takes the form say*

$$\begin{aligned}\|q_{n+1}-x^*\| &\leq \|q_0-x^*\|\delta^{2(n+1)}\prod_{m=0}^{n}(1-(1-\delta)\alpha_m\beta_m)\\ &= \|q_0-x^*\|\delta^{2(n+1)}(1-(1-\delta)\alpha\beta)^{n+1}.\end{aligned}$$

More so, it is easy to see that iterative process (7) *and* (6) *takes the form*

$$\begin{aligned}\|u_{n+1}-x^*\| &\leq \|u_0-x^*\|\delta^{2(n+1)}\prod_{m=0}^{n}(1-(1-\delta)\alpha_m)\\ &= \|u_0-x^*\|\delta^{2(n+1)}(1-(1-\delta)\alpha)^{n+1}\end{aligned}$$

and

$$\begin{aligned}\|c_{n+1}-x^*\| &\leq \|c_0-x^*\|\delta^{2(n+1)}\prod_{m=0}^{n}(1-(1-\delta)\alpha_m)\\ &= \|c_0-x^*\|\delta^{2(n+1)}(1-(1-\delta)\alpha)^{n+1}.\end{aligned}$$

Suppose

$$\begin{aligned}a_n &= \|x_0-x^*\|\delta^{2(n+1)}(1-(1-\delta)\alpha)^{n+1},\\ b_n &= \|p_0-x^*\|\delta^{2(n+1)}(1-(1-\delta)\alpha\beta)^{n+1},\\ r_n &= \|q_0-x^*\|\delta^{2(n+1)}(1-(1-\delta)\alpha\beta)^{n+1},\end{aligned}$$

we then have that

$$\begin{aligned}\frac{a_n}{b_n} &= \frac{\delta^{2(n+1)}(1-(1-\delta)\alpha)^{n+1}\|x_0-x^*\|}{\delta^{2(n+1)}(1-(1-\delta)\alpha\beta)^{n+1}\|p_0-x^*\|}\to 0 \quad as \quad n\to\infty,\\ \frac{a_n}{r_n} &= \frac{\delta^{2(n+1)}(1-(1-\delta)\alpha)^{n+1}\|x_0-x^*\|}{\delta^{2(n+1)}(1-(1-\delta)\alpha\beta)^{n+1}\|q_0-x^*\|}\to 0 \quad as \quad n\to\infty.\end{aligned}$$

Consequently, for the iterative process (8)*, we have an upper bound for the degree of convergence which is better than the found upper bounds of degree of convergence by the iterative processes* (5) *and* (4)*. It is also easy to see that the guaranteed upper bound for the degree of convergence of processes* (8)*,* (7) *and* (6) *converges to* 0 *faster than the upper found upper bounds for the degree of convergence by the iterative processes (*(5) *and* (4)*. In what follows, we support the analytical proof with a numerical examples.*

Example *Let $X = \mathbb{R}$ and $C = [0, 50]$. Let $T : C \to C$ be a mapping defined by $T(x) = \sqrt{x^2 - 9x + 54}$. It is clear that the fixed point of T is 6. Choose $\alpha_n = \beta_n = \frac{3}{4}$, with an initial value of $x_1 = 30$. The corresponding table shows the values of the iterative processes.*

Step	Our Algorithm	Ullah et al.	Karakaya et al.	Thakur et al.	Gürsoy et al.
1	30	30	30	30	30
2	19.732829	19.724261	19.727530	20.393512	20.395231
3	11.106978	11.069355	11.082393	12.090075	12.096497
4	6.619690	6.568135	6.587878	7.037310	7.049479
5	6.021424	6.018470	6.019856	6.051060	6.052872
6	6.000591	6.000508	6.000547	6.001874	6.001944
7	6.000016	6.000014	6.000015	6.000068	6.000070
8	6.000000	6.000000	6.000000	6.000002	6.000003
9	6.000000	6.000000	6.000000	6.000000	6.000000

Comparison shows that the iterative processes (8), (7) *and* (6) *converges faster than the iterative processes* (5) *and* (4)*. More so, the iterative processes* (8), (7) *and* (6) *converges at the same rate, likewise iterative processes* (5) *and* (4)*.*

