Bulletin of the *Transilvania* University of Braşov • Vol 11(60), No. 1 - 2018
Series III: Mathematics, Informatics, Physics, 85-98

# POLYNOMIAL OF SECOND DEGREE STRUCTURES ON BIG TANGENT BUNDLE

**Adelina MANEA**[1]

**Abstract**

We introduced the generalized $(a, b)$-structure on a Riemannain manifold $M$, notion which includes the generalized almost complex and generalized almost product structures. We studied the canonical generalized $(a, b)$-structure $J_g$ induced by the Riemannian metric on $M$, afterwards the generalized $(a, b)$-structure $\hat{J}$ induced by a similar structure on $M$. Considering a torsion-free linear connection $\nabla$ on $M$, we define the $\nabla$-integrability of a generalized $(a, b)$-structure and conditions for $\nabla$-integrability of $J_g$ and $\hat{J}$ are given.

2000 *Mathematics Subject Classification:* 53C15, 53D18.
*Key words:* polynomial structures, generalized geometry

## 1 Introduction

Polynomial structures on manifolds were introduced and studied in [6], closely related to known structures as almost product, almost complex and contact structures. A ploynomial structure $F$ of degree $d$ on a **C**$^\infty$ manifold $M$ is, [6], a $(1, 1)$-tensor field satisfying a polynomial equation

$$F^d + a_1 F^{d-1} + ... + a_{d-1} F + a_d I_d = 0,$$

where $a_1, ...a_d$ are real numbers and $d$ is the smallest integer on which $I_d, F, ..., F^d$ are dependent. Integrability conditions for polynomial structures, involving the Nijenhuis torsion of the structural endomorphism, are given in [17], under the assumption that the polynomial $X^d + a_1 X^{d-1} + ... + a_{d-1} X + a_d$ has only simple roots. Examples of polynomial structures of second degree are: the almost product structure, when $F^2 = I_d$; the almost complex structure with $F^2 = -I_d$ for $dimM = 2n$ and the metallic structure $F^2 = pF + qI_d$, with integers $p, q$ such that $p^2 + 4q$ is positive. The particular case of metallic structures was considered in the last decade, [7], [10]. For $dimM = 2n+1$, an almost contact structure is an example of polynomial structure of third degree, since the structural endomorphism satisfies $F^3 + F = 0$.

[1]Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania, e-mail: adelina.manea@unitbv.ro

In the last years many geometers passed from the tangent bundle of a manifold $M$ to the generalized tangent bundle $E = TM \oplus T^*M$ and extended different polynomial structures defined on $M$ to similar structures on the big tangent bundle $E$. Generalized geometry of a manifold $M$ is the geometry of structures of the big tangent bundle $TM \oplus T^*M$ endowed with the neutral metric. The generalized complex structure was investigated in [3], [4], [8], [18], the generalized para-complex structure was studied in [20], [22], since the generalized contact and para-contact structures were the topic for [1], [15], [16]. A particular assumption under the behaviour of the neutral metric of the big tangent bundle with respect to the generalized structure endomorphism determined the classes of generalized Hermitian, K$\ddot{a}$hler, para-Hermitian and para-K$\ddot{a}$hler structures, [19], [20].

A polynomial structure on a manifold is called integrable if the eigen-distributions of the structural endomorphism are involutive. The vanishing of Nijenhuis tensor gives a necessary and sufficient condition for integrability. Considering a linear connection $\nabla$ on the manifold $M$, the $\nabla$-brackets of sections of big tangent bundle $E$ and the concept of $\nabla$-integrability of a generalized complex structures on $M$ are defined in [13]: the eigen-distributions of the structural endomorphism are closed under the $\nabla$-brackets, . Corresponding Nijenhuis tensors are defined with respect to $\nabla$-brackets and necessary and sufficient conditions for the $\nabla$-integrability are given.

In this paper we start with two real numbers $a, b$ such that $\Delta = a^2 + 4b \neq 0$ and we define the notion of $(a, b)$-structure on a smooth manifold $M$. This notion generalized the almost complex, almost paracomplex structures on manifolds and includes metallic structures. Some properties of $(a, b)$-structures on manifolds are given in the second section of the paper. Integrability conditions are obtained.

In the third section we define the generalized $(a, b)$-structure, which is a polynomial of second degree structure on big tangent bundle of a Riemannian manifold $M$. We investigate the link between such a structure and the generalized almost product or generalized almost complex structures. The canonical generalized $(a, b)$-structure $J_g$ determined by a Riemannian metric on the base $M$ is studied. We show that a $(a, b)$ structure $\varphi$ on $M$ induced in a natural way a similar generalized structure $\hat{J}$.

The $\nabla$-integrability for generalized $(a, b)$-structures is defined in the last section. We also defined the Nijenhuis torsion $N_J^\nabla$ of a generalized $(a, b)$-structure $J$ and we proved that the $\nabla$-integrability of $J$ is equivalent to $N_J^\nabla = 0$. We obtained that a generalized $(a, b)$-structure is $\nabla$-integrable if and only if its associated generalized almost complex/-paracomplex structure is $\nabla$-integrable. Finally, we find conditions for the $\nabla$-integrability of the particular generalized $(a, b)$-structures $J_g$ and $\hat{J}$.

