Bulletin of the *Transilvania* University of Braşov • Vol 11(60), No. 1 - 2018
Series III: Mathematics, Informatics, Physics, 37-48

HYPERSURFACE OF A FINSLER SPACE WITH DEFORMED BERWALD-MATSUMOTO METRIC

V. K. CHAUBEY[1] **and Brijesh Kumar TRIPATHI**[*,2]

Abstract

In the present paper we have studied the Finslerian hypersurfaces of a Finsler space with the special deformed Berwald-Matsumoto metric. We also examined the hypersurfaces of this special metric as a hyperplane of first, second and third kinds.

2000 *Mathematics Subject Classification:* 53B40, 53C60.
Key words: Finslerian hypersurface, Matsumoto metric and Berwald metric.

1 Introduction

P. Finlser was the first who introduced the slope of a mountain with respect to time measurement but he thought that (as he mentioned in his letter [9] to Matsumoto) it was a typical model of Finsler mertric. Further in 1989, Matsumoto [9] worked on the above problem and introduced the concept of slope metric which is defined as $\frac{\alpha^2}{v\alpha - w\beta}$ where $\alpha^2 = a_{ij}(x)y^iy^j$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a $1-$ form on n-dimensional manifold M^n, v and w are the non-zero constants. In 1990 Aikou and coauthors [1] studied the above metric in detail and named it Matsumoto metric and obtained very interesting results for this metric in stand point of Finsler metric.

In 1929 Berwald [2] introduced a very famous Finsler metric which was defined on unit ball $B^n(1)$ with all the straight line segments. Its geodesics has constant flag curvature $K = 0$ in the form of

$$L = \frac{\{\sqrt{1 - |x|^2|y|^2 + \langle x, y\rangle^2} + \langle x, y\rangle\}^2}{(1 - |x|^2)^2\sqrt{1 - |x|^2|y|^2 + \langle x, y\rangle^2}} \tag{1}$$

From a modern point of view the above Berwald's metric belongs to a special kind of Finsler metric called Berwald type metric and defined as $\frac{(\alpha+\beta)^2}{\alpha}$[12] and the authors of

[1]Department of Applied Sciences, Buddha Institute of Technology, Sector-7, GIDA, Gorakhpur (U.P.)-273209 India, e-mail: vkcoct@gmail.com
[2]Department of Mathematics, L.E. College Morbi (Gujarat)-363642 India, e-mail: brijeshkumartripathi4@gmail.com [*]Corresponding Author

the papers introduced very interesting results in the field of Finsler geometry.

The concept of Finslerian hypersurface was first introduced by Matsumoto in 1985 and further he defined three types of hypersurfaces that were called a hyperplane of the first, second and third kinds. Further, many authors studied these hyperplanes in different changes of the Finsler metric [3, 4, 5, 6, 10, 11] and obtained different results. In the present paper we combine the Berwald and Matsumoto metric and obtain a new metric named deformed Berwald-Matsumoto metric. We also examine the hypersurfaces of this special metric as a hyperplane of first, second and third kinds.

2 Preliminaries

In the present paper we consider an n-dimensional Finsler space $F^n = \{M^n, L(\alpha,\beta)\}$, that is, a pair consisting of an n- dimensional differentiable manifold M^n equipped with a Fundamental function L as a special Finsler Space with the metric

$$L(\alpha,\beta) = \frac{(\alpha+\beta)^2}{\alpha} + \frac{\alpha^2}{(\alpha-\beta)} \tag{2}$$

i.e. the combination of Berwald and Matsumoto metric named deformed Berwald-Matsumoto metric and the Finsler space F^n equiped with this metric is known as Berwald-Matsumoto Finsler space.