Example *Let $X = \mathbb{R}$ and $C = [0, 1]$. Let $T : C \to C$ be a mapping defined by $T(x) = \frac{x}{2}$. It is clear that the fixed point of T is zero (0). Choose $\alpha_n = \beta_n = \frac{1}{2}$, with an initial value of $x_1 = 0.9$. The corresponding table shows the values of the iterative processes.*

Step	Our Algorithm	Ullah et al.	Karakaya et al.	Thakur et al.	Gursoy et al.
1	0.9	0.9	0.9	0.9	0.9
2	1.687500e-01	1.687500e-01	1.687500e-01	1.968750e-01	1.968750e-01
3	3.164063e-02	3.164063e-02	3.164063e-02	4.306641e-02	4.306641e-02
4	5.932617e-03	5.932617e-03	5.932617e-03	9.420776e-03	9.420776e-03
5	1.112366e-03	1.112366e-03	1.112366e-03	2.060795e-03	2.060795e-03
6	2.085686e-04	2.085686e-04	2.085686e-04	4.507989e-04	4.507989e-04
7	3.910661e-05	3.910661e-05	3.910661e-05	9.861225e-05	9.861225e-05
8	7.332489e-06	7.332489e-06	7.332489e-06	2.157143e-05	2.157143e-05
9	1.374842e-06	1.374842e-06	1.374842e-06	4.718750e-06	4.718750e-06
10	2.577828e-07	2.577828e-07	2.577828e-07	1.032227e-06	1.032227e-06
11	4.833428e-08	4.833428e-08	4.833428e-08	2.257996e-07	2.257996e-07
12	9.062677e-09	9.062677e-09	9.062677e-09	4.939366e-08	4.939366e-08
13	1.699252e-09	1.699252e-09	1.699252e-09	1.080486e-08	1.080486e-08
14	3.186097e-10	3.186097e-10	3.186097e-10	2.363564e-09	2.363564e-09
15	5.973933e-11	5.973933e-11	5.973933e-11	5.170296e-10	5.170296e-10

Comparison shows that the iterative processes (8), (7) *and* (6) *converges faster than the iterative processes* (5) *and* (4)*. More so, the iterative processes* (8), (7) *and* (6) *converges at the same rate, likewise iterative processes* (5) *and* (4)*.*

Theorem 5. *Let C be a nonempty closed and convex subset of a normed space X. Let T be a mapping satisfying* (9) *with $F(T) \neq \emptyset$. Let $\{x_n\}$ be defined by (8) with sequence $\{\alpha_n\}$ in $(a, 1)$, where $a \in (0, 1)$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the iteration (8) is T-stable.*

Proof. let $\{x_n\} \subset X$ be any arbitrary sequence in C and suppose that the sequence generated by (8) is $x_{n+1} = f(T, x_n)$ converging to a unique fixed point x^* and

that $\epsilon_n = \|x_{n+1} - f(T, x_n)\|$. To establish that T is stable, we need to prove that $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} x_n = x^*$.
Suppose that $\lim_{n\to\infty} \epsilon_n = 0$. Using triangular inequality and (13), we have that

$$\begin{aligned}\|x_{n+1} - x^*\| &\le \|x_{n+1} - f(T, x_n)\| + \|f(T, x_n) - x^*\| \\ &= \epsilon_n + \|(1-\alpha_n)T(Tx_n) + \alpha_n T(T(Tx_n)) - x^*\| \\ &\le \epsilon_n + \delta^2(1-(1-\delta)\alpha_n)\|x_n - x^*\| \\ &\le (1-(1-\delta)\alpha_n)\|x_n - x^*\| + \epsilon_n.\end{aligned}$$

Let $\Psi_n = \|x_n - x^*\|, \phi_n = (1-\delta)\alpha_n \in (0,1)$ and $\Phi_n = \epsilon_n$. By our hypothesis that, $\lim_{n\to\infty} \epsilon_n = 0$, it follows that $\lim_{n\to\infty} \frac{\epsilon_n}{(1-\delta)\alpha_n} = \lim_{n\to\infty} \frac{\Phi_n}{\phi_n} = 0$. Using Lemma (2), we have that $\lim_{n\to\infty} x_n = x^*$.

