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PROPERTIES OF AN INTEGRAL OPERATOR

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Abstract

In this paper we define an integral operator for analytic functions in the open unit disk and we determine some properties of this integral operator.

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1 Introduction

Let A be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

normalized by $f(0) = f'(0) - 1 = 0$, which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

We denote by $\mathcal{S}$ the subclass of A consisting of functions $f \in A$, which are univalent in U.

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U. For $a \in \mathbb{C}$ and $n \in \mathbb{N} - \{0\}$ we note

$$H[a,n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + \cdots\}$$

and

$$\mathcal{A}_n = \left\{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \cdots\right\},$$

with $\mathcal{A}_1 = A$.

In this paper we consider the integral operator $I_{\alpha,\beta,\gamma}$ defined by

$$I_{\alpha,\beta,\gamma}(z) = \left[\frac{\gamma+\beta}{z^\gamma} \int_0^z (f(t))^\beta (h'(t))^\alpha t^{\gamma-1} dt\right]^{\frac{1}{\beta}}, \quad (1)$$

$\alpha, \beta, \gamma \in \mathbb{C}$, $\beta \neq 0$, $f, h \in \mathcal{A}_n$.

We have the next remarks.

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k_1) For $n=1, \gamma=0, h(z)=z$ or $n=1, \alpha=\gamma=0$, we obtain the integral operator Miller-Mocanu [2].

k_2) If $\alpha=0$, we obtain an integral operator, which was studied by P.T. Mocanu, D. Ripeanu and I. Şerb [3].

2 Preliminaries

We need the following lemmas.

Lemma 1 (Pascu [5]). *Let α be a complex number, $Re\alpha>0$ and $f\in A$. If*

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right|\leq 1, \tag{2}$$

for all $z\in U$, then the function

$$F_\alpha(z)=\left[\alpha\int_0^z t^{\alpha-1}f'(t)dt\right]^{\frac{1}{\alpha}} \tag{3}$$

is regular and univalent in U.

Lemma 2 (Mocanu and Şerb, [4]). *Let $M_0=1,5936\ldots$ be the positive solution of equation*

$$(2-M)e^M=2.$$

If $f\in A$ and

$$\left|\frac{f''(z)}{f'(z)}\right|\leq M_0,\ z\in U, \tag{4}$$

then

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1, z\in U. \tag{5}$$

The edge M_0 is sharp.

Lemma 3 (General Schwarz Lemma, [1]). *Let f be the function regular in the disk $U_R=\{z\in\mathbb{C}:|z|<R\}$ with $|f(z)|<M$, M fixed. If the function f has in $z=0$ one zero with multiply $\geq m$, then*

$$|f(z)|\leq\frac{M}{R^m}|z|^m,\ z\in U_R, \tag{6}$$

the equality (in the inequality (6) for $z\neq 0$) can hold only if

$$f(z)=e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

In this paper we obtain properties of integral operator $I_{\alpha,\beta,\gamma}$.

3 Main results

Theorem 1. *Let α, β, γ be complex numbers, $\beta \neq 0$, $a = Re(\gamma + \beta) > 0$ and the functions $f, h \in \mathcal{A}_n$, L, M be the positive real numbers.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M,\ z \in U, \tag{7}$$

$$\left|\frac{zh''(z)}{h'(z)}\right| < L,\ z \in U \tag{8}$$

and

$$|\alpha|L + |\beta|M \leq \frac{(2a+n)^{\frac{n+2a}{2a}}}{2n^{\frac{n}{2a}}}\ , \tag{9}$$

then

$$I_{\alpha,\beta,\gamma}(z) = z\left(1 + b_2 z + b_3 z^2 + \cdots\right)^{\frac{\beta+\gamma}{\beta}},\ z \in D \cap U, \tag{10}$$

where D is domain of definition of the function $I_{\alpha,\beta,\gamma}(z)$ and $z^{\frac{\gamma}{\beta+\gamma}}\left[I_{\alpha,\beta,\gamma}(z)\right]^{\frac{\beta}{\beta+\gamma}}$ belongs to class S. If $\gamma = 0$, then $I_{\alpha,\beta,0}(z) \in S$, and $I_{\alpha,\beta,0}(z) = z + b_2 z^2 + \ldots$, $z \in U$.

Proof. From (1) we have

$$I_{\alpha,\beta,\gamma}(z) = \frac{1}{z^{\frac{\gamma}{\beta}}}\left\{\left[(\gamma+\beta)\int_0^z t^{\gamma+\beta-1}\left(\frac{f(t)}{t}\right)^{\beta}\left(h'(t)\right)^{\alpha} dt\right]^{\frac{1}{\gamma+\beta}}\right\}^{\frac{\gamma+\beta}{\beta}}, \tag{11}$$

for all $z \in U$.

We consider the function

$$K_{\alpha,\beta,\gamma}(z) = \left[(\gamma+\beta)\int_0^z t^{\gamma+\beta-1}\left(\frac{f(t)}{t}\right)^{\beta}\left(h'(t)\right)^{\alpha} dt\right]^{\frac{1}{\gamma+\beta}},\ z \in U. \tag{12}$$

Let the function

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\beta}\left(h'(t)\right)^{\alpha} dt,\ z \in U, \tag{13}$$

which is regular in U and $g(0) = g'(0) - 1 = 0$.

We have

$$\frac{zg''(z)}{g'(z)} = \beta\left(\frac{zf'(z)}{f(z)} - 1\right) + \alpha\frac{zh''(z)}{h'(z)},\ z \in U \tag{14}$$

and hence, we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1-|z|^{2a}}{a}\left[|\beta|\left|\frac{zf'(z)}{f(z)} - 1\right| + |\alpha|\left|\frac{zh''(z)}{h'(z)}\right|\right], \tag{15}$$

for all $z \in U$.

