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UNIVALENCE CRITERIA TO THE GENERALIZED SĂLĂGEAN AND RUSCHEWEYH OPERATOR

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Abstract

In this paper we obtain new sufficient conditions for univalence using the method of Loewner chains using the generalised Sălăgean and Ruscheweyh operator. In particular, we obtain some known univalence conditions due to Lewandowski, Becker, Kanas and Lecko.

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Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic and univalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Definition 1. *[7]*
For $f \in \mathcal{A}, n \in \mathbb{N}$, the Sălăgean differential operator $\mathscr{D}^n$ is defined by $\mathscr{D}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathscr{D}^0 f(z) = f(z),$$

$$\mathscr{D}^1 f(z) = z f'(z),$$

$$\ldots$$

$$\mathscr{D}^{n+1} f(z) = z\left(\mathscr{D}^n f(z)\right)', z \in U$$

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Remark 1. *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

$$\mathscr{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, z \in U.$$

Definition 2. *[8] For $f \in \mathcal{A}, n \in \mathbb{N}$, the operator $\mathscr{R}^n$ is defined by $\mathscr{R}^n : \mathcal{A} \to \mathcal{A}$,*

$$\mathscr{R}^0 f(z) = f(z),$$

$$\mathscr{R}^1 f(z) = z f'(z), \ldots$$

$$(n+1)\, \mathscr{R}^{n+1} f(z) = z \left(\mathscr{R}^n f(z)\right)' + n \mathscr{R}^n f(z), z \in U.$$

Remark 2. *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

$$\mathscr{R}^n f(z) = z + \sum_{k=2}^{\infty} \frac{(n+k-1)!}{n!\,(k-1)!} a_k z^k,\ z \in U.$$

Definition 3. *Let $\gamma \geq 0, n \in \mathbb{N}$. Denote by $\mathscr{L}^n$ the operator given by $\mathscr{L}^n : \mathcal{A} \to \mathcal{A}$,*

$$\mathscr{L}^n f(z) = (1-\gamma)\, \mathscr{R}^n f(z) + \gamma \mathscr{D}^n f(z), z \in U.$$

Remark 3. *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

$$\mathscr{L}^n f(z) = z + \sum_{k=2}^{\infty} \left\{ (1-\gamma) \frac{(n+k-1)!}{n!\,(k-1)!} + \gamma k^n \right\} a_k z^k,\ z \in U.$$

Definition 4. *Let f and g be analytic functions in U. We say that the function f is subordinate to the function g, if there exists a function w, which is analytic in U and $w(0) = 0; |w(z)| < 1; z \in U$, such that $f(z) = g(w(z)); \forall z \in U$. We denote by $\prec$ the subordination relation.*

In order to prove our main result we need the theory of Loewner chains.
Let $U_r = \{z \in \mathbb{C} : |z| < r,\ \ r \in (0,1]\}$, $I = [0, \infty)$ and $p \in \mathcal{P}$ (the class $\mathcal{P}$ is a Carath$\acute{e}$odory class of functions which are analytic with positive real part in U) be of the form $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$.

A function $L(z,t) : U \times I \to \mathbb{C}$ is said to be a Loewner chain if the following conditions are satisfied:

$i)$ $L(z,t)$ is analytic and univalent in U, $\forall t \in I$

$ii)$ $L(z,t) \prec L(z,s),\ \forall\ 0 \leq t \leq s < \infty.$

Lemma 1. *[6]*

Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots, \quad a_1(t) \neq 0$ be an analytic function in U_r, $\forall t \in I$. Suppose that:

$i)$ *$L(z,t)$ is a locally absolutely continuous function in I and locally uniformly with respect to U_r.*

$ii)$ *$a_1(t)$ is a complex valued continuous function on I such that $\lim_{t\to\infty} |a_1(t)| = \infty$ and $\left\{ \frac{L(z,t)}{a_1(t)} \right\}_{t\in I}$ is a normal family of functions in U_r.*

$iii)$ *There exists an analytic function $p : U_r \times I \to \mathbb{C}$ satisfying $\Re p(z,t) > 0$, $\forall (z,t) \in U \times I$ and*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \;\; z \in U_r, \;\; t \in I. \tag{2}$$

Then, for each $t \in I$, function $L(z,t)$ has an analytic and univalent extension to the whole unit disk U, i.e $L(z,t)$ is a Loewner chain.

The equation (2) is called the generalized Loewner differential equation.
If $a_1(t) = e^t$ then we say that $L(z,t)$ is a standard Loewner chain.
We follow Nistor [5], and we generalise her results.