Conversely, suppose that $\lim_{n\to\infty} x_n = x^*$. We have that

$$\begin{aligned}\epsilon_n &= \|x_{n+1} - f(T, x_n)\| \\ &\le \|x_{n+1} - x^*\| + \|x^* - f(T, x_n)\| \\ &\le \|x_{n+1} - x^*\| + \delta^2(1-(1-\delta)\alpha_n)\|x_n - x^*\|.\end{aligned}$$

Using our hypothesis that $\lim_{n\to\infty} x_n = x^*$, we then have that $\lim_{n\to\infty} \epsilon_n = 0$. Hence, iteration (8) is stable with respect to T. □

Acknowledgements: The authors are sincerely grateful to the reviewer and the editor for their careful reading, constructive comments and suggestions which have been useful for the improvement of this manuscript.

References

[1] Abbas, M., Kadelburg, Z. and Sahu, D. R., *Fixed point theorems for Lipschitzian type mappings in CAT(0) spaces,* Math. Comp. Modeling **55** (2012), 1418–1427.

[2] Abbas, M. and Nazir, T., *A new faster iteration process applied to constrained minimization and feasibility problems,* Matematiqki Vesnik **66** (2014), 223–234.

[3] Agarwal, R. P., O'Regan, D. and Sahu, D. R., *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings,* J. Convex Anal. **8** (2007), 61–79.

[4] Berinde, V., *Picard iteration converges faster than Mann iteration for a class of quasicontractive operators,* Fixed Point Theory and Applications **2** (2004), 97–105.

[5] Berinde, V., *On the approximation of fixed points of weak contractive mappings,* Carpath J. Math. **19** (2003), 7–22.

[6] Ishikawa, S., *Fixed points by new iteration method,* Proc. Am. Math. Soc. **149** (1974), 147–150.

[7] F. Gürsoy, F., Karakaya, V. and Rhoades, B. E., *Data dependence results of new multi-step and S-iterative schemes for contractive-like operators,* Fixed Point Theory Appl. (2013), 1–12.

[8] F. Gürsoy, F. and Karakaya, V., *A Picard-S hybrid type iteration method for solving a differential equation with retarded argument,* (2014), arXiv:1403.2546v2.

[9] Harder, A. M., *Fixed point theory and stability results for fixed point iteration procedures. PhD thesis,* University of Missouri-Rolla, Missouri, 1987.

[10] Mann, W. R., *Mean value methods in iteration,* Proc. Am. Math. Soc. **4** (1953, 506–510.

[11] V. Karakaya, V., Atalan, Y. and Dogan, K., *On fixed point result for a three steps iteration process in Banach space,* Fixed Point Theory, **18** (2017), no. 2, 625–640.

[12] Khan, A. R., Gïsoy, F. and Kumar, V., *Stability and data dependence results for Jungck-Khan iterative scheme,* Turk J. Math. **40** (2016), 631–640.

[13] Krasnosel'skii, M. A., *Two remarks on the method of successive approximations,* Usp. Mat. Nauk. **10** (1955), 123–127.

[14] Noor, M. A., *New approximation schemes for general variational inequalities,* J. Math. Anal. Appl. **251** (2000), 217–229.

[15] M. O. Olatinwo, *Some results on the continuous dependence of the fixed points in normed linear space,* Fixed Point Theory Appl. **10** (2009), 151–157.

[16] Soltuz, S. M. and Grosan, T., *Data dependence for Mann iteration,* Octogon Math Mag. **9** (2001), 825–828.

[17] Soltuz, S. M. and Grosan, T., *Data dependence for Ishikawa iteration when dealing with contractive like operators,* Fixed Point Theory Appl. doi:10.1155/2008/242916.

[18] Thakur, B. S., Thakur, D. and Postolache, M., *A new iterative scheme for numerical reckoning fixed points of Suzukis generalized nonexpansive mappings,* App. Math. Comp. **275** (2016), 147–155.

[19] Ullah, K. and Arshad, M., *Numerical reckoning fixed points for Suzuki generalized nonexpansive mappings via new iteration process,* Filomat. **32** (2018), no.1, 187–196.

[20] Weng, X., *Fixed point iteration for local strictly pseudocontractive mapping,* Proc Am Math Soc. **113** (1991), 727–731.

[21] Zamfirescu, T., *Fix point theorems in metric spaces,* Arch. Math. (Basel) **23** (1972), 292–298.