## 2 Polynomial of second degree structures on manifolds

Let $M$ be an $n$-dimensional **C**$^\infty$-manifold and $a, b$ two real numbers with $\Delta = a^2 + 4b \neq 0.$

We shall call a $(a, b)$-*structure* on $M$ a polynomial structure of second degree given by a $(1, 1)$-tensor field $\varphi$ which satisfies the equation,

$$\varphi^2 - a \cdot \varphi - b \cdot I_d = 0, \tag{1}$$

where $I_d$ is the identity on the vector fields space $\Gamma(TM)$. In this case the pair $(M,\varphi)$ will be called a $(a,b)$-manifold .

**Proposition 1.** *Let $(M,\varphi)$ be a $(a,b)$-manifold. If $\Delta = a^2+4b<0$, then the dimension $n$ of $M$ is an even number.*

*Proof.* The proof follows the ideas from the same assertion for almost complex manifolds. So, for every nonzero vector field $X$ on $M$, $X$ and $\varphi(X)$ are liniar independent. Indeed, if $\alpha\cdot X+\beta\cdot\varphi(X)=0$, and we apply $\varphi$, it results that the real numbers $\alpha,\beta$ satisfy

$$\alpha^2+a\alpha\beta-b\cdot\beta^2=0,$$

which implies $\alpha=\beta=0$, from $\Delta<0$.

Then, considering two nonzero vector fields $X$ and $Y$ such that $\{X,\varphi(X),Y\}$ are linear independent, we obtain $\{X,\varphi(X),Y,\varphi(Y)\}$ also linear independent. Indeed, if we suppose that there are real numbers $\alpha,\beta$ and $\gamma$ such that

$$\varphi(Y)=\alpha\cdot X+\beta\cdot\varphi(X)+\gamma\cdot Y,$$

and we apply the endomorphism $\varphi$, it results that the real number $\gamma$ is a root of $x^2-a\cdot x-b=0$, which is false from the hypotesis.

It follows that the dimension of $TM$ must be even. □

**Remark 1.** *For $a=0$, $b=1$, $\varphi$ is an almost product structure on $M$. If $a=0$, $b=-1$, then $\varphi$ is an almost complex structure on $M$. If $a=b=1$ then $\varphi$ is called a golden structure, and for $a,b$ integers such that $a^2+4b>0$, $\varphi$ is called a metallic structure on $M$.*

The main properties of $(a,b)$-structures are given in the following propositions and they are easy to prove by direct computation:

**Proposition 2.** *Let $\varphi$ a $(a,b)$-structure on the manifold $M$, with $b\neq 0$. Then $\varphi$ is an isomorphism on the tangent space $T_xM$, for every $x\in M$. Its inverse is $\varphi^{-1}=\frac{1}{b}\varphi-\frac{a}{b}I_d$, which is still a polynomial structure.*

**Proposition 3.** *To every $(a,b)$-structure $\varphi$ on $M$, given by* (1)*, we can associate another polynomial structure:*

$$F=\left(\frac{2}{\sqrt{|\Delta|}}\cdot\varphi-\frac{a}{\sqrt{|\Delta|}}\cdot I_d\right). \tag{2}$$

*If $\Delta>0$, then $F^2=I_d$ and $F$ is called the almost product structure associated to $\varphi$.*

*If $\Delta<0$, then $F^2=-I_d$ and $F$ is called the almost complex structure associated to $\varphi$.*

Now, let $F$ be a polynomial structure of second degree on $M$ and

$$\varphi=\frac{a}{2}\cdot I_d+\frac{\sqrt{|\Delta|}}{2}\cdot F. \tag{3}$$

By direct computation, we obtain

$$\varphi^2 - a\varphi - bI_d = -\frac{\Delta}{4} \cdot I_d + \frac{|\Delta|}{4} \cdot F^2.$$

The above relation shows that:

**Proposition 4.** *For any real numbers $a, b$ such that $a^2 + 4b > 0$, every almost product structure $F$ on $M$ induces a $(a, b)$-structure on $M$, given by* (3)*.*

**Proposition 5.** *For any real numbers $a, b$ such that $a^2 + 4b < 0$, every almost complex structure $F$ on $M$ induces a $(a, b)$-structure on $M$, given by* (3)*.*

Let $(M, g)$ be a Riemannian manifold endowed with the $(a, b)$-structure $\varphi$. We say that $\varphi$ is compatible with metric $g$ and that $M$ is a *Riemannian $(a, b)$-manifold*, if

$$g(\varphi X, Y) = g(X, \varphi Y), \tag{4}$$

for every $X, Y \in \Gamma(TM)$. An equivalent condition is

$$g(\varphi X, \varphi Y) = a \cdot g(X, \varphi Y) + b \cdot g(X, Y).$$

For a Riemannian $(a, b)$-manifold $(M, g, \varphi)$, let $F$ be the associated almost product/almost complex structure from (2). We obtain that $F$ is also $g$-symmetric:

$$g(FX, Y) = g(X, FY), \quad \forall X, Y \in \Gamma(TM). \tag{5}$$

The integrability of almost paracomplex/complex structure $F$ is usually expressed by the vanishing of the Nijenhuis tensor $N_F$:

$$N_F(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y],$$

which express the involutivity of eigenbundles of $F$. From relation (3) it is easy to see that the eigenbundles of $\varphi$ are exactly the eigenbundles of the associated structure $F$.

