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ZALCMAN FUNCTIONAL FOR CLASS OF $\lambda-$PSEUDO STARLIKE FUNCTIONS ASSOCIATED WITH SIGMOID FUNCTION

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Abstract

In our present investigation, we obtain the initial coeficients a_2, a_3, a_4 and a_5 for a class of univalent $\lambda-$pseudo starlike functions associated with Sigmoid function and then we find the Zalcman functional $|a_3^2 - a_5|$. Specializing the parameters, we improvise Zalcman functional for the class of starlike functions and also we point out Zalcman functional for the class of functions which are product combination of bounded turning and starlike functions.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}) \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and normalized $f(0) = f'(0) - 1 = 0$. Further, denote by $\mathcal{S}$ the class of analytic, normalized and univalent functions in $\mathbb{U}$.

Various subfamilies of the well-known subclass of $\mathcal{S}$ and Bazilevič functions $\mathcal{B}(\alpha)$ satisfying the geometric condition:

$$\Re\left(\frac{f(z)^{\alpha-1} f'(z)}{z^{\alpha-1}}\right) > 0,$$

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where α is greater than 1 ($\alpha \in \mathbb{R}$) were studied in recently and in the past (See: [4, 6, 10, 11]). The class includes the starlike and bounded turning functions as the cases $\alpha = 0$ and $\alpha = 1$ respectively. Further, the study has been extended by defining a subclass, Bazilevič functions of type α order β if and only if $\Re\left(\frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}}\right) > \beta$ denoted by $\mathcal{B}(\alpha, \beta)$.

Recently, Babalola [1] defined a new subclass $\lambda-$pseudo starlike function of order β $(0 \leq \beta < 1)$ satisfying the analytic condition

$$\Re\left(\frac{z(f'(z))^{\lambda}}{f(z)}\right) > \beta, \qquad (z \in \mathbb{U}, \lambda \geq 1 \in \mathbb{R}) \tag{2}$$

and denoted by $\mathcal{L}_{\lambda}(\beta)$.

Remark 1. *For $\lambda = 1,$ we have $\mathcal{L}_1(\beta) \equiv \mathcal{ST}(\beta)$ the class of starlike functions of order β (or $1-$pseudo starlike functions of order β), satisfying the condition*

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, \qquad (z \in \mathbb{U}) \tag{3}$$

By taking $\lambda = 1$ and $\beta = 0,$ we have $\mathcal{L}_1(0) \equiv \mathcal{ST}(0),$ simply write $\mathcal{L}_1 = \mathcal{ST}.$

Remark 2. *For $\lambda = 2,$ we note that functions in $\mathcal{L}_2(\beta) = \mathcal{G}(\beta)$ are defined by*

$$\Re\left(f'(z)\frac{zf'(z)}{f(z)}\right) > \beta, \qquad (z \in \mathbb{U}) \tag{4}$$

which is a product combination of geometric expressions for bounded turning and starlike functions.

Babalola [1] remarked that though for $\lambda > 1,$ these classes of $\lambda-$pseudo starlike functions clone the analytic representation of starlike functions, the possibility of any inclusion relations between them it is not yet known.

Activation function is an information process that is inspired by the way biological nervous systems such as the brain, process information. It consisted of a large number of neurons, highly interconnected processing elements working together to solve a specific task. This function works in a similar way as the brain does, it learns by examples and cannot be programmed to solve a specific task. The most popular activation function is the sigmoid function because of its gradient descendent learning algorithm. Sigmoid function can be evaluated in different ways, it can be done by truncated series expansion (for details see [5]).

The logistic sigmoid function

$$h(z) = \frac{1}{1 + e^{-z}} \tag{5}$$

is differentiable and has properties that it outputs real numbers between 0 and 1. It maps a large input domain to a small range of outputs. Also it never loses information as it is a one-to-one function. It increases monotonically. With all the properties mentioned in [5], Sigmoid function is perfectly useful in Geometric Function Theory. Recently, we examined the Fekete - Szegö results for $\mathcal{L}_\lambda(\beta)$ satisfying the condition

$$\Re\left(\frac{z(f'(z))^\lambda}{f(z)}\right) > \beta, \qquad (0 \leq \beta < 1, z \in \mathbb{U}, \lambda \geq 1 \in \mathbb{R})$$

associated with Sigmoid functions in [9].

