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MINIMUM COST FLOW IN A NETWORK WITH AN UNDERESTIMATED ARC CAPACITY

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Abstract

There are lot of real world problems that can be modeled and solved as minimum cost flow problems. Sometimes in this problems minor changes may occur as, for instance, the capacity of an arc may vary in time. In this paper we study the case in which the capacity of a given arc is augmented by a given value a. Supposing that a minimum cost flow has been already established in a network G, we focus on the problem of finding a minimum cost flow in a network having the same structure as G and the same arc costs and capacities excepting one: the capacity of a given arc (k, l) which has been increased by a units. We describe a method for obtaining a minimum cost flow in the modified network in $O(am)$ time starting from the minimum cost flow in the original network.

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1 Introduction

Network flow problems form a group of network optimization problems with widespread and diverse applications. Over the past 60 years researchers have made continuous improvements to algorithms for solving several classes of problems. From the late 1940s through the 1950s, researchers designed many of the fundamental algorithms for network flow, including methods for maximum flow and minimum cost flow problems. In the next decades, there are many research contributions concerning improving the computational complexity of network flow algorithms by using enhanced data structures, techniques of scaling the problem data etc.

The minimum cost flow problem, which combines the shortest paths with the maximum flows, is one of the complex problems in network flow theory and it has been studied extensively. The importance of the minimum cost flow problem is also due to the fact that it arises in almost all industries, including agriculture, communications, defense, education, energy, health care, medicine, manufacturing, retailing and transportation. And

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in these real world applications it is possible that the structure of the network changes in time.

Let $G = (N, A)$ be a directed graph, defined by a set N of n nodes and a set A of m arcs. Each arc $(x, y) \in A$ has a capacity $c(x, y)$ and a cost $b(x, y)$. We associate with each node $x \in N$ a number $v(x)$ which indicates its supply or demand depending on whether $v(x) > 0$ or $v(x) < 0$. In the directed network $G = (N, A, c, b, v)$, the minimum cost flow problem is to determine the flow $f(x, y)$ on each arc $(x, y) \in A$ which

$$minimize \sum_{(x,y)\in A} b(x, y)f(x, y) \tag{1}$$

subject to

$$\sum_{y|(x,y)\in A} f(x, y) - \sum_{y|(y,x)\in A} f(y, x) = v(x), \qquad \forall x \in N \tag{2}$$

$$0 \leq f(x, y) \leq c(x, y), \qquad \forall (x, y) \in A. \tag{3}$$

A flow f satisfying conditions 2 and 3 is a *feasible* flow.

The value $\sum_{(x,y)\in A} b(x, y)f(x, y)$ is the *cost* of the flow.

A *pseudoflow* is a function $f : A \to \Re_+$ satisfying only conditions 3. For any pseudoflow f the *imbalance* of node x is defined as $e(x) = f(N, x) - f(x, N)$, for all $x \in N$. If $e(x) > 0$ for some node x, we refer to $e(x)$ as the *excess* of node x; if $e(x) < 0$, we refer to $-e(x)$ as the *deficit* of node x. If $e(x) = 0$ for some node x, we refer to node x as *balanced*. Consequently, a flow is a particular case of psedoflow.

The residual network $G(f) = (N, A(f))$ corresponding to a pseudoflow f is defined as follows. For each arc $(x, y) \in A$ one creates two arcs (x, y) and (y, x). The arc (x, y) has the cost $b(x, y)$ and the residual capacity $r(x, y) = c(x, y) - f(x, y)$ and the arc (y, x) has the cost $b(y, x) = -b(x, y)$ and the residual capacity $r(y, x) = f(x, y)$. The residual network consists only of those arcs with positive residual capacity.

We shall assume that the minimum cost flow problem satisfies the following assumptions:

1. All data (cost, supply/demand and capacity) are integral.
2. The network contains no directed negative cost cycle of infinite capacity.
3. All arc costs are nonnegative.
4. The supplies/demands at the nodes satisfy the condition $\sum_{x\in N} v(x) = 0$ and the minimum cost flow problem has a feasible solution.
5. The network contains an uncapacitated directed path (i.e. each arc in the path has infinite capacity) between every pair of nodes.

All these assumptions can be made without any loss of generality (for details see [1]).