Moreover, considering the Nijenhuis tensor of $\varphi$

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y],$$

we obtain the following link between $N_\varphi$ and $N_F$:

$$N_\varphi = \frac{|\Delta|}{4} N_F + \frac{\Delta}{4}\left(Id - \frac{|\Delta|}{\Delta} F^2\right) = \frac{|\Delta|}{4} N_F.$$

**Definition 1.** *The $(a, b)$-structure $\varphi$ is called integrable if $N_\varphi = 0$.*

A direct consequence is that

**Proposition 6.** *The $(a, b)$-structure $\varphi$ is integrable if and only if the associated structure $F$ given by* (2) *is integrable. In this case the eigenbundles of $\varphi$ are involutive.*

## 3 Generalized polynomial of second degree structures

Generalized geometry of a manifold $M$ is the geometry of structures of the big tangent bundle $E = TM \oplus T^*M$ endowed with the neutral (or pairing) metric

$$g_0(X+\xi, Y+\eta) = \frac{1}{2}\left(\xi(Y) + \eta(X)\right), \tag{6}$$

for all vector fields $X, Y$ and 1-forms $\xi, \eta$ on $M$.

A *generalized almost paracomplex* structure of a manifold $M$ is an endomorphism $I \in End(E)$, which satisfies $I^2 = Id$, $I \neq \pm Id$. Such a structure was firstly considered in [22] and they unify symplectic forms, paracomplex structures, and Poisson structures. Generalized complex structures were firstly considered in [9] and unifies symplectic and complex geometry. A more general concept of generalized almost complex structures was introduced in [14], that is: a *generalized almost complex* structure of a manifold $M$ is an endomorphism $I \in End(E)$, which satisfies $I^2 = -Id$.

**Definition 2.** *Let $a, b$ be two real numbers with $a^2+4b \neq 0$. A* generalized $(a,b)$-structure *of a manifold $M$ is an endomorphism $J \in End(E)$, which satisfies $J^2 = aJ + bI_d$. Such a structure is compatible with neutral metric if $g_0(J(X+\xi), Y+\eta) = g_0(X+\xi, J(Y+\eta))$.*

In the block matrix form a general endomorphism $J \in End(E)$ can be written as

$$J = \begin{pmatrix} H & \alpha \\ \beta & K \end{pmatrix} \tag{7}$$

where $H : TM \to TM$ , $\alpha : T^*M \to TM$ , $\beta : TM \to T^*M$ , $K : T^*M \to T^*M$.

A straightforward computation proves that such an endomorphism is a generalized $(a,b)$-structure on a manifold $M$ if and only if the following conditions hold:

$$\begin{array}{cc} H^2 + \alpha \circ \beta = aH + bId, & H \circ \alpha + \alpha \circ K = a\alpha \\ \beta \circ H + K \circ \beta = a\beta, & \beta \circ \alpha + K^2 = aK + bId. \end{array} \tag{8}$$

Moreover, according to Definition 2, it is compatible with neutral metric if in addition to (8) we have

$$K = H^* , \; \alpha = \alpha^* , \; \beta = \beta^*, \tag{9}$$

where, $H^* : T^*M \to T^*M$ is the dual operator of $H$ defined by $H^*(\xi)(X) = \xi(HX)$, $\alpha = \alpha^*$ means $\eta(\alpha(\xi)) = \xi(\alpha(\eta))$ for all $\xi, \eta \in \Gamma(T^*M)$ and $\beta = \beta^*$ means $\beta(X)(Y) = \beta(Y)(X)$ for all $X, Y \in \Gamma(TM)$.

For a generalized $(a,b)$-structure $J$, the endomorphism

$$I = \frac{2}{\sqrt{|\Delta|}} \cdot J - \frac{a}{\sqrt{|\Delta|}} \cdot I_d, \tag{10}$$

is a generalized almost paracomplex structure if $\Delta > 0$, or a generalized almost complex structure if $\Delta < 0$, respectively. We shall call the endomorphism (10) the generalized almost paracomplex/complex structure associated with the generalized $(a,b)$-structure $J$.

### 3.1 Canonical generalized $(a, b)$-structures

Let $(M, g)$ be a Riemannian manifold. Let be $\natural_g : TM \to T^*M$ the bemolle musical isomorphism and $\sharp_g$ its inverse. There are the following relations:

$$\natural_g(X)(Y) = g(X, Y), \quad \forall X, Y \in \Gamma(TM),$$

$$\sharp_g(\xi) = X_\xi \quad \Leftrightarrow \quad g(X_\xi, Y) = \xi(Y), \quad \forall \xi \in \Gamma(T^*M), Y \in \Gamma(TM).$$

There are two canonical endomorphisms $P, C \in End(E)$ defined by Riemannian metric $g$ as it follows:

$$P = \begin{pmatrix} 0 & \sharp_g \\ \natural_g & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -\sharp_g \\ \natural_g & 0 \end{pmatrix}. \tag{11}$$