Main results

Theorem 1. *Let $f \in \mathcal{A}$ and p be an analytic function with $p(0) = 1$. If the inequalities*

$$\left| \frac{2}{p(z)+1} \cdot \frac{zf'(z)}{\mathscr{L}^{n+1}f(z) + (1-\gamma)\, n \left(\mathscr{R}^{n+1}f\,(z) - \mathscr{R}^{n}f\,(z)\right)} - 1 \right| \leq 1 \tag{3}$$

and

$$\Bigg| \left(\frac{2}{p(z)+1} \cdot \frac{zf'(z)}{\mathscr{L}^{n+1}f(z) + (1-\gamma)\, n \left(\mathscr{R}^{n+1}f(z) - \mathscr{R}^{n}f(z)\right)} - 1 \right) |z|^2 + \\ + \left(1 - |z|^2\right) \left(\frac{\mathscr{L}^{n+2}f(z) +}{\mathscr{L}^{n+1}f(z) + (1-\gamma)\, n \left(\mathscr{R}^{n+1}f(z) - \mathscr{R}^{n}f(z)\right)} - \\ \frac{(1-\gamma)\left[\left(n^2+3n+1\right)\mathscr{R}^{n+2}f(z) - \left(2n^2+3n+1\right)\mathscr{R}^{n+1}f(z) + n^2\mathscr{R}^{n}f(z)\right]}{\mathscr{L}^{n+1}f(z) + (1-\gamma)\, n \left(\mathscr{R}^{n+1}f(z) - \mathscr{R}^{n}f(z)\right)} - \\ -1 + \frac{zp'(z)}{p(z)+1} \right) \Bigg| \leq 1 \tag{4}$$

holds true for $z \in U$, then the function f is univalent in U.

Proof. Let $f \in \mathcal{A}$ and p be an analytic function with $p(0) = 1$. We want to prove that there exists a real number $r \in (0, 1]$ such that the function

$$L(z,t) := f\left(e^{-t}z\right) + \left(e^{t}z - e^{-t}z\right) \frac{p\left(e^{-t}z\right) + 1}{2} \left(\mathscr{L}^{n}\right)' f(e^{-t}z) \tag{5}$$

is analytic in U_r, $\forall t \in I$.
From (5) we observe that $L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$, where $a_1(t) = e^t$, $a_1(t) \neq 0$, $\forall t \in I$ and $\lim\limits_{t\to\infty} |a_1(t)| = \infty$.

From the form of the chain $L(z,t)$ it follows that $L(\cdot,t)$ is regular in U, $\forall t \in I$ and $L(z,\cdot)$ is locally absolutely continuous on I, $\forall z \in U$.
The limit function $g(z) = z$ belongs to the family $\left\{\dfrac{L(z,t)}{a_1(t)}\right\}$ then, in every closed disk U_r there exists a constant $K > 0$ such that

$$\left|\frac{L(z,t)}{a_1(t)}\right| / < K, \forall z \in U_r, t \in I$$

uniformly in this disk, in the case that t is sufficiently large. Following Montel's Theorem, $\left\{\dfrac{L(z,t)}{a_1(t)}\right\}$ is a normal family in U_r.
Let $p(z,t)$ be the function defined by

$$p(z,t) = \frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}}.$$

In order to prove that $p \in \mathcal{P}$, we will show that the function

$$w(z,t) = \frac{p(z,t) - 1}{p(z,t) + 1}$$

is analytic in U and $|w(z,t)| < 1$, $\forall z \in U$ and $\forall t \in I$.
From this we get

$$w(z,t) = e^{-2t}A(z,t) + \left(1 - e^{-2t}\right)B(z,t) \ , \tag{6}$$

where

$$A(z,t) = \frac{2}{p(e^{-t}z) + 1} \cdot \frac{e^{-t}zf'(e^{-t}z)}{\mathscr{L}^{n+1}f(e^{-t}z) + (1-\gamma)\,n\left(\mathscr{R}^{n+1}f\left(e^{-t}z\right) - \mathscr{R}^{n}f\left(e^{-t}z\right)\right)} - 1 \tag{7}$$

and

$$B(z,t) = \frac{e^{-t}zp'(e^{-t}z)}{p(e^{-t}z) + 1} - 1 + \frac{\mathscr{L}^{n+2}f(e^{-t}z) +}{\mathscr{L}^{n+1}f(e^{-t}z) + (1-\gamma)\,n\left(\mathscr{R}^{n+1}f(e^{-t}z) - \mathscr{R}^{n}f(e^{-t}z)\right)} +$$

$$+\frac{(1-\gamma)\left[\left(n^2+3n+1\right)\mathscr{R}^{n+2}f(e^{-t}z) - \left(2n^2+3n+1\right)\mathscr{R}^{n+1}f(e^{-t}z) + n^2\mathscr{R}^{n}f(e^{-t}z)\right]}{\mathscr{L}^{n+1}f(e^{-t}z) + (1-\gamma)\,n\left(\mathscr{R}^{n+1}f(e^{-t}z) - \mathscr{R}^{n}f(e^{-t}z)\right)}$$