Differentiating equation (2) partially with respect to α and β we get

$$L_\alpha = \frac{(\alpha+\beta)(\alpha-\beta)^3+\alpha^3(\alpha-2\beta)}{\alpha^2(\alpha-\beta)^2}, \qquad L_\beta = \frac{2(\alpha+\beta)(\alpha-\beta)^2+\alpha^3}{\alpha(\alpha-\beta)^2}$$

$$L_{\alpha\alpha} = \frac{2\left\{\alpha^3+(\alpha-\beta)^3\right\}\beta^2}{\alpha^3(\alpha-\beta)^3} \qquad L_{\beta\beta} = \frac{2\left\{\alpha^3+(\alpha-\beta)^3\right\}}{\alpha(\alpha-\beta)^3}$$

$$L_{\alpha\beta} = \frac{-2\left\{\alpha^3+(\alpha-\beta)^3\right\}\beta}{\alpha^2(\alpha-\beta)^3}$$

where $L_\alpha = \frac{\partial L}{\partial\alpha}, \quad L_\beta = \frac{\partial L}{\partial\beta}, \quad L_{\alpha\alpha} = \frac{\partial L\alpha}{\partial\alpha}, \quad L_{\beta\beta} = \frac{\partial L\beta}{\partial\beta}, \quad L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial\beta}.$

In Finsler space $F^n = \{M^n, L(\alpha,\beta)\}$ the normalized element of support $l_i = \dot{\partial_i}L$ and angular metric tensor h_{ij} are given by [7]:

$$l_i = \alpha^{-1}L_\alpha Y_i + L_\beta b_i$$

$$h_{ij} = pa_{ij} + q_0 b_i b_j + q_{-1}(b_i Y_j + b_j Y_i) + q_{-2} Y_i Y_j$$

where $Y_i = a_{ij}y^j$. For the fundamental metric function (2) the above constants are

$$p = \frac{4\alpha^7 + \beta^7 - 6\alpha^6\beta - 6\alpha^5\beta^2 - 6\alpha^4\beta^3 + 4\alpha^3\beta^4 - 3\alpha^2\beta^5 - \alpha\beta^6}{\alpha^4(\alpha-\beta)^3} \tag{3}$$

$$
\begin{aligned}
q_0 &= \frac{8\alpha^6 - 8\alpha\beta^5 + 2\alpha^2\beta^4 + 8\alpha^3\beta^3 - 2\alpha^4\beta^2 - 6\alpha^5\beta + 2\beta^6}{\alpha^2(\alpha-\beta)^4},\\
q_{-1} &= \frac{-8\alpha^6\beta + 8\alpha\beta^6 - 2\alpha^2\beta^5 - 8\alpha^3\beta^4 + 2\alpha^4\beta^3 + 6\alpha^5\beta^2 - 2\beta^7}{\alpha^4(\alpha-\beta)^4},\\
q_{-2} &= \\
&= \frac{-4\alpha^8 - 4\alpha^6\beta^2 + 17\alpha^3\beta^5 - 26\alpha^5\beta^3 + 12\alpha^7\beta + 10\alpha^4\beta^4 - 4\alpha^2\beta^6 - 6\alpha\beta^7 + 3\beta^8}{\alpha^6(\alpha-\beta)^4}
\end{aligned}
$$

Fundamental metric tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$ and its reciprocal tensor g^{ij} for $L = L(\alpha,\beta)$ are given by [7]

$$g_{ij} = pa_{ij} + p_0 b_i b_j + p_{-1}(b_i Y_j + b_j Y_i) + p_{-2} Y_i Y_j \tag{5}$$

where

$$
\begin{aligned}
p_0 &= q_0 + L_\beta^2, \\
p_{-1} &= q_{-1} + L^{-1} p L_\beta \\
p_{-2} &= q_{-2} + p^2 L^{-2}
\end{aligned}
\tag{6}
$$