In the field of Geometric Function Theory, one of the classical conjectures proposed by Lawrence Zalcman in 1960 is that the coefficients of class $\mathcal{S}$ satisfy the inequality,

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2. \tag{6}$$

The above form holds equality only for the famous Koebe function $k(z) = \frac{z}{(1-z)^2}$ and its rotation. When $n = 2$, the above inequality leads to Fekete - Szegö inequality. In literature, many researchers [2, 3, 7, 8] studied the Zalcman functional.

For $f \in \mathcal{A}$, Uralegaddi et. al [12] studied the subclass $\mathcal{M}(\beta)$ of $\mathcal{S}$ satisfying the analytic criteria

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < \beta, \qquad (z \in \mathbb{U})$$

for some real number β with $1 < \beta \leq \frac{4}{3}$. Motivated by Babalola [1] and Uralegaddi et. al [12], in this paper, we study a new subclass of $\lambda-$pseudo starlike functions of order β $(\beta > 1)$ satisfying the criterion

$$\Re\left(\frac{z(f'(z))^\lambda}{f(z)}\right) < \beta, \qquad (\beta > 1, z \in \mathbb{U}, \lambda \geq 1 \in \mathbb{R}) \tag{7}$$

denoted by $\mathcal{L}_\lambda^*$ and investigate how the sigmoid function is related to analytic univalent $\lambda-$pseudo starlike functions in terms of coefficients bounds with improvised results on Zalcman functional. Further, when λ takes the values 1 and 2, the class $\mathcal{L}_\lambda^*$ deduces to $\mathcal{ST}^*$ and $\mathcal{G}^*$, the subclasses related with sigmoid functions satisfying the analogous conditions given by (3) and (4) respectively. We obtain Zalcman functional for these classes too.

2 Initial coefficients

First we recall the following due to Joseph et al [5] in order to prove our main result.

Lemma 1. *[5] Let h be a sigmoid function given by (5) and*

$$\Phi(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right)^m, \tag{8}$$

then $\Phi(z) \in \mathcal{P}, |z| < 1$ where $\Phi(z)$ is a modified sigmoid function.

Lemma 2. *[5] Let*

$$\Phi_{n,m}(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right)^m, \tag{9}$$

then $|\Phi_{n,m}(z)| < 2$.

Lemma 3. *[5] If $\Phi(z) \in \mathcal{P}$ is starlike, then f is a normalized univalent function of the form (1).*

Taking $m = 1$, Joseph et al [5] remarked the following:

Remark 3. *Let $\Phi(z) = 1 + \sum_{n=1}^{\infty} C_n z^n$ where $C_n = \frac{-1(-1)^n}{2\ n!}$ then*

$$|C_n| \leq 2, n = 1, 2, 3, ...$$

this result is sharp for each n.

Theorem 1. *If $f \in \mathcal{A}$ and of the form (1) is belongs to $\mathcal{L}_\lambda^*$ $(\lambda \geq 1 \in \mathbb{R})$, then*

$$\begin{aligned}
|a_2| &\leq \frac{1}{4(2\lambda - 1)} \\
|a_3| &\leq \frac{|4\lambda - 2\lambda^2 - 1|}{16(3\lambda - 1)(2\lambda - 1)^2} \\
|a_4| &\leq \frac{1}{192|(4\lambda - 1)(3\lambda - 1)^3(2\lambda - 1)^6|} \\
&\quad \times \big|6368\lambda^9 - 31824\lambda^8 + 85384\lambda^7 - 147116\lambda^6 + 159120\lambda^5 \\
&\qquad -102776\lambda^4 + 37310\lambda^3 - 6969\lambda^2 + 524\lambda - 1\big| \\
|a_5| &\leq \frac{1}{3072|(5\lambda - 1)(4\lambda - 1)(3\lambda - 1)^3(2\lambda - 1)^7|} \\
&\quad \times \big|1102336\lambda^{11} - 5575808\lambda^{10} + 13365376\lambda^9 - 20347200\lambda^8 \\
&\qquad +21706976\lambda^7 - 16440728\lambda^6 + 8559328\lambda^5 - 2908052\lambda^4 \\
&\qquad +606222\lambda^3 - 70832\lambda^2 + 3906\lambda - 68\big| \text{ and} \\
|a_3^2 - a_5| &\leq \frac{1}{\big|1536(5\lambda - 1)(4\lambda - 1)(3\lambda - 1)^3(2\lambda - 1)^7\big|} \\
&\quad \times \big|551168\lambda^{11} - 2799424\lambda^{10} + 6755072\lambda^9 - 10360896\lambda^8 \\
&\qquad +11117824\lambda^7 - 8448820\lambda^6 + 4407488\lambda^5 - 1501330\lambda^4 \\
&\qquad +314619\lambda^3 - 37186\lambda^2 + 2109\lambda - 40\big|.
\end{aligned}$$