(8)

From (3) and (4) we deduce that $w(z,t)$ is analytic in U. In view of (3), (7) and (8), we have

$$|w(z,0)| = |A(z,0)| < 1 \text{ and } |w(0,t)| < 1.$$

Since $\left|e^{-t}z\right| \leq e^{-t} < 1$ for $t \in I$ and $z \in U, z \neq 0$, then $w(z,t)$ is analytic in $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$. From maximum modulus principle we know that there exists $\theta = \theta(t) \in \mathbb{R}$ such that

$$|w(z,t)| = \max_{|\zeta|=1} |w(\zeta,t)| = \left|w\left(e^{i\theta},t\right)\right|, \forall z \in U.$$

If we denote $u = e^{-t}e^{i\theta}$, then $|u| = e^{-t}$ and because $u \in U$, we get that $\left|w\left(e^{i\theta},t\right)\right| \leq 1$. If we take the upper relations we get that $|w(z,t)| < 1, \forall z \in U$ and $t \in I$ which means that $p(z,t)$ is regular in U and $\Re p(z,t) > 0, \forall t \in I, z \in U$. Therefore, in view of Lemma 1, $L(z,t)$ is a Loewner chain and hence the function $L(z,0) = f(z)$ is univalent in U. □

Remark 4. *If $\gamma = 1$ we get Theorem 1 from Nistor [5] and for $\gamma = 0$ we get Theorem 3 from the same article.*

Setting $n = 0$ in Theorem 1 we obtain the following corollary due to Lewandowski [4] :

Corollary 1. *Let $f \in \mathcal{A}$ and $p \in \mathcal{P}$. If*

$$\left|\frac{1-p(z)}{1+p(z)}|z|^2 + \left(1-|z|^2\right)\left(\frac{zf''(z)}{f'(z)} + \frac{zp'(z)}{1+p(z)}\right)\right| \leq 1, \quad z \in U,$$

then the function f is univalent in U.

For $p = 1$ the following criterion reduces to a well-known criterion found by Becker [1] (see also Duren et al. [2]).

Corollary 2. *Let $f \in \mathcal{A}$. If*

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \quad z \in U,$$

then the function f is univalent in U.

For $n = 1$, Theorem 1 results

Corollary 3. *Let $f \in \mathcal{A}$ and p be an analytic function with $p(0) = 1$. If*

$$\left|\frac{2}{p(z)+1} \cdot \frac{f'(z)}{f'(z) + zf''(z)} - 1\right| \leq 1$$

and

$$\left|\left(\frac{2}{p(z)+1}\cdot\frac{f'(z)}{f'(z)+zf''(z)}-1\right)|z|^2+\right.$$
$$\left.+\left(1-|z|^2\right)\left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}+\frac{zp'(z)}{p(z)+1}\right)\right|\le 1$$

holds true for $z\in U$ then the function f is univalent in U.

For the Loewner chain

$$L(z,t):=f\left(e^{-t}z\right)+\left(e^tz-e^{-t}z\right)\frac{p\left(e^{-t}z\right)+1}{2}\cdot\frac{\mathscr{L}^{n+1}f(e^{-t}z)}{\mathscr{L}^nf(e^{-t}z)} \tag{9}$$

identically with the proof of Theorem 1, we get:

