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ON SEVERAL INEQUALITIES IN AN INNER PRODUCT SPACE

Silvestru Sever DRAGOMIR[1] **and Nicuşor MINCULETE**[2]

Abstract

The aim of this article is to establish some identity in an inner product space and to prove new results related to several inequalities in an inner product space. Also, we obtain some applications of these equalities and inequalities.

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1 Introduction

The many inequalities in inner product spaces have important applications in Mathematics in various fields, as: Linear Operators Theory, Nonlinear Analysis, Approximation Theory, Optimization Theory, Geometry, Probability Theory, Statistics and other fields. An important inequality is the triangle inequality,

$$\|x+y\| \leq \|x\| + \|y\|,$$

for all $x, y \in X$, where X is a complex normed space. Several refinements of this inequality can be find in [6] and [13].

Another inequality which plays the central role in an inner product space is the inequality of Cauchy-Schwarz [3], namely:

$$|\langle x, y\rangle| \leq \|x\| \|y\|,$$

for all $x, y \in X$, where X is a complex inner product space.

A proof of the Cauchy-Schwarz inequality is given by Aldaz in [1]. Dragomir [5,8] studied the Cauchy-Schwarz inequality in the complex case. Many other proofs in the real case and in the complex case can be found in [2],[6], [7] and [12], [15]. Several improvements of this inequality can be found in [6] and [18].

[1]Victoria University, Mathematics, School of Engineering & Science, PO Box 14428, Melbourne City, MC 8001, Australia, e-mail: Sever.Dragomir@vu.edu.au

[2]Transilvania University of Braşov, Department of Mathematics and Informatics Iuliu Maniu Street no. 50, 500091, Braşov, Romania, e-mail: minculeten@yahoo.com

We obtained some reverses of the Cauchy-Schwarz inequality from [6], [13], [14] and [19].

Clarkson [4] gives the notion *the angular distance* $\alpha[x,y]$ between nonzero vectors x and y in X , by $\alpha[x,y]=\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|$. A simple norm inequality related to $\alpha[x,y]$ can be seen in [9]. The Cauchy-Schwarz inequality can be deducted from the following equality, as in Aldaz [1] and Niculescu [17], in terms of the angular distance between two vectors, thus

$$\langle x,y\rangle=\|x\|\,\|y\|\left(1-\frac{1}{2}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2\right),$$

for all $x,y\in X$, $x,y\neq 0$. Kirk and Smiley in [12] gave another characterization of inner product spaces by the angular distance between two vectors and improved a result from [14].

In [10], Ilišević and Varošaneć showed the Cauchy-Schwarz inequality and its reverse in semi-inner product $\mathbb{C}^*-$modules.

The Schwarz, triangle, Bessel, Gram and most recently, Grüss type inequalities have been frequently used as powerful tools in obtaining bounds or estimating the errors for various approximation formulae [7].

2 Main results

In this section of the article we obtain several results related to the identities for complex inner product spaces, and thus we obtain a proof of the Cauchy-Schwarz inequality in the complex case.

Let X be an inner product space over the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. The inner product $\langle\cdot,\cdot\rangle$ induces an associated norm, given by $\|x\|=\sqrt{<x,x>}$, for all $x\in X$, thus X is a normed vector space.

Theorem 1. *In an inner product space X over the field of complex numbers $\mathbb{C}$, we have*

$$\frac{1}{\|y\|^2}\langle\alpha y-x,x-\beta y\rangle=\left(\alpha-\frac{\langle x,y\rangle}{\|y\|^2}\right)\left(\frac{\overline{\langle x,y\rangle}}{\|y\|^2}-\overline{\beta}\right)-\frac{1}{\|y\|^2}\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2, \quad (1)$$

for all $x,y\in X$, $y\neq 0$, and for every $\alpha,\beta\in\mathbb{C}$.

