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RIEMANN-LAGRANGE GEOMETRY FOR DYNAMICAL SYSTEM CONCERNING MARKET COMPETITION

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Dedicated to Professor Mircea Lupu on his 75-th anniversary

Abstract

In this paper we study the Riemann-Lagrange geometry, in the sense of non-linear connection, d-torsions, d-curvatures and Yang-Mills-like energy, associated with the dynamical system concerning market competition. Some possible economic interpretations are derived.

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1 Introduction

Considering a generic market where two different types of firms produce and trade their own homogenous goods, we study the dynamical competition between these two economical sectors, via the first order differential system used by Udrişte and Postolache [6]

$$\begin{cases} \dfrac{dE_1}{dt} = g_1 E_1 \left(1 - \dfrac{E_1}{K_1} - \beta_1 \dfrac{E_2}{K_1}\right) \\ \dfrac{dE_2}{dt} = g_2 E_2 \left(1 - \dfrac{E_2}{K_2} - \beta_2 \dfrac{E_1}{K_2}\right), \end{cases} \tag{1}$$

where: • E_1 and E_2 are two populations of new firms born in the above economical sectors; • g_1 and g_2 are strictly positive constants representing the growth rates of the two economical sectors; • K_1 and K_2 are strictly positive constants representing the investments of capitals; • β_1 and β_2 are strictly positive constants representing the competitive interaction coefficients.

By differentiation, the dynamical system (1) can be extended to a dynamical system of order two coming from a first order Lagrangian of least square type. This extension is called in the literature in the field geometric dynamical system (see Udrişte [5]).

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2 The Riemann-Lagrange geometry

The system (1) can be regarded on the tangent space $T\mathbb{R}^2$, whose coordinates are

$$\left(x^1 = E_1,\ x^2 = E_2,\ y^1 = \frac{dE_1}{dt},\ y^2 = \frac{dE_2}{dt}\right).$$

Remark 1. *We recall that the transformations of coordinates on the tangent space $T\mathbb{R}^2$ are given by*

$$\widetilde{x}^i = \widetilde{x}^i(x^j), \quad \widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^j} y^j, \tag{2}$$

where $i, j = \overline{1,2}$.

In this context, the solutions of class C^2 of the system (1) are the global minimum points of the least square Lagrangian [5], [4]

$$L = \left(y^1 - X^1\left(E_1, E_2\right)\right)^2 + \left(y^2 - X^2\left(E_1, E_2\right)\right)^2 \geq 0, \tag{3}$$

where

$$X^1\left(E_1, E_2\right) = g_1 E_1 \left(1 - \frac{E_1}{K_1} - \beta_1 \frac{E_2}{K_1}\right),$$
$$X^2\left(E_1, E_2\right) = g_2 E_2 \left(1 - \frac{E_2}{K_2} - \beta_2 \frac{E_1}{K_2}\right).$$

Remark 2. *The solutions of class C^2 of the system (1) are solutions of the Euler-Lagrange equations attached to the least square Lagrangian (3), namely (geometric dynamics, in Udrişte's terminology)*

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) = 0,\ y^i = \frac{dx^i}{dt}, \quad \forall\, i = \overline{1,2}, \Leftrightarrow \tag{4}$$

$$\frac{d^2x^i}{dt^2} + 2G^i(x^k, y^k) = 0 \Leftrightarrow \frac{d^2x^i}{dt^2} + \frac{1}{2}\left(\frac{\partial^2 L}{\partial y^i \partial x^k} y^k - \frac{\partial L}{\partial x^i}\right) = 0 \Leftrightarrow$$

$$\frac{d^2x^i}{dt^2} = \left(\frac{\partial X^i}{\partial x^k} - \frac{\partial X^k}{\partial x^i}\right) y^k + \frac{\partial X^k}{\partial x^i} X^k,$$

where

$$\begin{aligned} G^i(x^k, y^k) &= \frac{1}{4}\left(\frac{\partial^2 L}{\partial y^i \partial x^k} y^k - \frac{\partial L}{\partial x^i}\right) = \\ &= -\frac{1}{2}\left[\left(\frac{\partial X^i}{\partial x^k} - \frac{\partial X^k}{\partial x^i}\right) y^k + \frac{\partial X^k}{\partial x^i} X^k\right] \end{aligned} \tag{5}$$