The reciprocal tensor g^{ij} of g_{ij} is given by

$$g^{ij} = p^{-1}a^{ij} - s_0 b^i b^j - s_{-1}(b^i y^j + b^j y^i) - s_{-2} y^i y^j \tag{7}$$

where $b^i = a^{ij} b_j \quad and \quad b^2 = a_{ij} b^i b^j$

$$
\begin{aligned}
s_0 &= \frac{1}{\tau p}\{pp_0 + (p_0 p_{-2} - p_{-1}^2)\alpha^2\},\\
s_{-1} &= \frac{1}{\tau p}\{pp_{-1} + (p_0 p_{-2} - p_{-1}^2)\beta\},\\
s_{-2} &= \frac{1}{\tau p}\{pp_{-2} + (p_0 p_{-2} - p_{-1}^2)b^2\},\\
\tau &= p(p + p_0 b^2 + p_{-1}\beta) + (p_0 p_{-2} - p_{-1}^2)(\alpha^2 b^2 - \beta^2)
\end{aligned}
\tag{8}
$$

The hv-torsion tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ is given by [11]

$$2pC_{ijk} = p_{-1}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k \tag{9}$$

where,

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_{-1}q_0, \qquad m_i = b_i - \alpha^{-2}\beta Y_i \tag{10}$$

Here m_i is a non-vanishing covariant vector orthogonal to the element of support y^i.

Let $\{^{\,i}_{jk}\}$ be the component of christoffel symbols of the associated Riemannian space R^n and $\bigtriangledown_k$ be the covariant derivative with respect to x^k relative to this christoffel symbol. Now we define,

$$2E_{ij} = b_{ij} + b_{ji}, \qquad 2F_{ij} = b_{ij} - b_{ji} \tag{11}$$

where $b_{ij} = \bigtriangledown_j b_i$.

Let $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, \Gamma^{i}_{jk})$ be the Cartan connection of F^n. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^{\ i}_{jk}\}$ of the special Finsler space F^n is given by

$$\begin{aligned} D^i_{jk} = B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} \\ -C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda^s (C^i_{jm} C^m_{sk} + \\ C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}) \end{aligned} \tag{12}$$

where

$$\begin{aligned} B_k = p_0 b_k + p_{-1} Y_k, \quad B^i = g^{ij} B_j, \quad F^k_i = g^{kj} F_{ji} \\ B_{ij} = \frac{\{p_{-1}(a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j\}}{2}, \quad B^k_i = g^{kj} B_{ji} \\ A^m_k = B^m_k E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k \\ \lambda^m = B^m E_{00} + 2 B_0 F^m_0, \quad B_0 = B_i y^i \end{aligned} \tag{13}$$

where $'0'$ denotes contraction with y^i except for the quantities p_0, q_0 and s_o.

3 Induced Cartan Connection

Let F^{n-1} be a hypersurface of F^n given by the equation $x^i = x^i(u^\alpha)$ where $\{\alpha = 1, 2, 3...(n-1)\}$. The element of support y^i of F^n is to be taken tangential to F^{n-1}, that is [8],

$$y^i = B^i_\alpha(u) v^\alpha \tag{14}$$

the metric tensor $g_{\alpha\beta}$ and hv-tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by

$$g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta, \qquad C_{\alpha\beta\gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma$$

and at each point (u^α) of F^{n-1}, a unit normal vector $N^i(u, v)$ is defined by

$$g_{ij}\{x(u,v), y(u,v)\} B^i_\alpha N^j = 0, \qquad g_{ij}\{x(u,v), y(u,v)\} N^i N^j = 1$$

Angular metric tensor $h_{\alpha\beta}$ of the hypersurface are given by

$$h_{\alpha\beta} = h_{ij} B^i_\alpha B^j_\beta, \qquad h_{ij} B^i_\alpha N^j = 0, \qquad h_{ij} N^i N^j = 1 \tag{15}$$

(B^α_i, N_i) inverse of (B^i_α, N^i) is given by

$$B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta, \qquad B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \qquad B^\alpha_i N^i = 0, \qquad B^i_\alpha N_i = 0$$

$$N_i = g_{ij} N^j, \quad B^k_i = g^{kj} B_{ji}, \qquad B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j$$