Proof. Using the axioms: conjugate symmetry, $\langle x,y\rangle=\overline{\langle y,x\rangle}$, and linearity in the first argument, $\langle\alpha x,y\rangle=\alpha\langle x,y\rangle$, $\langle x+y,z\rangle=\langle x,z\rangle+\langle y,z\rangle$, we obtain $\langle x,\beta y\rangle=\overline{\beta}\langle x,y\rangle$, and

$$\frac{1}{\|y\|^2}\langle\alpha y-x,x-\beta y\rangle=\alpha\frac{\overline{\langle x,y\rangle}}{\|y\|^2}+\overline{\beta}\frac{\langle x,y\rangle}{\|y\|^2}-\alpha\overline{\beta}-\frac{\|x\|^2}{\|y\|^2}=$$
$$\left(\alpha-\frac{\langle x,y\rangle}{\|y\|^2}\right)\left(\frac{\overline{\langle x,y\rangle}}{\|y\|^2}-\overline{\beta}\right)-\frac{1}{\|y\|^2}\left(\|x\|^2-\frac{|\langle x,y\rangle|^2}{\|y\|^2}\right).$$

But, it is easy to see that $\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2=\|x\|^2-\frac{|\langle x,y\rangle|^2}{\|y\|^2}$. Therefore, we obtain the relation of the statement. $\square$

Corollary 1. *In an inner product space X over the field of real numbers $\mathbb{R}$, we have*

$$\frac{1}{\|y\|^2}\langle\alpha y-x,x-\beta y\rangle=\left(\alpha-\frac{\langle x,y\rangle}{\|y\|^2}\right)\left(\frac{\langle x,y\rangle}{\|y\|^2}-\beta\right)-\frac{1}{\|y\|^2}\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2, \quad (2)$$

for all $x,y\in X$, $y\neq 0$, and for every $\alpha,\beta\in\mathbb{R}$.

Proof. Since $\overline{\langle x,y\rangle}=\langle x,y\rangle$ and $\overline{\beta}=\beta$, we apply Theorem 1 and we deduce equality (2). $\square$

Corollary 2. *Let E_3 be the Euclidean punctual space. Then*

$$\frac{1}{\|y\|^2}\langle\alpha y-x,x-\beta y\rangle=\left(\alpha-\frac{\langle x,y\rangle}{\|y\|^2}\right)\left(\frac{\langle x,y\rangle}{\|y\|^2}-\beta\right)-\frac{1}{\|y\|^4}\|x\times y\|^2, \quad (3)$$

for all $x,y\in E_3$, $y\neq 0$, and for every $\alpha,\beta\in\mathbb{R}$.

Proof. In relation (2), we use the relation

$$\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2=\|x\|^2-\frac{|\langle x,y\rangle|^2}{\|y\|^2}=\frac{1}{\|y\|^2}\left(\|x\|^2\|y\|^2-\langle x,y\rangle^2\right)$$

and the Lagrange identity, $\|x\|^2\|y\|^2-\langle x,y\rangle^2=\|x\times y\|^2$, and we obtain the relation of the statement. $\square$

Corollary 3. *In an inner product space X over the field of complex numbers $\mathbb{C}$, we have*

$$\|x-\alpha y\|^2=\left|\alpha\|y\|-\frac{\langle x,y\rangle}{\|y\|}\right|^2+\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2, \quad (4)$$

for all $x,y\in X$, $y\neq 0$, and for every $\alpha\in\mathbb{C}$.

Proof. We apply Theorem 1 for $\alpha=\beta$, and we deduce

$$\frac{1}{\|y\|^2}\|x-\alpha y\|^2=\left|\alpha-\frac{\langle x,y\rangle}{\|y\|^2}\right|^2+\frac{1}{\|y\|^2}\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2.$$

Consequently, we deduce the statement. $\square$

Remark 1. It is easy to see that

$$\inf_{\alpha\in\mathbb{C}}\|x-\alpha y\|=\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|=\sqrt{\|x\|^2-\frac{|\langle x,y\rangle|^2}{\|y\|^2}}.$$

Corollary 4. *In an inner product space X over the field of complex numbers $\mathbb{C}$, we have*

$$\|x-\alpha y\| \geq \left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|, \tag{5}$$

and

$$\|x-\alpha y\| \geq \left|\alpha\|y\|-\frac{\langle x,y\rangle}{\|y\|}\right|, \tag{6}$$

for all $x, y \in X$, $y \neq 0$, and for every $\alpha \in \mathbb{C}$.