The classical algorithms for determining a minimum cost flow, that have been developed between 1950 and 1960 and can be found in [1], are of two types:

1. those that maintain feasible solutions and strive toward optimality
2. those that maintain infeasible solutions that satisfy optimality conditions and strive toward feasibility.

The most known classical algorithms of the first type are cycle-canceling algorithm and out-of-kilter algorithm. The cycle-canceling algorithm maintains a feasible flow at every iteration, augments flow along negative cycle in the residual network and terminates when there is no more negative cycle in the residual network, which means that the flow is a minimum cost flow. The out-of-kilter algorithm maintains a feasible flow at every iteration and augments flow along shortest path in order to reach the optimality.

The most known classical algorithms of the second type are the successive shortest path algorithm and primal-dual algorithm. The successive shortest path algorithm maintains an optimal pseudoflow and augments the flow along shortest paths from excess nodes to deficit nodes in the residual network in order to convert the pseudoflow into a flow. The primal-dual algorithm also maintains an optimal pseudoflow and solves maximum flow problems in order to convert it into a flow.

Starting from the classical algorithms for minimum cost flow, several polynomial-time algorithms have been developed. Most of them have been obtained by using the scaling technique. By capacity scaling, by cost scaling or by capacity and cost scaling, the following polynomial-time algorithms have been developed: capacity scaling algorithm, cost scaling algorithm, double scaling algorithm, repeated capacity scaling algorithm and enhanced capacity scaling algorithm. Another way to obtain polynomial-time algorithms is by improving the running time of the cycle-canceling algorithm by imposing different rules for selecting the negative cycles, for instance selecting the negative cycle with maximum improvement or the negative cycle with minimum cost.

Another approach for solving minimum cost flow problems is to use linear programming methodologies because these problems are linear programs. In this manner relaxation algorithm and simplex method for minimum cost flow have been developed.

2 Determining a minimum cost flow in a network with an underestimated arc capacity

There are wide spread real world problems that can be modeled and solved as minimum cost flow problems in appropriate networks. Sometimes in these problems an input data change occurs. For instance, an usual change in real life applications can imply an augmentation of the capacity of one arc in the corresponding network, in which a minimum cost flow has been already determined (using one of the algorithms mentioned in the previous section). In this case, one can apply again a minimum cost algorithm in the modified network or, more efficient, one can use the minimum flow cost already established in the original network as a starting point. We focus on the second approach.

Suppose that we have already established a minimum cost flow f of cost c_f in the network $G = (N, A, c, b, v)$ and now we need to find a minimum cost flow in a network $G' = (N, A, c', b, v)$ having the same structure and the same capacities and cost, excepting one capacity, as G. So, the only difference between these two networks is the capac-

ity of a given arc (k, l), which is greater in G' than in G. So, $c'(x, y) = c(x, y), \forall (x, y) \in A \backslash \{(k, l)\}$ and $c'(k, l) = c(k, l) + a$, where $a > 0$.

Obviously, a greater capacity of the arc (k, l) may imply the existence of additional negative cost directed cycles in the residual network and, consequently, the existence of minimum cost flow in the network G' having a cost smaller than c_f. But it is not compulsory. Here two cases might appear:

Case 1: The residual network $G(f)$ with respect to the minimum cost flow f already contains the arc (k, l). In this case f is also a minimum cost flow in G'.

Case 2: The residual network $G(f)$ with respect to the minimum cost flow f does not contain the arc (k, l). In this case increasing the capacity of the arc (k, l) by a units implies to add an arc (k, l) having the residual capacity equal to a to the residual network. This means that new negative cost directed cycles might appear in the residual network, which implies that the minimum cost flow value in G' is at most c_f. If there are such cycles, they all contain the arc (k, l). In this case, for determining a minimum cost flow in G' starting from the minimum cost flow in G, which is a feasible flow in G', it is natural to develop an algorithm of the first type, i.e. an algorithm that maintains the feasibility of the flow and strives toward its optimality. If we apply the cycle-canceling algorithm (see [1]) in $G'(f)$, it performs at most a flow augmentations. Since the time complexity of a flow augmentation is $O(m)$, it follows that a minimum cost flow in G' can be obtained in $O(am)$ time starting from a minimum cost flow in G.

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