Proof. Let $f(z) \in \mathcal{L}_\lambda^*$, by definition there exists $\Phi(z) \in \mathcal{P}$ such that

$$\frac{z(f'(z))^\lambda}{f(z)} = \frac{1}{2}[3 - \Phi(z)] \quad (z \in \mathbb{U}, \lambda \geq 1 \in \mathbb{R}), \tag{10}$$

where the function $\Phi(z)$ is a modified sigmoid function given by

$$\Phi(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{1}{64}z^6 + \frac{779}{20160}z^7 - ... \quad . \tag{11}$$

Thus

$$2z(f'(z))^\lambda = f(z)[3 - \Phi(z)]. \tag{12}$$

In view of (10), (11) and (12), expanding in series forms we have

$$\begin{aligned}
&2z+4\lambda\, a_2z^2+[6\lambda a_3+4\lambda(\lambda-1)a_2^2]z^3\\
&\quad+\left[8\lambda a_4+12\lambda(\lambda-1)a_2a_3+\frac{8}{3}\lambda(\lambda-1)(\lambda-2)a_2^3\right]z^4+\\
&\quad+\Big[10\lambda a_5+9\lambda(\lambda-1)\, a_3^2+16\lambda(\lambda-1)\, a_2a_4\\
&\qquad+12\lambda(\lambda-1)(\lambda-2)a_2^2a_3+\frac{4}{3}\lambda(\lambda-1)(\lambda-2)(\lambda-3)a_2^4\Big]z^5...\\
&=2z+\left[2a_2-\frac{1}{2}\right]z^2+\left[2a_3-\frac{1}{2}a_2\right]z^3\\
&\quad+\left[2a_4-\frac{1}{2}a_3+\frac{1}{24}\right]z^4+\left[2a_5-\frac{1}{2}a_4+\frac{1}{24}\right]z^5... \qquad (13)
\end{aligned}$$

Comparing the coefficients of z^2, z^3, z^4 and z^5 in the above equation (13), we obtain

$$a_2 = -\frac{1}{4(2\lambda-1)}, \qquad (14)$$

$$a_3 = \frac{(4\lambda-2\lambda^2-1)}{16(3\lambda-1)(2\lambda-1)^2}, \qquad (15)$$

$$\begin{aligned}
2(4\lambda-1)a_4 &= \frac{1}{24}-\frac{1}{2}a_3-12\lambda(\lambda-1)a_2a_3-\frac{8}{3}\lambda(\lambda-1)(\lambda-2)a_2^3\\
a_4 &= \frac{1}{48(4\lambda-1)}+\frac{(2\lambda^2-4\lambda+1)}{2(4\lambda-1)(3\lambda-1)(2\lambda-1)^2}\\
&\quad\times\left(\frac{1}{32}-\frac{3\lambda(\lambda-1)}{16(2\lambda-1)}+\frac{8\lambda(\lambda-1)(\lambda-2)(2\lambda^2-4\lambda+1)^2}{3(3\lambda-1)^2(2\lambda-1)^4}\right)
\end{aligned}$$

By simple computation, we get

$$\begin{aligned}
a_4 &= \frac{1}{192(4\lambda-1)(3\lambda-1)^3(2\lambda-1)^6}\\
&\quad\times\big[6368\lambda^9-31824\lambda^8+85384\lambda^7-147116\lambda^6+159120\lambda^5\\
&\qquad-102776\lambda^4+37310\lambda^3-6969\lambda^2+524\lambda-1\big] \qquad (16)\\
2(5\lambda-1)a_5 &= \frac{1}{24}-\frac{1}{2}a_4\big[1+32\lambda(\lambda-1)a_2\big]-12\lambda(\lambda-1)(\lambda-2)a_2^2a_3\\
&\qquad-9\lambda(\lambda-1)a_3^2-\frac{4}{3}\lambda(\lambda-1)(\lambda-2)(\lambda-3)a_2^4 \qquad (17)
\end{aligned}$$