For endomorphism $P$ we have in (7) $H = 0$, $K = 0$, $\beta = \natural_g$ and $\alpha = \sharp_g$, and relations (8), (9) are satisfied for $a = 0$ and $b = 1$. For endomorphism $C$ we have in (7) $H = 0$, $K = 0$, $\beta = \natural_g$ and $\alpha = -\sharp_g$, and relations (8), (9) are satisfied for $a = 0$ and $b = -1$. We obtained:

**Proposition 7.** *The endomorphism $P$ is a generalized almost product structure, since the endomorphism $C$ is a generalized almost complex structure on $M$. Moreover, $P$ and $C$ are generalized structures compatible with the neutral metric $g_0$.*

**Remark 2.** *The endomorphisms $P$ and $C$ satisfy the following relation:*

$$P \circ C = -C \circ P,$$

*and $P \circ C$, $C \circ P$ are also generalized almost product structures on $M$.*

**Proposition 8.** *Let $a, b$ be two real numbers with $\Delta = a^2 + 4b \neq 0$. The endomorphism of $E$ defined by:*

$$J_g = \begin{pmatrix} \frac{a}{2} I_d & \frac{|\Delta|}{\Delta} \frac{\sqrt{|\Delta|}}{2} \cdot \sharp_g \\ \frac{\sqrt{|\Delta|}}{2} \cdot \natural_g & \frac{a}{2} I_d \end{pmatrix}, \tag{12}$$

*is a generalized $(a, b)$-structure on $M$ compatible with $g_0$. We shall call $J_g$ the generalized $(a, b)$-structure induced by Riemannian metric $g$.*

*Proof.* By direct computation, it results

$$J_g^2 - aJ_g - bI_d = 0,$$

which shows that $J_g$ is a generalized $(a, b)$-structure on $M$.

$$g_0\left(J_g(X+\xi), Y+\eta\right) = \frac{a}{2} g_0\left(X+\xi, Y+\eta\right) + \frac{\sqrt{|\Delta|}}{2} g_0\left(\frac{|\Delta|}{\Delta}\sharp_g(\xi) + \natural_g(X)), Y+\eta\right).$$

If $\Delta > 0$ we obtain

$$g_0\left(J_g(X+\xi), Y+\eta\right) = \frac{a}{2} g_0\left(X+\xi, Y+\eta\right) + \frac{\sqrt{\Delta}}{4} g_0\left(P(X+\xi), Y+\eta\right) =$$

$$= \frac{a}{2} g_0 \left(X + \xi, Y + \eta\right) + \frac{\sqrt{\Delta}}{4} g_0 \left(X + \xi, P(Y + \eta)\right) = g_0 \left(X + \xi, J_g(Y + \eta)\right).$$

If $\Delta < 0$, it results

$$g_0 \left(J_g(X + \xi), Y + \eta\right) = \frac{a}{2} g_0 \left(X + \xi, Y + \eta\right) + \frac{\sqrt{-\Delta}}{4} g_0 \left(C(X + \xi), Y + \eta\right) =$$

$$= \frac{a}{2} g_0 \left(X + \xi, Y + \eta\right) + \frac{\sqrt{-\Delta}}{4} g_0 \left(X + \xi, C(Y + \eta)\right) = g_0 \left(X + \xi, J_g(Y + \eta)\right).$$

Or, equivalent, in (7) we have $H = K = \frac{a}{2} I_d$, $\alpha = \frac{|\Delta|}{\Delta} \frac{\sqrt{|\Delta|}}{2} \sharp_g$, $\beta = \frac{\sqrt{|\Delta|}}{2} \natural_g$. By straightforward computation, conditions (8),(9) are satisfied. □

**Remark 3.** *The generalized structure* (10) *associated to* $J_g$ *is* $P$ *for* $\Delta > 0$ *and* $C$ *for* $\Delta < 0$*, respectively.*

### 3.2 Generalized $(a, b)$-structures induced on a $(a, b)$-manifold $(M, g, \varphi)$

Let $a, b$ be two real numbers with $\Delta = a^2 + 4b \neq 0$ and $(M, g, \varphi)$ a Riemannian $(a, b)$-manifold with structural endomorphism $\varphi$. Let us consider the endomorphism $\varphi^* \in End(T^*M)$, with

$$\varphi^*(\xi)(X) = \xi\left(\varphi(X)\right), \quad \forall \xi \in \Gamma(T^*M), X \in \Gamma(TM).$$

**Proposition 9.** *The endomorphism* $\hat{J} \in End(E)$ *defined by*

$$\hat{J} = \begin{pmatrix} \varphi & 0 \\ 0 & \varphi^* \end{pmatrix}, \tag{13}$$

*is a generalized* $(a, b)$*-structure on* $M$*, induced by the* $(a, b)$ *structure* $\varphi$ *on* $M$*. The generalized structure* $\hat{J}$ *is compatible with the neutral metric* $g_0$*.*

*Proof.* Using $\varphi^2 = a\varphi + bI_d$, from the definition of $\varphi^*$ it results

$$(\varphi^*)^2(\xi)(X) = \varphi^*(\varphi^*(\xi))(X) = \varphi^*(\xi)(\varphi(X)) =$$

$$= \xi(\varphi^2(X)) = a\xi(\varphi(X)) + b \cdot X = a\varphi^*(X) + b \cdot X,$$

so $(\varphi^*)^2 = a\varphi^* + bI_d$. For endomorphism $\hat{J}$ we have $H = \varphi$, $K = \varphi^*$, $\alpha = 0$, $\beta = 0$, in (7). Conditions (8) and (9) are satisfied. □

