Bulletin of the *Transilvania* University of Braşov • Vol 11(60), No. 1 - 2018
Series III: Mathematics, Informatics, Physics, 121-126

AN INTEGRAL LINKED TO THE ARITHMETIC-GEOMETRIC MEAN

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Abstract

An integral involving hyperbolic functions is linked to the arithmetic-geometric mean in the same way as in the Gauss formula and a numerical method to compute the real elliptic integral of first kind is presented.

2010 *Mathematics Subject Classification:* 33C99, 33F99.
Key words: arithmetic-geometric mean, elliptic integrals.

1 Introduction

If $M(a,b)$ denotes the arithmetic-geometric mean of two positive numbers, a and b, then the following result established by Carl Friedrich GAUSS (1777-1855) in 1799 occurs, [5]:

Theorem 1. *If a and b are positive reals then*

$$\frac{1}{M(a,b)} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sqrt{a^2\cos^2 x + b^2\sin^2 x}}. \tag{1}$$

We shall denote by $I(a,b,\alpha = \frac{\pi}{2})$ the integral of the right hand side of (1). The definition of $I(a,b,\alpha)$ is given in (3).

For $a > b > 0$ and $\alpha > 0$ we shall take care of the integral

$$J(a,b,\alpha) = \int_0^{\alpha} \frac{\mathrm{d}x}{\sqrt{a^2\cosh^2 x - b^2\sinh^2 x}}. \tag{2}$$

First, we shall express $J(a,b,i\frac{\pi}{2})$ through $I(a,b,\frac{\pi}{2})$ involving an elliptic integral and then we present a pure real approach of $J(a,b,\alpha)$. We obtain a relation that links the integral $J(a,b,\alpha)$ with $M(a,b)$. In this case the computation is similar to the method presented in [5]. A simpler proof of (1) is given in [1], p.6.

Finally, using the same method for $I(a,b,\alpha)$ we obtain a numerical method to compute the real elliptic integral of first kind. The method will require the iterative computation of three sequences. For $\alpha = \frac{\pi}{2}$ the result is given in [3]. In [2], [4] other approaches to compute an elliptic integral of first kind are presented.

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2 $I(a,b,\alpha)$ and $J(a,b,\alpha)$ as elliptic integrals

We recall the following elliptic integrals, [6],

$$F(\phi,m)=\int_0^{\phi}\frac{\mathsf{d}\theta}{\sqrt{1-m\sin^2\theta}},\quad \text{and}\quad K(m)=F(\frac{\pi}{2},m).$$

$K(\phi,m)$ is called the elliptic integral of first kind.

We have

$$I(a,b,\alpha)=\frac{1}{a}\int_0^{\alpha}\frac{\mathsf{d}x}{\sqrt{1-\left(1-\frac{b^2}{a^2}\right)\sin^2 x}}=\frac{1}{a}F\left(\alpha,1-\frac{b^2}{a^2}\right) \tag{3}$$

and

$$I(a,b,\frac{\pi}{2})=\frac{1}{a}K\left(1-\frac{b^2}{a^2}\right).$$

Thus, equality (2) may be rewritten as $\frac{1}{M(a,b)}=\frac{2}{a\pi}K\left(1-\frac{b^2}{a^2}\right)$.

Using the changing of variable $x=iy$ we obtain

$$J(a,b,i\frac{\pi}{2})=i\int_0^{\frac{\pi}{2}}\frac{\mathsf{d}y}{\sqrt{a^2\cos^2 y+b^2\sin^2 y}}=i\,I(a,b)=\frac{i}{a}K\left(1-\frac{b^2}{a^2}\right)$$

and thus $J(a,b,i\frac{\pi}{2})=\frac{i\pi}{2M(a,b)}$.