$$
\begin{aligned}
a_5 = & \frac{1}{48(5\lambda-1)} - \frac{9\lambda(\lambda-1)(2\lambda^2-4\lambda+1)^2}{512(5\lambda-1)(3\lambda-1)^2(2\lambda-1)^4} \\
& + \frac{3\lambda(\lambda-1)(\lambda-2)(2\lambda^2-4\lambda+1)}{128(5\lambda-1)(3\lambda-1)(2\lambda-1)^4} - \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{384(5\lambda-1)(2\lambda-1)^4} \\
& + \frac{8\lambda^2-10\lambda+1}{768(5\lambda-1)(4\lambda-1)(3\lambda-1)^3(2\lambda-1)^7} \times \Big[6368\lambda^9 - 31824\lambda^8 \\
& \quad + 85384\lambda^7 - 147116\lambda^6 + 159120\lambda^5 - 102776\lambda^4 + 37310\lambda^3 \\
& \quad - 6969\lambda^2 + 524\lambda - 1 \Big]
\end{aligned}
$$

Upon Simplification, the above equation deduces to

$$
\begin{aligned}
a_5 = & \frac{1}{3072(5\lambda-1)(4\lambda-1)(3\lambda-1)^3(2\lambda-1)^7} \\
& \times \Big[1102336\lambda^{11} - 5575808\lambda^{10} + 13365376\lambda^9 - 20347200\lambda^8 \\
& \quad + 21706976\lambda^7 - 16440728\lambda^6 + 8559328\lambda^5 - 2908052\lambda^4 \\
& \quad + 606222\lambda^3 - 70832\lambda^2 + 3906\lambda - 68 \Big]. \qquad (18)
\end{aligned}
$$

In order to find Zalcman functional, we use equations (15) and (18), then we obtain

$$
\begin{aligned}
a_3^2 - a_5 = & -\frac{1}{1536(5\lambda-1)(4\lambda-1)(3\lambda-1)^3(2\lambda-1)^7} \\
& \times \Big[551168\lambda^{11} - 2799424\lambda^{10} + 6755072\lambda^9 - 10360896\lambda^8 \\
& \quad + 11117824\lambda^7 - 8448820\lambda^6 + 4407488\lambda^5 - 1501330\lambda^4 \\
& \quad + 314619\lambda^3 - 37186\lambda^2 + 2109\lambda - 40 \Big]. \qquad (19)
\end{aligned}
$$

That is,

$$
\begin{aligned}
|a_3^2 - a_5| \leq & \frac{1}{\left|1536(5\lambda-1)(4\lambda-1)(3\lambda-1)^3(2\lambda-1)^7\right|} \\
& \times \big| 551168\lambda^{11} - 2799424\lambda^{10} + 6755072\lambda^9 - 10360896\lambda^8 \\
& \quad + 11117824\lambda^7 - 8448820\lambda^6 + 4407488\lambda^5 - 1501330\lambda^4 \\
& \quad + 314619\lambda^3 - 37186\lambda^2 + 2109\lambda - 40 \big|. \qquad (20)
\end{aligned}
$$

□

Corollary 1. *If $f(z) \in \mathcal{A}$ given by (1) belongs to $\mathcal{L}_1^* \equiv \mathcal{ST}^*$, then*

$$|a_2| \leq \frac{1}{4}, \quad |a_3| \leq \frac{1}{32}, \quad |a_4| \leq \frac{5}{1152}, \quad |a_5| \leq \frac{91}{18432} \quad \text{and}$$

$$|a_3^2 - a_5| \leq \frac{73}{18432}.$$

Corollary 2. *If $f(z) \in \mathcal{A}$ given by (1) belongs to $\mathcal{L}_2^* \equiv \mathcal{G}^*$, then*

$$|a_2| \leq \frac{1}{12}, \quad |a_3| \leq \frac{1}{720}, \quad |a_4| \leq \frac{19}{6720}, \quad |a_5| \leq \frac{2407}{907200} \quad \text{and}$$

$$|a_3^2 - a_5| \leq \frac{1069}{403200}.$$

Concluding Remarks: We remark that, the Zalcman fuctional obtained in Corollary 1 is the improvised result given in [2]. Further, we note that the coefficient estimates a_2, a_3 and a_4 obtained in Corollary 1 and 2 are betterment to the results given in Corollary 2.8 and 2.9 of [9]. Using the coefficient estimates a_2, a_3 and a_4, we can easily find the Fekete-Szegö results $|a_3 - a_2^2|$ and $|a_2a_4 - a_3^2|$, for the function classes $\mathcal{L}_\lambda^*, \mathcal{ST}^*$ and $\mathcal{G}^*$.

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