Proof. Using relation (4) and that $\left|\alpha-\frac{\langle x,y\rangle}{\|y\|^2}\right|^2 \geq 0$ and $\frac{1}{\|y\|^2}\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2 \geq 0$, we obtain the relations of the statement. □

Remark 2. From relation (6), for $\alpha = 0$, we obtain the inequality of Cauchy-Schwarz: $\|x\|\|y\| \geq |\langle x,y\rangle|$.

Corollary 5. *In an inner product space X over the field of complex numbers $\mathbb{C}$, we have*

$$\|x\|^2\|y\|^2 = |\langle x,y\rangle|^2 + \left\|\|y\|x-\frac{\langle x,y\rangle}{\|y\|}y\right\|^2, \tag{7}$$

$$\|x\|^2\|y\|^2\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2 = |\|x\|\|y\|-\langle x,y\rangle|^2 + \|x\|^2\|y\|^2 - |\langle x,y\rangle|^2, \tag{8}$$

for all $x, y \in X$, $y \neq 0$, and

$$\text{Re}\langle x,y\rangle = \frac{1}{2}\left(\|x\|^2+\|y\|^2-\|x-y\|^2\right), \tag{9}$$

$$\text{Im}\langle x,y\rangle = \frac{1}{2}\left(\|x\|^2+\|y\|^2-\|x-iy\|^2\right), \tag{10}$$

for all $x, y \in X$.

Proof. From relation (4), for $\alpha = 0$, we obtain relation (7).

From relation (4), for $\alpha = \frac{\|x\|}{\|y\|}$, we obtain

$$\|x\|y\|-\|x\|y\|^2 = |\|x\|\|y\|-\langle x,y\rangle|^2 + \left\|\|y\|x-\frac{\langle x,y\rangle}{\|y\|}y\right\|^2,$$

which is equivalent to

$$\|x\|^2\|y\|^2\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2 = |\|x\|\|y\|-\langle x,y\rangle|^2 + \left\|\|y\|x-\frac{\langle x,y\rangle}{\|y\|}y\right\|^2.$$

But, we have the equality $\left\| \|y\| \, x - \frac{\langle x,y\rangle}{\|y\|} y \right\|^2 = \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2$. Therefore, we deduce the equality of the statement.

For $y = 0$, equality (9) is true. In the case $y \neq 0$, from relation (4), for $\alpha = 1$, we obtain

$$\begin{aligned} \|x-y\|^2 &= \left| \|y\| - \frac{\langle x,y\rangle}{\|y\|} \right|^2 + \left\| x - \frac{\langle x,y\rangle}{\|y\|^2} y \right\|^2 = \\ &= \left(\|y\| - \frac{\text{Re}\,\langle x,y\rangle}{\|y\|} \right)^2 + \frac{\text{Im}^2\,\langle x,y\rangle}{\|y\|^2} + \left\| x - \frac{\langle x,y\rangle}{\|y\|^2} y \right\|^2 . \end{aligned}$$

But, we have

$$\begin{gathered} \frac{\text{Im}^2\,\langle x,y\rangle}{\|y\|^2} + \left(\|y\| - \frac{\text{Re}\,\langle x,y\rangle}{\|y\|} \right)^2 = \frac{\text{Im}^2\,\langle x,y\rangle}{\|y\|^2} + \|y\|^2 - 2\text{Re}\,\langle x,y\rangle + \frac{\text{Re}^2\,\langle x,y\rangle}{\|y\|^2} = \\ \|y\|^2 - 2\text{Re}\,\langle x,y\rangle + \frac{\text{Re}^2\,\langle x,y\rangle + \text{Im}^2\,\langle x,y\rangle}{\|y\|^2} = \|y\|^2 - 2\text{Re}\,\langle x,y\rangle + \frac{|\langle x,y\rangle|^2}{\|y\|^2}, \end{gathered}$$

and $\left\| x - \frac{\langle x,y\rangle}{\|y\|^2} y \right\|^2 = \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2}$. It follows that $\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\text{Re}\,\langle x,y\rangle$, which prove relation (9).