Generally

$$J(a,b,\alpha)=i\int_0^{-i\alpha}\frac{\mathsf{d}y}{\sqrt{a^2\cos^2 y+b^2\sin^2 y}}=\frac{i}{a}\int_0^{-i\alpha}\frac{\mathsf{d}y}{\sqrt{1-\left(1-\frac{b^2}{a^2}\right)\sin^2 y}}=$$

$$=\frac{i}{a}F\left(-i\alpha,1-\frac{b^2}{a^2}\right)=-\frac{i}{a}F\left(i\alpha,1-\frac{b^2}{a^2}\right)$$

and consequently $I(a,b,i\alpha)=i\,J(a,b,\alpha)$.

3 A pure real approach of $J(a,b,\alpha)$

If $a_0=a$ and $b_0=b$ then the sequences $(a_k)_{k\in\mathbb{N}},(b_k)_{k\in\mathbb{N}}$ defined by the recurrences

$$a_{k+1}=\frac{a_k+b_k}{2},\qquad b_{k+1}=\sqrt{a_k b_k},\qquad k\in\mathbb{N},$$

converge to $M(a,b)$.

In order to compute (2) the main ingredient is the changing of the variable

$$\sinh x=\frac{2a\sinh\varphi}{a+b-(a-b)\sinh^2\varphi}. \tag{4}$$

From (4) it results

$$\cosh^2 x = 1 + \sinh^2 x =$$
$$= \frac{(a+b)^2 + 2(a^2+b^2)\sinh^2\varphi + (a-b)^2\sinh^4\varphi}{(a+b-(a-b)\sinh^2\varphi)^2} \tag{5}$$

and then

$$a^2\cosh^2 x - b^2\sinh^2 x = \frac{a^2\left((a+b)^2 + 2(a^2-b^2)\sinh^2\varphi + (a-b)^2\sinh^4\varphi\right)}{(a+b-(a-b)\sinh^2\varphi)^2} =$$
$$= a^2\left(\frac{a+b+(a-b)\sinh^2\varphi}{a+b-(a-b)\sinh^2\varphi}\right)^2.$$

It follows that

$$\sqrt{a^2\cosh^2 x - b^2\sinh^2 x} = a\frac{a+b+(a-b)\sinh^2\varphi}{a+b-(a-b)\sinh^2\varphi}. \tag{6}$$

In (5), using the relations

$$(a+b)^2 = 4a_1^2, \qquad a^2+b^2 = 4a_1^2 - 2b_1^2, \qquad (a-b)^2 = 4a_1^2 - 4b_1^2.$$

the numerator expressions are linked to a_1 and b_1 and we get

$$\cosh^2 x = \frac{4\left((1+2\sinh^2\varphi+\sinh^4\varphi)a_1^2 - (\sinh^2\varphi+\sinh^4\varphi)b_1^2\right)}{(a+b-(a-b)\sinh^2\varphi)^2} =$$
$$= \frac{4\cosh^2\varphi(a_1^2\cosh^2\varphi - b_1^2\sinh^2\varphi)}{(a+b-(a-b)\sinh^2\varphi)^2},$$

or

$$\cosh x = \frac{2\cosh\varphi\sqrt{a_1^2\cosh^2\varphi - b_1^2\sinh^2\varphi}}{a+b-(a-b)\sinh^2\varphi}. \tag{7}$$

Denoting $f(t) = \frac{2at}{a+b-(a-b)t^2}$ we have

$$f'(t) = 2a\frac{a+b+(a-b)t^2}{(a+b-(a-b)t^2)^2} > 0,$$

which means that $f(t)$ is increasing.

The new variable φ will belong to the interval $[0, \alpha_1]$, where α_1 is given by the equation

$$\sinh\alpha = \frac{2a\sinh\alpha_1}{a+b-(a-b)\sinh^2\alpha_1}.$$

Below we will return to this equation.