*is endowed with the geometrical meaning of **semispray** of L (for more geometrical details, see book of Miron and Anastasiei [3] and Udrişte's book [5]).*

But, the least square Lagrangian (3), together with its Euler-Lagrange equations (4), provide us with an entire Riemann-Lagrange geometry on the tangent space $T\mathbb{R}^2$, in the sense of nonlinear connection, d-torsions, d-curvatures and Yang-Mills-like energy. These geometrical objects are naturally associated with the economical dynamical system (1).

Let us recall the main geometrical ideas developed in Miron and Anastasiei's book [3]. The canonical nonlinear connection $N = \left(N_j^i\right)_{i,j=\overline{1,2}}$ produced by the semispray (5) is given by the components

$$N_j^i = \frac{\partial G^i}{\partial y^j} = -\frac{1}{2}\left(\frac{\partial X^i}{\partial x^j} - \frac{\partial X^j}{\partial x^i}\right).$$

Remark 3. *We recall that, under a transformation of coordinates (2), the local components of the nonlinear connection obey the rules [3]*

$$\widetilde{N}_l^k = N_i^j \frac{\partial x^i}{\partial \widetilde{x}^l}\frac{\partial \widetilde{x}^k}{\partial x^j} - \frac{\partial x^i}{\partial \widetilde{x}^l}\frac{\partial \widetilde{y}^k}{\partial x^i}.$$

From a geometrical point of view, we point out that the coefficients N_j^i of the above nonlinear connection do not have a global character on $T\mathbb{R}^2$.

Remark 4. *Using the Cartan-Kosambi-Chern (KCC) theory exposed in the paper of Böhmer et al. [1], we can remark that the **deviation curvature tensor** associated with the dynamical system (1) is given by the formula*

$$P_j^i = -2\frac{\partial G^i}{\partial x^j} - 2G^l \frac{\partial N_j^i}{\partial y^l} + \frac{\partial N_j^i}{\partial x^l} y^l + N_l^i N_j^l.$$

It is important to note that the solutions of the Euler-Lagrange equations (4) are Jacobi stable iff the real parts of the eigenvalues of the deviation tensor P_j^i are strictly negative everywhere, and Jacobi unstable, otherwise. For more details, see [1] and references therein.

The canonical nonlinear connection defines the adapted bases of vector fields and covector fields on the tangent space $T\mathbb{R}^2$, namely

$$\left\{\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\right\}, \quad \left\{dx^i, \delta y^i = dy^i + N_j^i dx^j\right\}.$$

The adapted local components of the Cartan N-linear connection

$$C\Gamma(N) = \left(L_{jk}^i, C_{jk}^i\right)$$

are given by the formulas

$$L_{jk}^i = \frac{g^{ir}}{2}\left(\frac{\delta g_{rk}}{\delta x^j} + \frac{\delta g_{rj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^r}\right), \quad C_{jk}^i = \frac{g^{ir}}{2}\left(\frac{\partial g_{rk}}{\partial y^j} + \frac{\partial g_{rj}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^r}\right),$$

where
$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j} = \delta_{ij}.$$
The only non-vanishing d-torsion adapted component associated with the Cartan N-linear connection $C\Gamma(N)$ is given by the coefficient
$$R^r_{ij} = \frac{\delta N^r_i}{\delta x^j} - \frac{\delta N^r_j}{\delta x^i} = \frac{\partial N^r_i}{\partial x^j} - \frac{\partial N^r_j}{\partial x^i}.$$
At the same time, all the adapted components of the curvature attached to the Cartan N-linear connection $C\Gamma(N)$ are zero (for all curvature formulas, see [3]).