The induced connection $IC\Gamma = (\Gamma^{*\alpha}_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ of F^{n-1} from the Cartan's connection $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^{*i}_{jk})$ is given by [8].

$$\Gamma^{*\alpha}_{\beta\gamma} = B^{\alpha}_{i}(B^{i}_{\beta\gamma} + \Gamma^{*i}_{jk} B^{j}_{\beta} B^{k}_{\gamma}) + M^{\alpha}_{\beta} H_{\gamma}$$

$$G^{\alpha}_{\beta} = B^{\alpha}_{i}(B^{i}_{0\beta} + \Gamma^{*i}_{0j} B^{j}_{\beta}), \quad C^{\alpha}_{\beta\gamma} = B^{\alpha}_{i} C^{i}_{jk} B^{j}_{\beta} B^{k}_{\gamma}$$

where

$$M_{\beta\gamma} = N_i C^{i}_{jk} B^{j}_{\beta} B^{k}_{\gamma}, \quad M^{\alpha}_{\beta} = g^{\alpha\gamma} M_{\beta\gamma}, \quad H_{\beta} = N_i(B^{i}_{0\beta} + \Gamma^{*i}_{oj} B^{j}_{\beta})$$

and

$$B^{i}_{\beta\gamma} = \frac{\partial B^{i}_{\beta}}{\partial u^{\gamma}}, \qquad B^{i}_{0\beta} = B^{i}_{\alpha\beta} v^{\alpha}$$

The quantities $M_{\beta\gamma}$ and H_{β} are called the second fundamental v-tensor and normal curvature vector respectively [8]. The second fundamental h-tensor $H_{\beta\gamma}$ is defined as [8]

$$H_{\beta\gamma} = N_i(B^{i}_{\beta\gamma} + \Gamma^{*i}_{jk} B^{j}_{\beta} B^{k}_{\gamma}) + M_{\beta} H_{\gamma} \tag{16}$$

where

$$M_{\beta} = N_i C^{i}_{jk} B^{j}_{\beta} N^{k} \tag{17}$$

The relative h and v-covariant derivatives of projection factor B^{i}_{α} with respect to $IC\Gamma$ are given by

$$B^{i}_{\alpha|\beta} = H_{\alpha\beta} N^{i}, \qquad B^{i}_{\alpha}|_{\beta} = M_{\alpha\beta} N^{i}$$

It is obvious form equation (15) that $H_{\beta\gamma}$ is generally not symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_{\beta} H_{\gamma} - M_{\gamma} H_{\beta} \tag{18}$$

The above equation yields

$$H_{0\gamma} = H_{\gamma}, \quad H_{\gamma 0} = H_{\gamma} + M_{\gamma} H_{0} \tag{19}$$

We shall use the following lemmas which are due to Matsumoto [8] in the coming section

Lemma 1. *The normal curvature $H_0 = H_{\beta} v^{\beta}$ vanishes if and only if the normal curvature vector H_{β} vanishes.*

Lemma 2. *A hypersurface $F^{(n-1)}$ is a hyperplane of the first kind with respect to connection $C\Gamma$ if and only if $H_{\alpha} = 0$.*

Lemma 3. *A hypersurface $F^{(n-1)}$ is a hyperplane of the second kind with respect to connection $C\Gamma$ if and only if $H_{\alpha} = 0$ and $H_{\alpha\beta} = 0$.*

Lemma 4. *A hypersurface $F^{(n-1)}$ is a hyperplane of the third kind with respect to connection $C\Gamma$ if and only if $H_{\alpha} = 0$ and $H_{\alpha\beta} = M_{\alpha\beta} = 0$.*

4 Hypersurface $F^{(n-1)}(c)$ of a deformed Berwald-Matsumoto Finsler space

Let us consider a Finsler space with the deformed Berwald-Matsumoto metric $L(\alpha, \beta) = \frac{(\alpha+\beta)^2}{\alpha} + \frac{\alpha^2}{(\alpha-\beta)}$, where, $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and vector field $b_i(x) = \frac{\partial b}{\partial x^i}$ is a gradient of some scalar function $b(x)$. Now we consider a hypersurface $F^{(n-1)}(c)$ given by equation $b(x) = c$, a constant [11].