For $y = 0$, equality (10) is true. Now, for $y \neq 0$, using relation (4), for $\alpha = i$, we obtain

$$\begin{aligned} \|x-iy\|^2 &= \left| i\,\|y\| - \frac{\langle x,y\rangle}{\|y\|} \right|^2 + \left\| x - \frac{\langle x,y\rangle}{\|y\|^2} y \right\|^2 = \\ &\quad \frac{\text{Re}^2\,\langle x,y\rangle}{\|y\|^2} + \left(\|y\| - \frac{\text{Im}\,\langle x,y\rangle}{\|y\|} \right)^2 + \left\| x - \frac{\langle x,y\rangle}{\|y\|^2} y \right\|^2 . \end{aligned}$$

Similarly as above, we have

$$\begin{gathered} \frac{\text{Re}^2\,\langle x,y\rangle}{\|y\|^2} + \left(\|y\| - \frac{\text{Im}\,\langle x,y\rangle}{\|y\|} \right)^2 = \frac{\text{Re}^2\,\langle x,y\rangle}{\|y\|^2} + \|y\|^2 - 2\text{Im}\,\langle x,y\rangle + \frac{\text{Im}^2\,\langle x,y\rangle}{\|y\|^2} = \\ = \|y\|^2 - 2\text{Im}\,\langle x,y\rangle + \frac{\text{Re}^2\,\langle x,y\rangle + \text{Im}^2\,\langle x,y\rangle}{\|y\|^2} = \|y\|^2 - 2\text{Im}\,\langle x,y\rangle + \frac{|\langle x,y\rangle|^2}{\|y\|^2}, \end{gathered}$$

and $\left\| x - \frac{\langle x,y\rangle}{\|y\|^2} y \right\|^2 = \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2}$. It follows that $\|x-iy\|^2 = \|x\|^2 + \|y\|^2 - 2\text{Im}\,\langle x,y\rangle$, which implies relation (10). □

Remark 3. From relation (8) applied in an inner product space X over the field of real numbers $\mathbb{R}$, this becomes

$$\frac{1}{2}\left\|x\right\|\left\|y\right\|\left\|\frac{x}{\left\|x\right\|}-\frac{y}{\left\|y\right\|}\right\|^{2}=\left\|x\right\|\left\|y\right\|-\left\langle x,y\right\rangle , \tag{11}$$

for all $x, y \in X, x, y \neq 0.$ This implies the identity given in [1].

Remark 4. In relations (9) and (10), we make the substitutions $x \longrightarrow \frac{x}{\|x\|}$, $y \longrightarrow \frac{y}{\|y\|}$, and, we deduce the relations, given by Aldaz in [1]:

$$\text{Re}\left\langle x,y\right\rangle =\left\|x\right\|\left\|y\right\|\left(1-\frac{1}{2}\left\|\frac{x}{\left\|x\right\|}-\frac{y}{\left\|y\right\|}\right\|^{2}\right), \tag{12}$$

$$\text{Im}\left\langle x,y\right\rangle =\left\|x\right\|\left\|y\right\|\left(1-\frac{1}{2}\left\|\frac{x}{\left\|x\right\|}-\frac{iy}{\left\|y\right\|}\right\|^{2}\right), \tag{13}$$

for all non-zero vectors $x, y \in X$.

Remark 5. Adding equalities (9) and (10), and using the parallelogram identity, $\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$, we deduce

$$2\left(\text{Re}\left\langle x,y\right\rangle +\text{Im}\left\langle x,y\right\rangle \right)=\left\|x+y\right\|^{2}-\left\|x-iy\right\|^{2}, \tag{14}$$

for all $x, y \in X$.

Finally, we present several applications of these identities and inequalities.

Theorem 2. *In an inner product space X over the field of complex numbers $\mathbb{C}$, we have*

$$\left\|x-\left\langle x,e\right\rangle e\right\|^{2}=\left(\alpha-\left\langle x,e\right\rangle \right)\left(\overline{\left\langle x,e\right\rangle }-\overline{\beta}\right)-\left\langle \alpha e-x,x-\beta e\right\rangle , \tag{15}$$

for all $x, e \in X$, $\|e\| = 1$, and for every $\alpha, \beta \in \mathbb{C}$.