The electromagnetic-like distinguished 2-form attached to the Lagrangian L, defined via its deflection d-tensors (for more details, see Miron and Anastasiei's book [3]), is given by $\mathbb{F} = F_{ij}\delta y^i \wedge dx^j$, where
$$F_{ij} = \frac{1}{2}\left(g_{ir}N^r_j - g_{jr}N^r_i\right) = \frac{1}{2}\left(N^i_j - N^j_i\right) = N^i_j.$$

In this context, using the notation $J(X) = \left(\dfrac{\partial X^i}{\partial E_j}\right)_{i,j=\overline{1,2}} =$
$$= \begin{pmatrix} g_1 - 2g_1\dfrac{E_1}{K_1} - g_1\beta_1\dfrac{E_2}{K_1} & -g_1\beta_1\dfrac{E_1}{K_1} \\ -g_2\beta_2\dfrac{E_2}{K_2} & g_2 - 2g_2\dfrac{E_2}{K_2} - g_2\beta_2\dfrac{E_1}{K_2} \end{pmatrix}$$
and following the above Miron and Anastasiei's geometrical ideas, we obtain the following geometrical results:

Theorem 1. (i) *The canonical nonlinear connection on* $T\mathbb{R}^2$*, produced by the system (1), has the local components* $N = \left(N^i_j\right)_{i,j=\overline{1,2}}$*, where* N^i_j *are the entries of the skew-symmetric matrix*
$$\begin{aligned} N &= \left(N^i_j\right)_{i,j=\overline{1,2}} = -\frac{1}{2}\left[J(X) - {}^T J(X)\right] = \\ &= \begin{pmatrix} 0 & \dfrac{1}{2}\left(g_1\beta_1\dfrac{E_1}{K_1} - g_2\beta_2\dfrac{E_2}{K_2}\right) \\ \dfrac{1}{2}\left(-g_1\beta_1\dfrac{E_1}{K_1} + g_2\beta_2\dfrac{E_2}{K_2}\right) & 0 \end{pmatrix}. \end{aligned}$$

(ii) *All adapted components of the canonical Cartan connection* $C\Gamma(N)$*, produced by the system (1), are zero.*

(iii) *The effective adapted components* R^i_{jk} *of the torsion d-tensor* **T** *of the canonical Cartan connection* $C\Gamma(N)$*, produced by the system (1), are the entries of the skew-symmetric matrices*
$$R_1 = \left(R^i_{j1}\right)_{i,j=\overline{1,2}} = \frac{\partial N}{\partial E_1} = \begin{pmatrix} 0 & \dfrac{g_1\beta_1}{2K_1} \\ -\dfrac{g_1\beta_1}{2K_1} & 0 \end{pmatrix},$$

$$R_2 = \left(R^i_{j2}\right)_{i,j=\overline{1,2}} = \frac{\partial N}{\partial E_2} = \begin{pmatrix} 0 & -\dfrac{g_2\beta_2}{2K_2} \\ \dfrac{g_2\beta_2}{2K_2} & 0 \end{pmatrix}.$$

(iv) *All adapted components of the curvature d-tensor* **R** *of the canonical Cartan connection $C\Gamma(N)$, produced by the system (1), vanish.*

(v) *The geometric electromagnetic-like distinguished 2-form, produced by the system (1), is given by $\mathbb{F} = F_{ij}\delta y^i \wedge dx^j$, where the adapted components F_{ij} are the entries of the skew-symmetric matrix $F = (F_{ij})_{i,j=\overline{1,2}} = N =$*

$$= \frac{1}{2}\begin{pmatrix} 0 & g_1\beta_1\dfrac{E_1}{K_1} - g_2\beta_2\dfrac{E_2}{K_2} \\ -g_1\beta_1\dfrac{E_1}{K_1} + g_2\beta_2\dfrac{E_2}{K_2} & 0 \end{pmatrix}.$$

(vi) *The geometric Yang-Mills-like energy, produced by the system (1), is given by the formula*

$$\mathcal{EYM}(t) = F_{12}^2 = \frac{1}{4}\left(g_1\beta_1\frac{E_1}{K_1} - g_2\beta_2\frac{E_2}{K_2}\right)^2.$$