From the parametric equation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we get

$$\frac{\partial b(x)}{\partial u^\alpha} = 0$$

$$\frac{\partial b(x)}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha} = 0$$

$$b_i B^i_\alpha = 0$$

Above it is shown that $b_i(x)$ are covarient components of a normal vector field of hypersurface $F^{n-1}(c)$. Further, we have

$$b_i B^i_\alpha = 0 \quad and \quad b_i y^i = 0 \quad i.e \quad \beta = 0 \tag{20}$$

and induced matric $L(u, v)$ of $F^{n-1}(c)$ is given by

$$L(u, v) = a_{\alpha\beta} v^\alpha v^\beta, a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta \tag{21}$$

which is a Riemannian metric.

Writing $\beta = 0$ in equations (3), (4) and (6) we get

$$\begin{gathered} p = 4, \quad q_0 = 8, \quad q_{-1} = 0 \quad q_{-2} = -4\alpha^{-2} \\ p_0 = 17 \quad p_{-1} = 6\alpha^{-1} \quad p_{-2} = 0 \quad \tau = 16(1+2b^2), \\ s_0 = \frac{1}{2(1+2b^2)} \quad s_{-1} = \frac{3}{8\alpha(1+2b^2)} \quad s_{-2} = \frac{-9b^2}{16\alpha^2(1+2b^2)} \end{gathered} \tag{22}$$

from (5) we get,

$$g^{ij} = \frac{1}{4}a^{ij} - \frac{1}{2(1+2b^2)}b^i b^j - \frac{3}{8\alpha(1+2b^2)}(b^i y^j + b^j y^i) + \frac{9b^2}{16\alpha^2(1+b^2)}y^i y^j \tag{23}$$

thus along $F^{n-1}(c)$, (22) and (19) lead to

$$g^{ij} b_i b_j = \frac{b^2}{4(1+2b^2)}$$

So we get

$$b_i(x(u)) = \sqrt{\frac{b^2}{4(1+2b^2)}} N_i, \qquad b^2 = a^{ij} b_i b_j \tag{24}$$

where b is the length of vector b^i

Again from (22) and (23), we get

$$b^i = a^{ij}b_j = \sqrt{4b^2(1+2b^2)}N^i + \frac{3b^2y^i}{2\alpha} \tag{25}$$

thus we have,

Theorem 1. *In a deformed Berwald-Matsumoto Finsler hypersurface $F^{(n-1)}(c)$, the Induced Riemannian metric is given by (20) and the scalar function b(x)is given by (23) and (24).*

Now the angular metric tensor h_{ij} and metric tensor g_{ij} of F^n are given by

$$h_{ij} = 4a_{ij} + 8b_ib_j - \frac{4}{\alpha^2}Y_iY_j \quad and \quad g_{ij} = 4a_{ij} + 17b_ib_j + \frac{6}{\alpha}(b_iY_j + b_jY_i) \tag{26}$$

From equation (19), (25) and (14) it follows that if $h^{(a)}_{\alpha\beta}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$, then we have along $F^{n-1}_{(c)}$, $h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$.