Taking into account that $M$ is a Riemannian $(a, b)$-manifold, hence $g(\varphi X, Y) = g(X, \varphi Y)$, we have the following relations:

$$\varphi \circ \sharp_g = \sharp_g \circ \varphi^*, \quad \natural_g \circ \varphi = \varphi^* \circ \natural_g. \tag{14}$$

**Proposition 10.** *Let $(M, g, \varphi)$ be a Riemannian $(a, b)$-manifold. The following relation between the canonical generalized $(a, b)$- structure $J_g$ and the induced generalized $(a, b)$-structure $\hat{J}$ holds:*

$$\hat{J} \circ J_g = J_g \circ \hat{J}.$$

*Moreover, if $a = 0$, then the endomorphism $\hat{J} \circ J_g$ is a generalized $(0, b^2)$-structure, while if $b = 0$, then $\hat{J} \circ J_g$ is a generalized $(a^2, 0)$-structure.*

*Proof.* Firstly we have to remark that relations (14) prove that the induced $(a, b)$ structure $\hat{J}$ commutes with the canonical generalized structures $P$ and $C$ from the previous subsection. Taking into account the definition of $J_g$ from Proposition 8, it results that the generalized $(a, b)$ structures $\hat{J}$ and $J_g$ commute. We denote $J = \hat{J} \circ J_g$ and compute

$$J^2 = a^2 J + ab(\hat{J} + J_g) + b^2 I_d.$$

Considering $a = 0$ or $b = 0$, we obtain that $J$ is a generalized $(0, b^2)$-, or $(a^2, 0)$-structure. □

**Remark 4.** *The generalized structure* (10) *associated to $\hat{J}$ is*

$$I_{\hat{J}} = \begin{pmatrix} F & 0 \\ 0 & F^* \end{pmatrix}, \tag{15}$$

*where $F$ is the polynomial structure* (2) *associated to $\varphi$ and $F^* : T^*M \to T^*M$ is defined by $F^*(\xi)(X) = \xi(FX)$.*

Another generalized $(a, b)$-structure, induced by $\varphi$ and $g$, on a Riemannian $(a, b)$-manifold $(M, g, \varphi)$ is

$$J_{\varphi,g} = \begin{pmatrix} \varphi & 0 \\ \frac{\sqrt{\Delta}}{2} \cdot \natural_g & aI_d - \varphi^* \end{pmatrix}, \tag{16}$$

Indeed, for $H = \varphi$, $K = aI_d - \varphi^*$, $\alpha = 0$ and $\beta = \frac{\sqrt{\Delta}}{2} \cdot \natural_g$, in (7), conditions (8) are verified, but (9) are not satified, so $J_{\varphi,g}$ is not compatible with $g_0$.

The generalized structure (10) associated to $J_{\varphi,g}$ is

$$I_{\varphi,g} = \begin{pmatrix} F & 0 \\ \natural_g & F^* \end{pmatrix}. \tag{17}$$

## 4 $\nabla$-integrability of generalized $(a, b)$-structures

The integrability of generalized structures can be defined by using a linear connection $\nabla$ on $M$. It defines, in a canonical way, a bracket in $E = TM \oplus T^*M$ by

$$[X + \xi, Y + \eta]_\nabla = [X, Y] + \nabla_X \eta - \nabla_Y \xi, \tag{18}$$

for all $X + \xi, Y + \eta \in \Gamma(E)$. Moreover, $\nabla$-bracket defined by (18) satisfies (see [11, 13]):

1. $[X + \xi, Y + \eta]_\nabla = -[Y + \eta, X + \xi]_\nabla$,

2. $[f(X+\xi), Y+\eta]_\nabla = f\,[X+\xi, Y+\eta]_\nabla - Y(f)(X+\xi)$,

3. Jacobi's identity holds for $[\cdot,\cdot]_\nabla$ if and only if $\nabla$ has zero curvature.

Let $J$ be a generalized $(a,b)$-structure on $M$ and $\mathcal{L}_+$ and $\mathcal{L}_-$ be the eigen distributions of $J$, that is

$$\mathcal{L}_+ = Ker\,(J-\lambda_1 I_d) = \{X+\xi \in \Gamma(E) | \quad J(X+\xi) = \lambda_1(X+\xi)\}, \tag{19}$$

$$\mathcal{L}_- = Ker\,(J-\lambda_2 I_d) = \{X+\xi \in \Gamma(E) | \quad J(X+\xi) = \lambda_2(X+\xi)\},$$

where $\lambda_{1,2} = \frac{a\pm\sqrt{\Delta}}{2}$.