From (4) we find

$$\cosh x\,\mathrm{d}x = 2a\frac{a+b+(a-b)\sinh^2\varphi}{(a+b-(a-b)\sinh^2\varphi)^2}\cosh\varphi\,\mathrm{d}\varphi.$$

Using (6) the above equality may be rewritten as

$$\cosh x\,\mathrm{d}x = \sqrt{a^2\cosh^2 x - b^2\sinh^2 x}\frac{2\cosh\varphi}{a+b-(a-b)\sinh^2\varphi}\,\mathrm{d}\varphi,$$

otherwise

$$\frac{\mathrm{d}x}{\sqrt{a^2\cosh^2 x - b^2\sinh^2 x}} = \frac{2\cosh\varphi}{\cosh x(a+b-(a-b)\sinh^2\varphi)}\,\mathrm{d}\varphi.$$

Finally, using (7), from the right hand side we obtain

$$\frac{\mathrm{d}x}{\sqrt{a^2\cosh^2 x - b^2\sinh^2 x}} = \frac{\mathrm{d}\varphi}{\sqrt{a_1^2\cosh^2\varphi - b_1^2\sinh^2\varphi}}$$

and then

$$\int_0^{\alpha}\frac{\mathrm{d}x}{\sqrt{a^2\cosh^2 x - b^2\sinh^2 x}} = \int_0^{\alpha_1}\frac{\mathrm{d}\varphi}{\sqrt{a_1^2\cosh^2\varphi - b_1^2\sinh^2\varphi}}. \tag{8}$$

Iterating (8) it results

$$J(a,b,\alpha) \stackrel{def}{=} J_0(a_0,b_0,\alpha_0) = J_1(a_1,b_1,\alpha_1) = J_2(a_2,b_2,\alpha_2) = \ldots \tag{9}$$

where

$$J_k(a_k,b_k.\alpha_k) = \int_0^{\alpha_k}\frac{\mathrm{d}\varphi}{\sqrt{a_k^2\cosh^2\varphi - b_k^2\sinh^2\varphi}}.$$

The integration limit α_k is given by the equation

$$\sinh\alpha_{k-1} = \frac{2a_{k-1}\sinh\alpha_k}{a_{k-1}+b_{k-1}-(a_{k-1}-b_{k-1})\sinh^2\alpha_k}. \tag{10}$$

Rewriting (10) we deduce that

$$\frac{\sinh\alpha_{k-1}}{\sinh\alpha_k} = \frac{2a_{k-1}}{a_{k-1}+b_{k-1}-(a_{k-1}-b_{k-1})\sinh^2\alpha_k} > 1 \quad\Leftrightarrow\quad 1+\sinh^2 a_k > 0.$$

Consequently, the sequence $(\sinh\alpha_k)_{k\in\mathbb{N}}$ is decreasing and therefore the sequence $(\alpha_k)_{k\in\mathbb{N}}$ is decreasing, too. Because $\alpha_k > 0$, the sequence converges to some α_∞.

The limit in (10) does not generate an equation for α_∞. In order to compute an approximation of α_∞ the elements of the sequence must be sequentially computed using a stopping rule which assures that the last computed element is near the limit.

From (10) we get

$$\sinh\alpha_k = \frac{\sqrt{a_{k-1}^2\cosh^2\alpha_{k-1} - b_{k-1}^2\sinh^2\alpha_{k-1}} - a_{k-1}}{(a_{k-1}-b_{k-1})\sinh\alpha_{k-1}} = y_k \tag{11}$$

and

$$\alpha_k = \ln\left(y_k + \sqrt{y_k^2+1}\right).$$

If $a=5,\ b=3$ and $\alpha=\alpha_0=1$ then after 4 iterations we obtained $\alpha_\infty \approx 0.71093896$. The stopping rule was $|\alpha_k-\alpha_{k-1}|<10^{-7}$ or $a_{k-1}-b_{k-1}<10^{-10}$.