Remark 5. *In the author's opinion, from an economical point of view the zero level of the geometric Yang-Mills-like energy produced by the economical system (1) is important. The geometric Yang-Mills-like economical energy produced by the system (1) is zero on the straight line*

$$g_1\beta_1\frac{E_1}{K_1} = g_2\beta_2\frac{E_2}{K_2} \Leftrightarrow \frac{E_1}{E_2} = \frac{g_2\beta_2K_1}{g_1\beta_1K_2}.$$

This means that E is directly proportional with the ratio $K/(g\beta)$, that is the population of new firms born in the corresponding economical sector increases with the growth of the investments of capital and decreases with the growth of the competitive interaction coefficient and with the growth rate of the corresponding economical sector.

At the same time, we consider that the constant level curves of the geometric Yang-Mills-like economical energy

$$\mathcal{EYM}(t) = \frac{1}{4}\left(g_1\beta_1\frac{E_1}{K_1} - g_2\beta_2\frac{E_2}{K_2}\right)^2 = C^2, \quad C \geq 0,$$

could contain important economic connotations. These curves are in the system of axes OE_1E_2 exactly the parallel straight lines

$$g_1\beta_1\frac{E_1}{K_1} - g_2\beta_2\frac{E_2}{K_2} = 2C,$$

whose gradients are again $(g_2\beta_2K_1)/(g_1\beta_1K_2)$.

Remark 6. *The zero level set of the Yang-Mills-like energy is produced on the straight line*

$$E_2 = \frac{g_1\beta_1K_2}{g_2\beta_2K_1}E_1, \tag{6}$$

so, in this case, replacing (6) into the equations (1), we get the Bernoulli differential equation

$$\frac{dE_1}{dt} = g_1 E_1 \left[1 - \left(\frac{1}{K_1} + \frac{g_1 \beta_1^2 K_2}{g_2 \beta_2 K_1^2}\right) E_1\right],$$

which is completely integrable and it can be solved by using the changing of variable $z = E_1^{-1}$. *The solution of the above Bernoulli differential equation is*

$$E_1(t) = \frac{1}{a \exp(-g_1 t) + b},$$

where $a \in \mathbb{R}$ *is an arbitrary constant, and we have*

$$b = \frac{1}{K_1} + \frac{g_1 \beta_1^2 K_2}{g_2 \beta_2 K_1^2}.$$

Remark 7. *The deviation curvature tensor components* P_j^i *can be obtained by contracting with* y^k *the nonzero components of the torsion tensor* R_{jk}^i, *that is* $P_j^i = R_{jk}^i y^k = \left(\partial N_j^i / \partial x^k\right) y^k$. *Consequently, the matrix of the deviation curvature tensor is given by*

$$P = R_k y^k = \begin{pmatrix} 0 & \dfrac{g_1\beta_1}{2K_1} \\ -\dfrac{g_1\beta_1}{2K_1} & 0 \end{pmatrix} y^1 + \begin{pmatrix} 0 & -\dfrac{g_2\beta_2}{2K_2} \\ \dfrac{g_2\beta_2}{2K_2} & 0 \end{pmatrix} y^2 =$$

$$= \begin{pmatrix} 0 & \dfrac{g_1\beta_1}{2K_1} y^1 - \dfrac{g_2\beta_2}{2K_2} y^2 \\ -\dfrac{g_1\beta_1}{2K_1} y^1 + \dfrac{g_2\beta_2}{2K_2} y^2 & 0 \end{pmatrix}.$$

The eigenvalues of the matrix P *are*

$$\lambda_{1,2} = \pm i \left(\frac{g_1\beta_1}{2K_1} y^1 - \frac{g_2\beta_2}{2K_2} y^2\right) \Rightarrow \Re(\lambda_{1,2}) = 0.$$

In conclusion, the behavior of neighboring solutions of the Euler-Lagrange equations (4) is Jacobi unstable.