Proof. If we take $y = e$, with $\|e\| = 1$, then (1), becomes

$$\left\langle \alpha e-x,x-\beta e\right\rangle =\left(\alpha-\left\langle x,e\right\rangle \right)\left(\overline{\left\langle x,e\right\rangle }-\overline{\beta}\right)-\left\|x-\left\langle x,e\right\rangle e\right\|^{2},$$

which implies the statement. □

If we take Re in relation (15), we get the well known identity:

$$\begin{aligned}\left\|x\right\|^{2}-\left|\left\langle x,e\right\rangle \right|^{2}&=\left\|x-\left\langle x,e\right\rangle e\right\|^{2}= \\ &= \text{Re}\left[\left(\alpha-\left\langle x,e\right\rangle \right)\left(\overline{\left\langle x,e\right\rangle }-\overline{\beta}\right)\right]-\text{Re}\left[\left\langle \alpha e-x,x-\beta e\right\rangle \right],\end{aligned} \tag{16}$$

which was used to prove various Grüss type inequalities (see [6]).

Remark 6. From a different perspective, if we take the modulus in relation (15), we have

$$\begin{aligned}\left\|x-\left\langle x,e\right\rangle e\right\|^{2} &= \left|\left(\alpha-\left\langle x,e\right\rangle \right)\left(\overline{\left\langle x,e\right\rangle }-\overline{\beta}\right)-\left\langle \alpha e-x,x-\beta e\right\rangle \right|\geq \\ & \left|\left(\alpha-\left\langle x,e\right\rangle \right)\left(\overline{\left\langle x,e\right\rangle }-\overline{\beta}\right)\right|-\left|\left\langle \alpha e-x,x-\beta e\right\rangle \right|,\end{aligned}$$

which implies that

$$\|x-\langle x,e\rangle e\|^2+|\langle \alpha e-x,x-\beta e\rangle|\geq\left|(\alpha-\langle x,e\rangle)\left(\overline{\langle x,e\rangle}-\overline{\beta}\right)\right|, \tag{17}$$

or

$$\|x-\langle x,e\rangle e\|^2+\left|(\alpha-\langle x,e\rangle)\left(\overline{\langle x,e\rangle}-\overline{\beta}\right)\right|\geq|\langle \alpha e-x,x-\beta e\rangle|. \tag{18}$$

Theorem 3. *In an inner product space X over the field of complex numbers $\mathbb{C}$, we have*

$$\|x\|^2-|\langle x,e\rangle|^2\leq\frac{1}{4}|\alpha-\beta|^2+\left\|x-\frac{\alpha+\beta}{2}e\right\|^2. \tag{19}$$

for all $x,e\in X$, $\|e\|=1$, and for every $\alpha,\beta\in\mathbb{C}$.

Proof. From (16) we also get

$$\|x\|^2-|\langle x,e\rangle|^2=\text{Re}\left[(\alpha-\langle x,e\rangle)\left(\overline{\langle x,e\rangle}-\overline{\beta}\right)\right]+\text{Re}\left[\langle \alpha e-x,\beta e-x\rangle\right]. \tag{20}$$

But

$$\text{Re}\left(a\bar{b}\right)\leq\frac{1}{4}|a+b|^2, \text{for any } a,b\in\mathbb{C},$$

and

$$\text{Re}\,\langle u,v\rangle\leq\frac{1}{4}\|u+v\|^2, \text{for any } u,v\in X.$$

So, we obtain

$$\text{Re}\left[(\alpha-\langle x,e\rangle)\left(\overline{\langle x,e\rangle}-\overline{\beta}\right)\right]\leq\frac{1}{4}|\alpha-\beta|^2 \tag{21}$$

and

$$\text{Re}\left[\langle \alpha e-x,\beta e-x\rangle\right]\leq\frac{1}{4}\|\alpha e+\beta e-2x\|^2=\left\|\frac{\alpha+\beta}{2}e-x\right\|^2. \tag{22}$$

Taking into account relations (20), (21) and (22), we get the relation of the statement. □

Remark 7. Inequality (19) is of interest since if we take,

$$\left\|x-\frac{\alpha+\beta}{2}e\right\|\leq\delta,$$

then we have the reverse inequality for Cauchy-Schwarz's inequality

$$\|x\|^2-|\langle x,e\rangle|^2\leq\frac{1}{4}|\alpha-\beta|^2+\delta^2.$$

Theorem 4. *In an inner product space X over the field of complex numbers $\mathbb{C}$, we have*