Every $X+\xi \in \Gamma(E)$ could be written $X+\xi = p_+(X+\xi)+p_-(X+\xi)$, where $p_+$ and $p_-$ are the projections $p_+ : \Gamma(E) \to \mathcal{L}_+, \quad p_- : \Gamma(E) \to \mathcal{L}_-$ defined by

$$p_+(X+\xi) = \frac{1}{\sqrt{\Delta}}\left(J(X+\xi) - \lambda_2(X+\xi)\right)$$
$$p_-(X+\xi) = \frac{1}{\sqrt{\Delta}}\left(-J(X+\xi) + \lambda_1(X+\xi)\right),$$

and we also have

$$E = \mathcal{L}_+ \oplus \mathcal{L}_-.$$

**Definition 3.** *The endomorphism $J$ is called $\nabla$-integrable if the eigen-distributions $\mathcal{L}_+$ and $\mathcal{L}_-$ are closed under the $\nabla$-brackets, that is*

$$p_\mp[p_\pm(\sigma), p_\pm(\tau)]_\nabla = 0\,,\ \forall\,\sigma = X+\xi,\ \tau = Y+\eta \in \Gamma(E). \tag{20}$$

Similar with [13], we define the Nijenhuis torsion with respect to $\nabla$ of the endomorphism $J$ being the antisymmetric tensor

$$N_J^\nabla : \Gamma(E)\times\Gamma(E) \to \Gamma(E),$$

$$N_J^\nabla(\sigma,\tau) = [J\sigma, J\tau]_\nabla - J\left([J\sigma,\tau]_\nabla + [\sigma, J\tau]_\nabla - a\cdot[\sigma,\tau]_\nabla\right) + b\cdot[\sigma,\tau]_\nabla\,, \tag{21}$$

for every $\sigma = X+\xi$ and $\tau = Y+\eta$, and we have

$$p_\mp[p_\pm(\sigma), p_\pm(\tau)]_\nabla = \mp\frac{1}{\Delta}p_\mp\left(N_J^\nabla\right). \tag{22}$$

A direct consequence is

**Proposition 11.** *The generalized $(a,b)$-structure $J$ is $\nabla$-integrable if and only if $N_J^\nabla = 0$.*

**Proposition 12.** *Let $\nabla$ be a linear connection on $M$, $J$ a generalized $(a,b)$-structure on $M$ and $I$ its associated generalized almost complex or paracomplex structure, defined by* (10)*. Between the Nijenhuis torsions $N_I^\nabla$ and $N_J^\nabla$ there is the followig relation:*

$$N_J^\nabla = \frac{|\Delta|}{4}N_I^\nabla. \tag{23}$$

*The structure $J$ is $\nabla$-integrable if and only if $I$ is $\nabla$-integrable.*

*Proof.* By a straigthforward computation we obtain relation (23), where $N_I^\nabla$ is defined by (21) replacing $a$ with $0$ and $b$ with $\epsilon^2$, where $\epsilon = \sqrt{\frac{\Delta}{|\Delta|}}$. If $\Delta > 0$, $\epsilon = 1$ and if $\Delta < 0$, then $\epsilon = i$. We obtain

$$\mathcal{L}_+ = \{X+\xi \in E| \quad I(X+\xi) = \epsilon(X+\xi)\}, \tag{24}$$
$$\mathcal{L}_- = \{X+\xi \in E| \quad I(X+\xi) = -\epsilon(X+\xi)\},$$

and we have $I^2 = \epsilon^2 \cdot I_d$.

The $\nabla$-integrability of a generalized almost complex/paracomplex structure is defined by the involutivity of its eigen distributions with respect to $\nabla$-brackets. From relations (24), the eigen distributions $\mathcal{L}_+$ and $\mathcal{L}_-$ of $J$ are exactly the eigen distributions of the associated generalized almost paracomplex/complex structure $I$, so $J$ is integrable if and only if $I$ is integrable.

Moreover, from Proposition 11, the $\nabla$-integrability of a generalized $(a,b)$-structure is expressed by the vanishing of the Nijenhuis torsion $N_J^\nabla$. Then, relation (23) proves that $\nabla$-integrability of $J$, defined by $N_J^\nabla = 0$, is equivalent to $N_I^\nabla = 0$, that means $I$ is $\nabla$-integrable. □

In the folllowing we study the $\nabla$-integrability in the particular cases of generalized $(a,b)$-structures $J_g$ and $\hat{J}$ introduced in Section 3.

**Theorem 1.** *Let $\nabla$ be a torsion-free linear connection on the Riemannian manifold $(M,g)$, $a, b$ two real numbers with $\Delta = a^2 + 4b \neq 0$, and $J_g$ the generalized $(a,b)$-structure determined by $g$, defined by* (12). *The endomorphism $J_g$ is $\nabla$-integrable if and only if the metric $g$ is a Codazzi tensor, i.e.*

$$(\nabla_X g)\, Y = (\nabla_Y g)\, X, \quad \forall X, Y \in \Gamma(TM).$$

*Proof.* According to Remark 3, the associated generalized almost paracomplex/complex structure $I$ of $J_g$ is $P$, canonical generalized almost product structure determined by $g$, if $\Delta > 0$, or $C$, canonical generalized almost complex structure determined by $g$, if $\Delta < 0$, respectively. Taking into account Proposition 12, investigating $\nabla$-integrability of $J_g$ is equivalent to investigating the $\nabla$-integrability of $I$.