3 A realistic example

Let us analyse the population competition for a realistic market described by the following data (see the Ferrara and Niglia's paper [2]): • $g_1 = 0.3$ and $g_2 = 0.2$ represent the growth rates of the two economic sectors; • $K_1 = 0.005$ and $K_2 = 0.007$ represent the investments of capitals; • $\beta_1 = 0.001$ and $\beta_2 = 0.003$ are the competitive interaction coefficients.

So, we study the geometric dynamics associated with the least square Lagrangian (3), together with the Riemann-Lagrange geometry attached to this Lagrangian. In our particular context, the first order differential system (1) becomes

$$\begin{cases} \dfrac{dE_1}{dt} = 0.3E_1\left(1 - \dfrac{E_1}{0.005} - 0.001\dfrac{E_2}{0.005}\right) \\ \dfrac{dE_2}{dt} = 0.2E_2\left(1 - \dfrac{E_2}{0.007} - 0.003\dfrac{E_1}{0.007}\right) \end{cases} \Leftrightarrow \tag{7}$$

$$\Leftrightarrow \begin{cases} \dfrac{dE_1}{dt} = 0.3E_1\left(1 - 200E_1 - 0.2E_2\right) \\ \dfrac{dE_2}{dt} = 0.2E_2\left(1 - 142.85E_2 - 0.43E_1\right), \end{cases}$$

and the following geometrical results are true:

Corollary 1. (i) *The canonical nonlinear connection on $T\mathbb{R}^2$, produced by the system (7), has the local components $N = \left(N^i_j\right)$, where N^i_j are the entries of the skew-symmetric matrix*

$$N = \left(N^i_j\right)_{i,j=\overline{1,2}} = \begin{pmatrix} 0 & 0.03E_1 - 0.045E_2 \\ -0.03E_1 + 0.045E_2 & 0 \end{pmatrix}.$$

(ii) *All adapted components of the canonical Cartan connection $C\Gamma(N)$, produced by the system (7), are zero.*

(iii) *The effective adapted components R^i_{jk} of the torsion d-tensor* **T** *of the canonical Cartan connection $C\Gamma(N)$, produced by the system (7), are the entries of the skew-symmetric matrices*

$$R_1 = \left(R^i_{j1}\right)_{i,j=\overline{1,2}} = \frac{\partial N}{\partial E_1} = \begin{pmatrix} 0 & 0.03 \\ -0.03 & 0 \end{pmatrix},$$

$$R_2 = \left(R^i_{j2}\right)_{i,j=\overline{1,2}} = \frac{\partial N}{\partial E_2} = \begin{pmatrix} 0 & -0.045 \\ 0.045 & 0 \end{pmatrix}.$$

(iv) *All adapted components of the curvature d-tensor* **R** *of the canonical Cartan connection $C\Gamma(N)$, produced by the system (7), vanish.*

(v) *The geometric electromagnetic-like distinguished 2-form, produced by the system (7), is given by $\mathbb{F} = F_{ij}\delta y^i \wedge dx^j$, where the adapted components F_{ij} are the entries of the skew-symmetric matrix*

$$F = \left(F_{ij}\right)_{i,j=\overline{1,2}} = N = \begin{pmatrix} 0 & 0.03E_1 - 0.045E_2 \\ -0.03E_1 + 0.045E_2 & 0 \end{pmatrix}.$$

(vi) *The geometric Yang-Mills-like energy, produced by the system (7), is given by the formula*

$$\mathcal{EYM}(t) = F_{12}^2 = \left(0.03E_1 - 0.045E_2\right)^2.$$

Remark 8. *From an economic point of view the constant level curves of the geometric Yang-Mills-like energy produced by the economic system (7) are the parallel straight lines*

$$0.03E_1 - 0.045E_2 = C \Leftrightarrow E_1 = 1.5E_2 + C, \quad C \geq 0,$$

whose gradients are $3/2$*. Consequently, according to Ferrara and Niglia, we emphasize that the market competition has heteroclinic connections (see [2]) but, moreover, it has even affine connections.*

Open problem. The economic interpretations associated with the geometrical objects constructed in this paper still represent an open problem.

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