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| \leq \tag{23}$$
$$\left(\frac{1}{4}|\alpha-\beta|^2 + \left\|x - \frac{\alpha+\beta}{2}e\right\|^2\right)^{1/2}\left(\frac{1}{4}|\lambda-\mu|^2 + \left\|y - \frac{\lambda+\mu}{2}e\right\|^2\right)^{1/2}.$$

for all $x,y,e \in X$, $\|e\| = 1$, and for every $\alpha,\beta,\lambda,\mu \in \mathbb{C}$.

Proof. It is well known [6] that for all $x,y \in X$, and $e \in X$, $\|e\| = 1$ we have

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| \leq \left(\|x\|^2 - |\langle x,e\rangle|^2\right)^{1/2}\left(\|y\|^2 - |\langle y,e\rangle|^2\right)^{1/2}. \tag{24}$$

Combining inequality (24) with inequality (19), we deduce inequality (23). □

Remark 8. In particular, $x,y,e \in X$, $\|e\| = 1$, and for every $\alpha,\beta,\lambda,\mu \in \mathbb{C}$, if

$$\left\|x - \frac{\alpha+\beta}{2}e\right\| \leq \delta \text{ and } \left\|y - \frac{\lambda+\mu}{2}e\right\| \leq \varepsilon,$$

with $\delta,\varepsilon > 0$, then we have the Grüss type inequality

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| \leq \left(\frac{1}{4}|\alpha-\beta|^2 + \delta^2\right)^{1/2}\left(\frac{1}{4}|\lambda-\mu|^2 + \varepsilon^2\right)^{1/2}. \tag{25}$$

3 Applications

1. If $\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| = \sqrt{2}$ and using relation (12),

$$\text{Re}\,\langle x,y\rangle = \|x\|\,\|y\|\left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2\right),$$

which is equivalent to

$$\text{Re}\,\langle x,y\rangle = \|x\|\,\|y\|\left(1 - \frac{1}{\sqrt{2}}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|\right)\left(1 + \frac{1}{\sqrt{2}}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|\right),$$

then we deduce $\text{Re}\,\langle x,y\rangle = 0$, and if $\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \leq \sqrt{2}$, then $\text{Re}\,\langle x,y\rangle \geq 0$ and

$$\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \geq \sqrt{2}\left(1 - \frac{\text{Re}\,\langle x,y\rangle}{\|x\|\,\|y\|}\right). \tag{26}$$

2. We present some applications of the above theorems to S_n numbers. Recall that if $(X;\langle\cdot,\cdot\rangle)$ is a inner product space and $\{e_1,e_2,...,e_n\}$ is an orthonormal system of vectors of X, then for any vectors $x,y \in X$, we define as in [6, 11]:

$$S_n(x,y) = \langle x,y\rangle - \sum_{k=1}^{n} \langle x,e_k\rangle \langle e_k,y\rangle .$$

Kechriniotis and Delibasis [11] gave a generalization of Grüss inequality in inner product spaces.

For all $x,y \in X$, and for every $\alpha,\beta,\lambda,\mu \in \mathbb{C}$, if

$$\left\| x - \frac{\alpha+\beta}{2} e_k \right\| \le \delta \text{ and } \left\| \lambda - \frac{\lambda+\mu}{2} e_k \right\| \le \varepsilon, k = \overline{1,n}$$

with $\delta,\varepsilon > 0$, then from relation (25), we have the Grüss type inequality

$$\left|\langle x,y\rangle - \langle x,e_k\rangle \langle e_k,y\rangle\right| \le \left(\frac{1}{4}|\alpha-\beta|^2 + \delta^2\right)^{1/2} \left(\frac{1}{4}|\lambda-\mu|^2 + \varepsilon^2\right)^{1/2} .$$