Applying lemma 3, from (7) and (8) we have

$$\left|\frac{zf'(z)}{f(z)}-1\right|\le M|z|^n,\ z\in U, \tag{16}$$

$$\left|\frac{zh''(z)}{h'(z)}\right|\le L|z|^n,\ z\in U. \tag{17}$$

From (15) and (16), (17) we get

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right|\le\frac{1-|z|^{2a}}{a}|z|^n\left(|\beta|M+|\alpha|L\right),\ z\in U. \tag{18}$$

We consider the function $Q:[0,1]\to\mathbb{R}, Q(x)=\frac{(1-x^{2a})x^n}{a}$, where $x=|z|$, $x\in[0,1]$.

We have

$$\max_{x\in[0,1]}Q(x)=\frac{2n^{\frac{n}{2a}}}{(2a+n)^{\frac{n+2a}{2a}}}. \tag{19}$$

By (9), (19) and (18) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right|\le 1, \tag{20}$$

for all $z\in U$.

Now, from (20) and Lemma 1, it results that

$$K_{\alpha,\beta,\gamma}\in S,\ K_{\alpha,\beta,\gamma}(z)=z+b_2z^2+\cdots, \tag{21}$$

and by (11) and (21) we obtain

$$I_{\alpha,\beta,\gamma}(z)=z\left(1+b_2z+b_3z^2+\cdots\right)^{\frac{\gamma+\beta}{\beta}},\ z\in D\cap U. \tag{22}$$

From (22) and (21), we have $z^{\frac{\gamma}{\beta+\gamma}}\left[I_{\alpha,\beta,\gamma}(z)\right]^{\frac{\beta}{\beta+\gamma}}$ belongs to class S.

For $\gamma=0$, we obtain $I_{\alpha,\beta,0}(z)\equiv K_{\alpha,\beta,0}(z)$, for all $z\in U$, $I_{\alpha,\beta,0}(z)\in S$, $I_{\alpha,\beta,0}(z)=z+b_2z^2+\dots,\ z\in U.$ □

Theorem 2. *Let α,β,γ be complex numbers, $\beta\neq 0$, $a=Re(\gamma+\beta)>0$, the functions $f,h\in\mathcal{A}_n$, $M_0=1,5936\dots$ the positive solution of equation $(2-M)e^M=2$. If*

$$\left|\frac{f''(z)}{f'(z)}\right|\le M_0,\ z\in U, \tag{23}$$

$$\left|\frac{h''(z)}{h'(z)}\right|\le M_0,\ z\in U \tag{24}$$

and

$$(2a+n)^{\frac{n+2a}{2a}}|\beta|+2an^{\frac{n}{2a}}|\alpha|M_0\le a(2a+n)^{\frac{n+2a}{2a}}, \tag{25}$$

then $z^{\frac{\gamma}{\beta+\gamma}}\left[I_{\alpha,\beta,\gamma}(z)\right]^{\frac{\beta}{\beta+\gamma}}$ *belongs to class* S*, where*

$$I_{\alpha,\beta,\gamma}(z)=z(1+c_2z+c_3z^2+\cdots)^{\frac{\beta+\gamma}{\beta}},\ z\in D\cap U, \tag{26}$$

D *is domain of definition of the function* $I_{\alpha,\beta,\gamma}(z)$*. If* $\gamma=0$*, then* $I_{\alpha,\beta,0}(z)\in S$ *and* $I_{\alpha,\beta,0}(z)=z+c_2z^2+\dots,\ z\in U.$

Proof. We use functions $K_{\alpha,\beta,\gamma}(z)$ and $g(z)$ defined in Theorem 1.

By (23), applying Lemma 2 we obtain $\left|\frac{zf'(z)}{f(z)}-1\right|<1$, $z\in U$ and by (24), applying Lemma 3 we have $\left|\frac{h''(z)}{h'(z)}\right|\le M_0|z|^{n-1}$, $z\in U$.

Using (15) we get

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right|\le\frac{1-|z|^{2a}}{a}|\beta|+\frac{1-|z|^{2a}}{a}|z|^n|\alpha|M_0, \tag{27}$$

for all $z\in U$.

From (27) and (19) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right|\le\frac{|\beta|}{a}+\frac{2n^{\frac{n}{2a}}}{(2a+n)^{\frac{n+2a}{2a}}}|\alpha|M_0,\ z\in U. \tag{28}$$

By (28) and (25) we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right|\le 1,\ z\in U. \tag{29}$$

From (29) and Lemma 1 we have

$$K_{\alpha,\beta,\gamma}\in S, K_{\alpha,\beta,\gamma}(z)=z+c_2z^2+\cdots, \tag{30}$$

and by (11) and (30) we obtain

$$I_{\alpha,\beta,\gamma}(z)=z\left(1+c_2z+c_3z^2+\cdots\right)^{\frac{\gamma+\beta}{\beta}},\ z\in D\cap U. \tag{31}$$

From (31) and (30) we obtain $z^{\frac{\gamma}{\beta+\gamma}}\left[I_{\alpha,\beta,\gamma}(z)\right]^{\frac{\beta}{\beta+\gamma}}$ belongs to class S.

For $\gamma=0$, we have $I_{\alpha,\beta,0}(z)\equiv K_{\alpha,\beta,0}(z)$, for all $z\in U$, $I_{\alpha,\beta,0}(z)\in S$, $I_{\alpha,\beta,0}(z)=z+c_2z^2+\dots$, $z\in U$. $\square$

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