thus along $F^{n-1}_{(c)}$, $\quad \frac{\partial p_0}{\partial \beta} = \frac{26}{\alpha}$

from equation (9) we get

$$r_1 = \frac{-10}{\alpha}, \qquad m_i = b_i$$

then hv-torsion tensor becomes

$$C_{ijk} = \frac{3}{4\alpha}(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j) - \frac{5}{4\alpha}b_ib_jb_k \tag{27}$$

in the deformed Berwald-Matsumoto Finsler hypersurface $F^{(n-1)}_{(c)}$. Due to facts from (14), (15), (17), (19) and (26) we have

$$M_{\alpha\beta} = \frac{3}{4\alpha}\sqrt{\frac{b^2}{4(1+2b^2)}}h_{\alpha\beta} \quad and \quad M_\alpha = 0 \tag{28}$$

Therefore, from equation (18) it follows that $H_{\alpha\beta}$ is symmetric. Thus we have

Theorem 2. *The second fundamental v-tensor of the deformed Berwald-Matsumoto Finsler hypersurface $F^{(n-1)}_{(c)}$ is given by (27) and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.*

Now from (19) we have $b_iB^i_\alpha = 0$. Then we have

$$b_{i|\beta}B^i_\alpha + b_iB^i_{\alpha|\beta} = 0$$

Therefore, from (17) and using $b_{i|\beta} = b_{i|j}B^j_\beta + b_i|_j N^j H_\beta$, we have

$$b_{i|j}B^i_\alpha B^j_\beta + b_{i|j}B^i_\alpha N^j H_\beta + b_i H_{\alpha\beta} N^i = 0 \tag{29}$$

since $b_i|_j = -b_h C^h_{ij}$, we get

$$b_{i|j}B^i_\alpha N^j = 0$$

Therefore from equation (28) we have,

$$\sqrt{\frac{b^2}{4(1+2b^2)}}H_{\alpha\beta} + b_{i|j}B^i_\alpha B^j_\beta = 0 \tag{30}$$

because $b_{i|j}$ is symmetric. Now contracting (29) with v^β and using (13) we get

$$\sqrt{\frac{b^2}{4(1+2b^2)}}H_\alpha + b_{i|j}B^i_\alpha y^j = 0 \tag{31}$$

Again contracting by v^α equation (4.12) and using (13), we have

$$\sqrt{\frac{b^2}{4(1+2b^2)}}H_0 + b_{i|j}y^i y^j = 0 \tag{32}$$

From lemma (1) and (2), it is clear that the deformed Berwald-Matsumoto hypersuface $F^{(n-1)}_{(c)}$ is a hyperplane of first kind if and only if $H_0 = 0$. Thus from (31) it is obvious that $F^{n-1}_{(c)}$ is a hyperplane of first kind if and only if $b_{i|j}y^i y^j = 0$. This $b_{i|j}$ being the covariant derivative with respect to $C\Gamma$ of F^n defined on y^i, but $b_{ij} = \bigtriangledown_j b_i$ is the covariant derivative with respect to Riemannian connection $\{^{\,i}_{jk}\}$ constructed from $a_{ij}(x)$. Hence b_{ij} does not depend on y^i. We shall consider the difference $b_{i|j} - b_{ij}$ where $b_{ij} = \bigtriangledown_j b_i$ in the following. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^{\,i}_{jk}\}$ is given by (11). Since b_i is a gradient vector, then from (10) we have

$$E_{ij} = b_{ij} \qquad F_{ij} = 0 \qquad and \qquad F^i_j = 0$$

Thus (11) reduces to

$$\begin{aligned} D^i_{jk} = B^i b_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m}g^{im}B_{jk} - C^i_{jm}A^m_k - \\ C^i_{km}A^m_j + C_{jkm}A^m_s g^{is} + \lambda^s(C^i_{jm}C^m_{sk} + \\ C^i_{km}C^m_{sj} - C^m_{jk}C^i_{ms}) \end{aligned} \tag{33}$$

where

$$B_i = 17b_i + 6\alpha^{-1}Y_i, \quad B^i = (\frac{2}{1+2b^2})b^i + \frac{3}{2\alpha(1+2b^2)}y^i, \tag{34}$$