But, we see that

$$S_n(x,y) - (n-1)\langle x,y\rangle = \sum_{k=1}^{n} \left(\langle x,y\rangle - \langle x,e_k\rangle \langle e_k,y\rangle\right),$$

which implies

$$\left|S_n(x,y) - (n-1)\langle x,y\rangle\right| \le \sum_{k=1}^{n} \left|\langle x,y\rangle - \langle x,e_k\rangle \langle e_k,y\rangle\right| \le n\left(\frac{1}{4}|\alpha-\beta|^2 + \delta^2\right)^{1/2} \left(\frac{1}{4}|\lambda-\mu|^2 + \varepsilon^2\right)^{1/2} .$$

Therefore, we obtain

$$\left|S_n(x,y) - (n-1)\langle x,y\rangle\right| \le n\left(\frac{1}{4}|\alpha-\beta|^2 + \delta^2\right)^{1/2} \left(\frac{1}{4}|\lambda-\mu|^2 + \varepsilon^2\right)^{1/2} . \quad (27)$$

3. If we take the vectors $x = \overrightarrow{AB}, y = \overrightarrow{AC}$ in relation (3), we obtain the following inequality:

$$\frac{1}{\left\|\overrightarrow{AC}\right\|^2} \left\langle \alpha\overrightarrow{AC} - \overrightarrow{AB}, \overrightarrow{AB} - \beta\overrightarrow{AC} \right\rangle = \\ = \left(\alpha - \frac{\left\langle \overrightarrow{AB}, \overrightarrow{AC}\right\rangle}{\left\|\overrightarrow{AC}\right\|^2}\right) \left(\frac{\left\langle \overrightarrow{AB}, \overrightarrow{AC}\right\rangle}{\left\|\overrightarrow{AC}\right\|^2} - \beta\right) - \frac{1}{\left\|\overrightarrow{AC}\right\|^4} \left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\|^2, \quad (28)$$

which implies the relation

$$\left\langle \alpha\overrightarrow{AC}-\overrightarrow{AB},\overrightarrow{AB}-\beta\overrightarrow{AC}\right\rangle =$$
$$=\left(\alpha\left\|\overrightarrow{AC}\right\|-\left\|\overrightarrow{AB}\right\|\cos A\right)\left(\left\|\overrightarrow{AB}\right\|\cos A-\beta\left\|\overrightarrow{AC}\right\|\right)-\frac{4\Delta^2}{\left\|\overrightarrow{AC}\right\|^2}, \quad (29)$$

where Δ is the area of the triangle ABC.

If $\alpha=\beta$, then equality (29) becomes

$$\left\|\overrightarrow{AB}-\alpha\overrightarrow{AC}\right\|^2=\left(\left\|\overrightarrow{AB}\right\|\cos A-\alpha\left\|\overrightarrow{AC}\right\|\right)^2+\frac{4\Delta^2}{\left\|\overrightarrow{AC}\right\|^2}, \quad (30)$$

For $\alpha=1$ in equality (30), we deduce

$$\left\|\overrightarrow{BC}\right\|^2=\left(\left\|\overrightarrow{AB}\right\|\cos A-\left\|\overrightarrow{AC}\right\|\right)^2+\frac{4\Delta^2}{\left\|\overrightarrow{AC}\right\|^2}, \quad (31)$$

If we take $\left\|\overrightarrow{AB}\right\|=c,\left\|\overrightarrow{AC}\right\|=b,\left\|\overrightarrow{BC}\right\|=a$ in relation (31), we deduce

$$a^2=(c\cos A-b)^2+\frac{4\Delta^2}{b^2}. \quad (32)$$

Therefore, we use the cosine law, cosA=$\frac{b^2+c^2-a^2}{2bc}$ and we find the relation

$$4a^2b^2=\left(c^2-a^2-b^2\right)^2+16\Delta^2. \quad (33)$$

From relation (33), we find two relations for the area of the triangle ABC.

First, by squared, we deduce the following formula [16]:

$$16\Delta^2=2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4. \quad (34)$$

Second, equality (33) can be written as $4a^2b^2-\left(c^2-a^2-b^2\right)^2=16\Delta^2$, which means that $\left(2ab-c^2+a^2+b^2\right)\left(2ab+c^2-a^2-b^2\right)=16\Delta^2$. It follows that $\left((a+b)^2-c^2\right)\left(c^2-(a-b)^2\right)=16\Delta^2$, and if s is the semi-perimeter, then

$$\Delta=\sqrt{s\,(s-a)\,(s-b)\,(s-c)}, \quad (35)$$

which is the well known Heron formula(see e.g. [13], p.54).