Since $\nabla$ is a torsion-free connection, we can write $[X,Y] = \nabla_X Y - \nabla_Y X$, and the $\nabla$-bracket on $E$ could be expressed by

$$[X+\xi, Y+\eta]_\nabla = \nabla_X(Y+\eta) - \nabla_Y(X+\xi).$$

We calculate, in the case $\Delta > 0$,

$$[P(X+\xi), P(Y+\eta)]_\nabla = \nabla_{\sharp_g(\xi)}(\sharp(\eta) + \natural_g(Y)) - \nabla_{\sharp_g(\eta)}(\sharp_g(\xi) + \natural_g(X)),$$
$$P\,[P(X+\xi), Y+\eta]_\nabla = \sharp_g\left(\nabla_{\sharp_g(\xi)}(\eta) - \nabla_Y(\natural_g(X)\right) + \natural_g\left(\nabla_{\sharp_g(\xi)}Y - \nabla_Y(\sharp_g(\xi))\right),$$
$$P\,[X+\xi, P(Y+\eta)]_\nabla = \sharp_g\left(\nabla_X(\natural_g(Y) - \nabla_{\sharp_g(\eta)}(\xi)\right) + \natural_g\left(\nabla_X(\sharp_g(\eta)) - \nabla_{\sharp_g(\eta)}X\right).$$

Using $(\nabla_X g)Y = \nabla_X \natural_g(Y) - \natural_g(\nabla_X Y)$, we obtain

$$\sharp_g\left(\nabla_Y(\natural_g(X)\right) = \sharp_g\left((\nabla_Y g)\, X\right) + \nabla_Y X,$$

$$\flat_g\left(\nabla_{\sharp_g(\xi)}Y\right) = -\left(\nabla_{\sharp_g(\xi)}g\right)Y + \nabla_{\sharp_g(\xi)}\flat_g(Y),$$

$$\flat_g\left(\nabla_Y \sharp_g(\xi)\right) = -\left(\nabla_Y g\right)\sharp_g(\xi) + \nabla_Y \xi,$$

$$\sharp_g\left(\nabla_{\sharp_g(\eta)}\xi\right) = \sharp_g\left(\left(\nabla_{\sharp_g(\eta)}g\right)\sharp_g(\xi)\right) + \nabla_{\sharp_g(\eta)}\sharp_g(\xi).$$

Hence,

$$N_P^{\nabla}(X+\xi, Y+\eta) = \sharp_g\left(\left(\nabla_{\sharp_g(\eta)}g\right)\sharp_g(\xi) - \left(\nabla_{\sharp_g(\xi)}g\right)\sharp_g(\eta)\right) +$$

$$+\sharp_g\left(\left(\nabla_Y g\right)X - \left(\nabla_X g\right)Y\right) + \left(\nabla_{\sharp_g(\xi)}g\right)Y - \left(\nabla_Y g\right)\sharp_g(\xi) +$$

$$+\left(\nabla_X g\right)\sharp_g(\eta) - \left(\nabla_{\sharp_g(\eta)}g\right)X.$$

If $(\nabla_X g)\,Y = (\nabla_Y g)\,X$, $\forall X, Y \in \Gamma(TM)$, then the above relation becomes $N_P^{\nabla} = 0$, so $P$ is $\nabla$-integrable.

Conversely, if $N_P^{\nabla}(X+\xi, Y+\eta) = 0$ for all vector fields $X, Y$ and $1$-forms $\xi, \eta$, we consider $\xi = 0$, $\eta = 0$ and obtain $(\nabla_X g)\,Y = (\nabla_Y g)\,X$, $\forall X, Y \in \Gamma(TM)$.

According to Proposition 12, we obtain that the generalized $(a,b)$-structure $J_g$ induced by the Riemannian metric $g$ is $\nabla$-integrable if and only if the torsion-free linear connection $\nabla$ satisfies

$$(\nabla_X g)\,Y = (\nabla_Y g)\,X, \quad \forall X, Y \in \Gamma(TM).$$

A similar computation gives the same result in the case $\Delta < 0$. □

Now, let $(M, g, \varphi)$ be a Riemannian $(a,b)$-manifold and let $\hat{J}$ be the generalized $(a,b)$-structure induced by $\varphi$. We study the $\nabla$-integrability of $\hat{J}$, where $\nabla$ is a linear torsion-free connection on the $(a,b)$-manifold $(M, g, \varphi)$.

**Proposition 13.** *Let $\nabla$ be a linear connection on $M$. The Nijenhuis torsion with respect to $\nabla$ of the generalized $(a,b)$-structure $\hat{J}$ is*

$$\begin{aligned} & N_{\hat{J}}^{\nabla}(X+\xi, Y+\eta) \\ = \;& N_\varphi(X,Y) + \left(\nabla_{\varphi X}\varphi^*\right)\eta - \varphi^*\left(\left(\nabla_X \varphi\right)\eta\right) - \left(\nabla_{\varphi Y}\varphi^*\right)\xi + \varphi^*\left(\left(\nabla_Y \varphi\right)\xi\right) \end{aligned} \quad (25)$$

*for all $X+\xi, Y+\eta \in \Gamma(E)$.*

*Proof.* By direct computation, using $\left(\nabla_{\varphi X}\varphi^*\right)\eta = \nabla_{\varphi X}(\varphi^*\eta) - \varphi^*(\nabla_{\varphi X}\eta)$. □