$$\lambda^m = B^m b_{00}, \quad B_{ij} = \frac{3}{\alpha}(a_{ij} - \frac{Y_iY_j}{\alpha^2}) + \frac{13}{\alpha}b_ib_j,$$

$$B^i_j = \frac{3}{4\alpha}(\delta^i_j - \alpha^{-2}y^iY_j) + \frac{7}{4\alpha(1+2b^2)}b^ib_j - \frac{(9+39b^2)}{8\alpha^2(1+2b^2)}b_jy^i,$$

$$A^m_k = B^m_k b_{00} + B^m b_{k0}$$

In view of (21) and (22), the relation in (12) becomes equation (33). Further we have $B^i_0 = 0,\ B_{i0} = 0$ which leads $A^m_0 = B^m b_{00}$.

Now contracting (32) by y^k we get

$$D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - B^m C^i_{jm} b_{00}$$

Again contracting the above equation with respect to y^j and we have

$$D^i_{00} = B^i b_{00} = \{(\tfrac{2}{1+2b^2})b^i + \tfrac{3}{2\alpha(1+2b^2)}y^i\}b_{00}$$

Paying attention to (19), along $F^{(n-1)}_{(c)}$, we get

$$b_i D^i_{j0} = \frac{2b^2}{(1+2b^2)}b_{j0} + \frac{(3+13b^2)}{4\alpha(1+2b^2)}b_j b_{00} + \frac{2}{(1+2b^2)}b_i b^m C^i_{jm} b_{00} \tag{35}$$

Now we contract (34) by y^j we have

$$b_i D^i_{00} = \frac{2b^2}{(1+2b^2)}b_{00} \tag{36}$$

From (15), (23), (24), (27) and $M_\alpha = 0$, we have

$$b_i b^m C^i_{jm} B^j_\alpha = b^2 M_\alpha = 0$$

Thus the relation $b_{i|j} = b_{ij} - b_r D^r_{ij}$ equations (34) and (35) give

$$b_{i|j} y^i y^j = b_{00} - b_r D^r_{00} = \tfrac{1}{1+2b^2} b_{00}$$

Consequently (30) and (31) may be written as

$$\sqrt{\frac{b^2}{4(1+2b^2)}}H_\alpha + \frac{1}{1+2b^2}b_{i0}B^i_\alpha = 0, \tag{37}$$
$$\sqrt{\frac{b^2}{4(1+2b^2)}}H_0 + \frac{1}{1+2b^2}b_{00} = 0$$

Thus condition $H_0 = 0$ is equivalent to $b_{00} = 0$. Using the fact that $\beta = b_i y^i = 0$ the condition $b_{00} = 0$ can be written as $b_{ij}y^i y^j = b_i y^i b_j y^j$ for some $c_j(x)$. Thus we can write,

$$2b_{ij} = b_i c_j + b_j c_i \tag{38}$$

Now from (19) and (37) we get

$$b_{00} = 0, \quad b_{ij}B^i_\alpha B^j_\beta = 0, \quad b_{ij}B^i_\alpha y^j = 0$$

Hence from (36) we get $H_\alpha = 0$, again from (37) and (33) we get $b_{i0}b^i = \frac{c_0 b^2}{2},\ \lambda^m = 0,\ A^i_j B^j_\beta = 0$ and $B_{ij}B^i_\alpha B^j_\beta = \frac{3}{\alpha}h_{\alpha\beta}$.

Now we use equations (15), (22), (23), (24), (27) and (32) then we have

$$b_r D^r_{ij} B^i_\alpha B^j_\beta = -\frac{3c_0 b^2}{8\alpha(1+2b^2)} h_{\alpha\beta} \tag{39}$$

Thus equation (29) reduces to

$$\sqrt{\frac{b^2}{4(1+2b^2)}} H_{\alpha\beta} + \frac{3b^2}{8\alpha(1+2b^2)} h_{\alpha\beta} = 0 \tag{40}$$

Hence the hypersurface $F^{n-1}_{(c)}$ is umbilic.