Equality (30) becomes, for $\alpha=-1$, thus,

$$\left\|\overrightarrow{AB}+\overrightarrow{AC}\right\|^2=\left(\left\|\overrightarrow{AB}\right\|\cos A+\left\|\overrightarrow{AC}\right\|\right)^2+\frac{4\Delta^2}{\left\|\overrightarrow{AC}\right\|^2}. \quad (36)$$

If we take $\left\|\overrightarrow{AB}\right\|=c,\left\|\overrightarrow{AC}\right\|=b$ and m_a, the length of the median from A, in relation (36), we obtain

$$4m_a^2=(c\cos A+b)^2+\frac{4\Delta^2}{b^2}. \quad (37)$$

References

[1] Aldaz, J. M., *Strengthened Cauchy-Schwarz and Hölder inequalities*, JIPAM **10** (2009). iss. 4, art. 116.

[2] Alzer, H, A., *Refinement of the Cauchy-Schwarz inequality*, Journal of Mathematical Analysis and Applications **168** (1992), 596-604.

[3] Cauchy, A.-L., *Cours d'Analyse de l'École Royale Polytechnique, I ère partie, Analyse Algébrique*, Paris, 1821. Reprinted by Ed. Jacques Gabay, Paris, 1989.

[4] Clarkson, J. A., *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), 396-414.

[5] Dragomir, S.S., *A potpourri of Schwarz related inequalities in inner product spaces (II)*, J. Inequal. Pure Appl. Math. **7** (2006), no. 1, Article 14. (2006)

[6] Dragomir, S.S., *Advances in inequalities of the Schwarz, Grüss and Bessel type in inner product spaces*, Nova Science Publishers, Inc., New York 2005.

[7] Dragomir, S.S., *Advances in inequalities of the Schwarz, triangle and Heisenberg type in inner product spaces*, Nova Science Publishers, Inc., New York, 2007.

[8] Dragomir, S.S., *Refinements of Buzano's and Kurepa's inequalities in inner product spaces*, Facta Universitatis (Niš) Ser. Math. Inform. **20** (2005). 65-73.

[9] Dunkl, C. F. and Wiliams, K. S., *A simple norm inequality*, The American Mathematical Monthly **71** (1964), 53-54.

[10] Ilišević and D.,Varošaneć, S., *On the Cauchy-Schwarz inequality and its reverse in semi-inner product* $\mathbb{C}^*$*-modules*, Banach J. Math. Anal. **1** (2007), no. 1, 78–84.

[11] Kechriniotis A.I. and Delibasis, K.K., *On generalization of Grüss inequality in inner product spaces and applications*, J. Inequal. Appl., vol. 2010 (2010) art. ID167091.

[12] Kirk, W.A. and Smiley, M.F., *Mathematical Notes: Another characterization of inner product spaces*, Amer. Math. Monthly **71** (1964), no. 8, 890–891.

[13] Maligranda, L., *Simple norm inequalities*, The American Mathematical Monthly, **113** (2006), March, 256-260.

[14] Massera, J. L. and Schäffer, J. J., *Linear differential equations and functional analysis*, I., Ann. of Math. **67** (1958), 517-573.

[15] Minculete, N. and Păltănea, R., *Improved estimates for the triangle inequality*, J. Inequal. Appl. vol. 2017:17 (2017).

[16] Mitrinović, D. S., Pečarić, J. E. and Volenec, V., *Recent adnances in geometric inequalities*, Kluwer Academic Publishers, Dordrecht 1989.

[17] Niculescu, C. P. *Inequalities and identities*, A Journey into Mathematics., Mostar, (2015) November 11-15.

[18] Ostrowski, A., *Vorlesungen über differential-und integralrechnung*, Vol. 2, Birkhauser, Basel 1951.

[19] Stoica, E., Minculete, N. and Barbu, C., *New aspects of Ionescu–Weitzenböck's inequality*, Balkan Journal of Geometry and Its Applications, **21** (2016), no. 2, 95-101.