If $\nabla$ is a torsion-free connection, then we obtain

$$N_\varphi(X,Y) = \left(\nabla_{\varphi X}\varphi\right)Y - \left(\nabla_{\varphi Y}\varphi\right)X + \varphi\left((\nabla_Y\varphi)X - (\nabla_X\varphi)Y\right). \quad (26)$$

**Theorem 2.** *If $\nabla$ is a torsion free connection on $M$ such that $\nabla\varphi = 0$, then the generalized $(a,b)$-structure $\hat{J}$ is $\nabla$-integrable.*

*Proof.* Condition $\nabla\varphi$=0 implies from (26) that $N_\varphi = 0$. Then, the same condition gives us $\nabla_X(\varphi Y) = \varphi(\nabla_X Y)$ for all vector fields $X, Y$. We also compute

$$(\nabla_{\varphi X}(\varphi^*\eta))\, Z = (\varphi X)(\eta(\varphi Z)) - \eta\left(\varphi(\nabla_{\varphi X} Z)\right) =$$

$$= (\varphi X)(\eta(\varphi Z)) - \eta(\nabla_{\varphi X}\varphi Z) = (\nabla_{\varphi X}\eta(\varphi Z) = \varphi^*\left(\nabla_{\varphi X}\eta\right)(Z),$$

for every vector field $Z$. We obtain

$$\nabla_{\varphi X}(\varphi^*\eta) = \varphi^*\left(\nabla_{\varphi X}\eta\right),$$

and then $(\nabla_{\varphi X}\varphi^*)\,\eta = 0$, for every vector field $X$ and $1$-form $\eta$. This relation and $\nabla\varphi = 0$ in (25) imply that $N_{\hat{J}}^{\nabla} = 0$, so $\hat{J}$ is $\nabla$-integrable. □

## References


[1] Blaga, A.M. and Ida, C., *Generalized almost paracontact structures*, An. Şt. Univ. Ovidius Constanţa, **23**,2015, 53-64.

[2] V. Cruceanu, V., Fortuny, P. and Gadea, P.M., *A survey on paracomplex geometry*, Rocky Mount. J. Math. **26** (1996), 83-115.

[3] Cruceanu, V., Gadea, P.M. and Munoz Masque, J., *Para-Hermitian and para-Kähler manifolds*. Quaderni Inst. Mat. Univ. Messina **1** (1995), 1-72.

[4] Crainic, M., *Generalized complex structures and Lie brackets*, arXiv:math/0412097v2, 2004.

[5] Courant, T., *Dirac manifolds*, Trans. Amer. Math. Soc., **319** (1990), 631-661.

[6] Goldberg S.I., and Yano K., *Polynomial structures on manifolds*. Kodai Math Sem Rep. **22**(1970); 22: 199-218.

[7] Gezer A. and Karaman C., *On metallic Riemannian structures*. Turk J Math. **39**(2015), 954-962.

[8] Gualtieri, M., *Generalized complex geometry*, PhD Thesis, 2004, arXiv:math/0401221v1.

[9] Hitchin, N., *Generalized Calabi-Yau manifolds*, Q. J. Math., **54**( 2003), 281-308.

[10] Hreţcanu C.E. and Crâşmăreanu M., *Metallic structures on Riemannian Manifolds*, Rev Un. Mat. Argentina **54** (2013), 15-27.

[11] Nannicini, A., *Calibrated complex structures on the generalized tangent bundle of a Riemannian manifold*, J. of Geom. and Phys. **56** (2006), 903-916.

[12] Nannicini, A., *Almost complex structures on cotangent bundles and generalized geometry*, J. of Geom. and Phys. **60** (2010), 1781-1791.

[13] A. Nannicini, A., *Generalized geometry of Norden manifolds*, J. of Geom. and Phys. **99** (2016), 244-255.

[14] Nannicini, A., *Almost complex structures on cotangent bundles and generalized geometry*, J. of Geom. and Phys. **60** (2010), 1781-1791.

[15] Y. S. Poon, Y.S. and Wade, A., *Generalized Contact Structures*, 2009, arXiv:math/0912.5314v1.

[16] B. Şahin, B. and Şahin, F., *Generalized almost para-contact manifolds*, arXiv:1401.5304v1. math. GT.

[17] Vanzura J., *Integrability conditions for polynomial structures*. Kodai Math Sem Rep. **27**(1976), 42-50.

[18] Vaisman, I., *Reduction and submanifolds of generalized complex manifolds*, Diff. Geom. and Appl. **25** (2007), 147-166.

[19] Vaisman, I., *From Generalized Kähler to Generalized Sasakian Structures*, J. of Geom. and Symmetry in Physics **18** (2010), 63-86.

[20] Vaisman, I., *Generalized para-Kahler manifolds*, 2015, arXiv:1503.01252v1[math.DG]

[21] Yano, K., *Differential geometry on complex and almost complex spaces*, Pure and Applied Math. **49**, New York, Pergamon Press Book, 1965.

[22] Wade, A., *Dirac structures and paracomplex manifolds*, Comptes Rendus Mathematique, **11** (2004), 889-894.

[23] Zabzine, M., *L*ectures On Generalized Complex Geometry and Supersymmetry, Archivum Mathematicum(BRNO) **42** (2006), 119 – 146.