From (9) it results that

$$J(a,b,\alpha) = \lim_{k\to\infty} J_k(a_k,b_k,\alpha_k) = \frac{\alpha_\infty}{M(a,b)}. \tag{12}$$

4 Numerical computation of $F(\alpha,m)$

For $0<b<a$ and $0<\alpha<1$, as in [5], for $I(a,b,\alpha)$ the changing of variables

$$\sin x = \frac{2a\sin\varphi}{a+b+(a-b)\sin^2\varphi}$$

leads to the sequence

$$I(a,b,\alpha) \overset{def}{=} I_0(a_0,b_0,\alpha_0) = I_1(a_1,b_1,\alpha_1) = I_2(a_2,b_2,\alpha_2) = \ldots \tag{13}$$

where

$$I_k(a_k,b_k,\alpha_k) = \int_0^{\alpha_k} \frac{\mathrm{d}\varphi}{\sqrt{a_k^2\cos^2\varphi + b_k^2\sin^2\varphi}}$$

and the upper integration limits are generated by the sequence

$$\sin\alpha_{k-1} = \frac{2a_{k-1}\sin\alpha_k}{a_{k-1}+b_{k-1}+(a_{k-1}-b_{k-1})\sin^2\alpha_k}.$$

The sequence $(\alpha_k)_{k\in\mathbb{N}}$ is convergent and

$$\begin{aligned}\sin\alpha_k &= \frac{a_{k-1}-\sqrt{a_{k-1}^2\cos^2\alpha_{k-1}+b_{k-1}^2\sin^2\alpha_{k-1}}}{(a_{k-1}-b_{k-1})\sin\alpha_{k-1}} = y_k \\ \alpha_k &= \arcsin y_k.\end{aligned} \tag{14}$$

From (13) it results

$$I(a,b,\alpha) = \lim_{k\to\infty} I_k(a_k,b_k,\alpha_k) = \frac{\alpha_\infty}{M(a,b)},$$

with $\alpha_\infty = \lim_{k\to\infty}\alpha_k$. Using (3) we get

$$I(a,b,\alpha) = \frac{1}{a}F\left(\alpha, 1-\frac{b^2}{a^2}\right) = \frac{\alpha_\infty}{M(a,b)}$$

and consequently

$$F\left(\alpha,1-\frac{b^2}{a^2}\right) = \frac{a\alpha_\infty}{M(a,b)} = \frac{\alpha_\infty}{\frac{1}{a}M(a,b)} = \frac{\alpha_\infty}{M(1,\frac{b}{a})}.$$

Denoting $m = 1 - \frac{b^2}{a^2}, (a > b > 0 \ \Leftrightarrow\ 0 < m < 1)$, the above equation becomes

$$F(\alpha, m) = \frac{\alpha_\infty}{M(1, \sqrt{1-m})}.$$

Therefore, the computation of $F(\alpha, m)$ returns to generate iteratively the sequences $(a_k)_k,\ (b_k)_k,\ (\alpha_k)_k$ until a stopping condition is fulfilled. The initial values are $a_0 = 1,\ b_0 = \sqrt{1-m}$ and $\alpha_0 = \alpha$. For $a_0 = 1$, instead of the sequences $(a_k)_k,\ (b_k)_k$ we may compute the sequences, [3],

$$\begin{array}{ll} s_0 = b_0 & p_0 = \frac{1}{2}(1+s_0) \\ s_{k+1} = \frac{2\sqrt{s_k}}{1+s_k} & p_{k+1} = \frac{1}{2}(1+s_k)p_k \end{array}.$$

Then $\lim_{k\to\infty} p_k = M(1, b_0)$.

If $\alpha = \frac{\pi}{2}$ then from the relation of recurrence (14) of α_k it follows that $\alpha_k = \frac{\pi}{2}$, for any $k \in \mathbb{N}$, and hence $\alpha_\infty = \frac{\pi}{2.}$ Consequently $K(m) = \frac{\pi}{2M(1,\sqrt{1-m})}$.

As a drawback from a practical point of view the method is not applicable when α is small, e.g. $0 < \alpha < 10^{-5}$.

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