Theorem 3. *The necessary and sufficient condition for a deformed Berwald-Matsumoto hypersurface $F^{(n-1)}_{(c)}$ to be a hyperplane of first kind is (37). In this case the second fundamental tensor of $F^{n-1}_{(c)}$ is proportional to its angular metric tensor.*

Now from lemma (3), $F^{(n-1)}_{(c)}$ is a hyperplane of second kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$. Thus from (38), we get

$$c_0 = c_i(x)y^i = 0$$

Therefore there exists a function $\psi(x)$ such that

$$c_i(x) = \psi(x)b_i(x)$$

Therefore from (37) we get

$$2b_{ij} = b_i(x)\psi(x)b_j(x) + b_j(x)\psi(x)b_i(x)$$

This can also be written as

$$b_{ij} = \psi(x)b_i b_j$$

Theorem 4. *The necessary and sufficient condition for a deformed Berwald-Matsumoto hypersurface $F^{(n-1)}_{(c)}$ to be a hyperplane of second kind is (39).*

Again lemma (4) and with equation (27) and $M_\alpha = 0$ show that $F^{n-1}_{(c)}$ does not become a hyperplane of third kind.

Theorem 5. *The deformed Berwald-Matsumoto hypersurface $F^{(n-1)}_{(c)}$ is not a hyperplane of the third kind.*

References

[1] **Aikou, T., Hashiguchi, M. and Yamauchi, K.:** *On Matsumoto's Finsler space with time measure*, Rep. Fac. Sci., Kagoshima Univ., (Math., Phys. and Chem.) **23**, (1990), 1-12.

[2] **Berwald, L.:** *$\ddot{U}$ber die n-dimensionalen Geometrien konstanter Kr$\ddot{u}$mmung*, in denen die Geraden die k$\ddot{u}$rzesten sind. Math. Z. **30**, (1929), 449-469.

[3] **Chaubey, V. K. and Mishra, A.:** *Hypersurface of a Finsler Space with Special (α, β)-metric*, Journal of Contemporary Mathematical Analysis **52** (2017), 1-7.

[4] **Chaubey, V. K. and Tripathi, B. K..:** *Finslerian Hypersurface of a Finsler Spaces with Special (γ, β)-metric*, Journal of Dynamical System and Geometric Theories **12** (2014), no. 1, 19-27.

[5] **Kitayama, M.:** *On Finslerian hypersurfaces given by $\beta-$ change*, Balkan Journal of Geometry and Its Applications **7** (2002), no. 2, 49-55.

[6] **Lee, I. Y., Park, H. Y. and Lee, Y. D.:** *On a hypersurface of a special Finsler spaces with a metric $(\alpha + \frac{\beta^2}{\alpha})$*, Korean J. Math. Sciences **8** (2001), 93-101.

[7] **Matsumoto, M.:** *Theory of Finsler spaces with (α, β)-metric*, Rep. on Math, Phys. **31** (1992), 43-83.

[8] **Matsumoto, M.:** *The induced and intrinsic Finsler connections of a hypersurface and Finslerian projective geometry*, J. Math. Kyoto Univ. **25** (1985), 107-144.

[9] **Matsumoto, M.:** *A slope of Mountain is a Finsler surface with respect to time measure*, J. Math. Kyoto Univ. **29** (1989), no. 1, 17–25.

[10] **Pandey, T. N. and Tripathi, B. K. :** *On a hypersurface of a Finsler Space with Special (α, β)-Metric*, Tensor, N. S., **68** (2007), 158-166.

[11] **Singh, U. P. and Kumari, Bindu:** *On a hypersurface of a Matsumoto space*, Indian J. pure appl. Math. **32** (2001), 521-531.

[12] **Shen, Z. M. and Yu, C. T.:** *On Einstein square metrics*, Publ. Math. Debr. **85** (2014), no. 3-4, 413–424.