Theorem 2. *Let $f\in\mathcal{A}$ and p be an analytic function with $p(0)=1$. If the inequalities*

$$\left|\frac{2}{p(z)+1}\cdot f'(z)\cdot\frac{\mathscr{L}^nf(z)}{\mathscr{L}^{n+1}f(z)}-1\right|\le 1 \tag{10}$$

and

$$\left|\left(\frac{2f'(z)}{p(z)+1}\cdot\frac{\mathscr{L}^nf(z)}{\mathscr{L}^{n+1}f(z)}-1\right)|z|^2+\left(1-|z|^2\right)\left(\frac{\mathscr{L}^{n+2}f(z)}{\mathscr{L}^{n+1}f(z)}-\frac{\mathscr{L}^{n+1}f(z)}{\mathscr{L}^nf(z)}+(1-\gamma)\right.\right.$$
$$\frac{\left[(n+1)\mathscr{L}^nf(z)\left(\mathscr{R}^{n+2}f(z)-\mathscr{R}^{n+1}f(z)\right)-n\mathscr{L}^{n+1}f(z)\left(\mathscr{R}^{n+1}f(z)-\mathscr{R}^nf(z)\right)\right]}{\mathscr{L}^nf(z)\cdot\mathscr{L}^{n+1}f(z)}+$$
$$\left.\left.+\frac{zp'(z)}{p(z)+1}\right)\right|\le 1 \tag{11}$$

holds true for $z\in U$, then the function f is univalent in U.

Remark 5. *If $\gamma=1$ we get Theorem 2 from Nistor [5] and for $\gamma=0$ we get Theorem 4 from the same article.*

Setting $n=0$ in Theorem 2 the result is:

Corollary 4. *Let $f\in\mathcal{A}$ and p be an analytic function with $p(0)=1$. If*

$$\left|\frac{2}{p(z)+1}\cdot\frac{f'(z)}{z}-1\right|\le 1$$

and

$$\left|\left(\frac{2}{p(z)+1}\cdot\frac{f'(z)}{z}-1\right)|z|^2+\left(1-|z|^2\right)\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}+\frac{zp'(z)}{p(z)+1}\right)\right|\le 1$$

holds true for $z\in U$, then the function f is univalent in U.

If we put $p = 1$ in the corollary above we get the result of Kanas and Lecko [3].

Setting $p(z) = \dfrac{f(z)}{z}$ we obtain:

Corollary 5. *Let $f \in \mathcal{A}$ with $\Re\dfrac{f(z)}{z} > 0$. If*

$$\left|\left(\frac{f(z)}{z} - 1\right)|z|^2 + \left(1 - |z|^2\right)\left[1 + \frac{zf''(z)}{f'(z)}\left(\frac{f(z)}{z} + 1\right) - \frac{zf'(z)}{f(z)}\right]\right| \leq \left|\frac{f(z)}{z} + 1\right|$$

holds true for $z \in U$, then the function f is univalent in U.

For $p(z) = \dfrac{zf'(z)}{f(z)}$ in Corollary 4, the result is:

Corollary 6. *Let $f \in \mathcal{A}$. If*

$$\left|2\frac{f(z)}{z} - \frac{zf'(z)}{f(z)} - 1\right| \leq \left|1 + \frac{zf'(z)}{f(z)}\right|$$

and

$$\left|\left(2\frac{f(z)}{z} - \frac{zf'(z)}{f(z)} - 1\right)|z|^2 + \left(1 - |z|^2\right)\frac{2zf'(z)}{f(z)+1}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| \leq$$

$$\leq \left|1 + \frac{zf'(z)}{f(z)}\right|$$

holds true for $z \in U$, then the function f is univalent in U.

References

[1] J. Becker, *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen,* J. Reine Angew. Math. **255** (1972), 23-43.

[2] Duren, V. H., Shapiro, S. and Shields, A.L., *Singular measures and domains not for Smirnov type,* Duke Math. J. **33** (1966), 242-254.

[3] Kanas, S. and Lecko, A., *Univalence criteria connected with arithmetic and geometric means,* II, Proceedings of the Second Int. Workshop of Transform Methods and Special Functions, Varna '96, Bulgar. Acad. Sci. (Sofia) (1996), 201-209.

[4] Lewandowski, Z., *On a univalence criterion,* Bull. Acad. Polon. Sci. Ser. Sci. Math. **29** (1981), 123-126.

[5] Nistor-Şerban, A. E., *Univalence criteria related with Sălăgean and Ruscheweyh operators,* Bulletin of the Transilvania University of Brasov, Series III: Mathematics, Informatics, Physics **9(58)** (2016), no. 2, 41-48.

[6] Pommerenke, Ch., *Über die Subordination analytischer Funktionen,* J. Reine Angew Math. **218** (1965), 159-173.

[7] Sălăgean, G. S., *Subclasses of univalent functions,* Lecture Notes in Math. (Springer Verlag), **1013** (1983), 362-372.

[8] Ruscheweyh, St. *New criteria for univalent functions,* Proc. Amer. Math. Soc. **49** (1975), 109-115.