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ON THE QUATERNIONIC W-CURVES IN THE SEMI-EUCLIAN SPACES

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Abstract

In this paper, the position vector of the semi-real spatial quaternionic W-curve in E_1^3 and the position vector of the semi-real quaternionic W-curve in E_2^4 are given and we obtain some characterizations for the semi-real spatial quaternionic W-curve in E_1^3 and the semi-real quaternionic W-curve in E_2^4 by using position vectors. Also, we characterize unit semi-real quaternionic curves with respect to second curvature $k(s)$ and third curvature $(r - \varepsilon_t \varepsilon_T \varepsilon_{N_1} K)(s)$.

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1 Introduction

The quaternion was defined by Hamilton in 1843. His initial attempt was to generalize the complex numbers by introducing a three-dimensional object failed in the sense that the algebra he constructed for these three dimensional object did not have the desired properties. Hamilton discovered that the appropriate generalization is one in which the scalar (real) axis is left unchanged whereas the vector (imaginary) axis is supplemented by adding two further vector axes. There are different types of quaternions, namely: real, complex dual quaternions. A real quaternion is defined as $q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$ is composed of four units $\{1, e_1, e_2, e_3\}$ where e_1, e_2, e_3 are orthogonal unit spatial vectors, q_i $(i = 0, 1, 2, 3)$ are real numbers and this quaternion can be written as a linear combination of a real part (scalar) and vectorial part (a spatial vector) [4, 8, 17].

In 1985, the Serret-Frenet formulas for a quaternionic curve in Euclidean spaces E^3 and E^4 are given by Bharathi and Nagaraj [1]. By using these formulas Karadağ and Sivridağ gave some characterizations for quaternionic inclined curves in terms of the harmonic curvatures in Euclidean spaces E^3 and E^4 [9] . The Serret-Frenet formulae for the quaternionic curves in the semi-Euclidean space E_2^4 are given in [4]. The semi-real quaternionic B_2 slant helices in four dimensional space E_2^4 are studied in [8].

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A curve α is called W-curve (or a helix), if it has constant Frenet curvatures. W-curves in the Euclidean space E^n have been studied intensively. The simplest examples are circles as planar W-curves and helices as non-planar W-curves in E^3.

All W-curves in the Minkowski 3-space are completely classified by Walrave in [16]. For example, the only planar spacelike W-curves are circles and hyperbolas. All spacelike W-curves in the Minkowski space-time E_1^4 are studied in [13]. The examples of null W-curves in the Minkowski space-time are given in [2]. Timelike W-curves in the same space have been studied in [15]. The position vectors of a spacelike W-curve (or a helix), i.e., curve with constant curvatures, with spacelike, timelike and null principal normal in the Minkowski 3-space E_1^3 are given in [6]. The position vectors of a timelike and a null helix in Minkowski 3-space are studied in [7].

In this paper, we obtain position vectors of the spatial semi-real quaternionic W-curves in E_1^3 and by using position vectors we give some characterizations for semi-real spatial quaternionic W-curves whose image lies on the semi-real spatial quaternionic sphere S_1^2, semi-real spatial quaternionic hyperbolical space H_0^2 in E_1^3. Then we obtain the position vector of the semi-real quaternionic W-curve in E_2^4 and by using the position vector we give some characterizations for semi-real quaternionic W-curve whose image lies on the semi-quaternionic sphere S_2^3, semi-quaternionic hyperbolical space H_1^3 and semi-real quaternionic null cone $C(m)$ in E_2^4. Also, we characterize unit semi-real quaternionic curves with respect to second curvature $k(s)$ and third curvature $(r - \varepsilon_t\varepsilon_T\varepsilon_{N_1}K)(s)$.

2 Preliminaries

To meet the requirements in the next sections, the basic elements of the theory of quaternions in the Euclidean space are briefly presented in this section. A more complete elementary treatment can be found in [17].

A real quaternion is defined with $q = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3} + d\overrightarrow{e_4}$ or ($q = S_q + \overrightarrow{V_q}$ where the symbols $S_q = d$ and $\overrightarrow{V_q} = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3}$ denote scalar and vector part of q) such that

$$\begin{array}{rrcl} i) & \overrightarrow{e_i} \times \overrightarrow{e_i} & = & -\varepsilon(\overrightarrow{e_i}), \quad 1 \leq i \leq 3, \quad \varepsilon(\overrightarrow{e_i}) = \pm 1 \\ ii) & \overrightarrow{e_i} \times \overrightarrow{e_j} & = & \varepsilon(\overrightarrow{e_i})\varepsilon(\overrightarrow{e_j})\overrightarrow{e_k}, \quad \text{in } R_1^3 \\ & \overrightarrow{e_i} \times \overrightarrow{e_j} & = & -\varepsilon(\overrightarrow{e_i})\varepsilon(\overrightarrow{e_j})\overrightarrow{e_k}, \quad \text{in } R_2^4 \end{array}$$

where $\varepsilon(\overrightarrow{e_i}) = h_\nu(\overrightarrow{e_i}, \overrightarrow{e_i})$ and (ijk) is an even permutation of (123) [4]. Notice here that we denote the set of all semi-real quaternions by Q_ν

$$Q_\nu = \{q \mid q = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3} + d\overrightarrow{e_4}; \quad a, b, c, d \in R, \quad \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3} \in R_1^3\}$$

where ν is an index $\nu = 1, 2$. If $\overrightarrow{e_i}$ is spacelike or timelike vector, then $\varepsilon(\overrightarrow{e_i}) = +1$ or -1 respectively.

Using these basic products we can now expand the product of two quaternions (assuming for the moment that the product is distributive with respect to addition):

$$p \times q = S_p S_q + \left\langle \overrightarrow{V_p}, \overrightarrow{V_q} \right\rangle + S_p \overrightarrow{V_q} + S_q \overrightarrow{V_p} + \overrightarrow{V_p} \wedge \overrightarrow{V_q}, \quad \forall p, q \in \mathbb{Q}_v,$$

where we have used the scalar and cross products in semi-Euclidean space E_1^3 [4]. The conjugate of the semi-quaternion q is denoted by q and defined $\alpha q = S_q - \overrightarrow{V_q}$. This defines the symmetric real-valued, non-degenerate, bilinear form h_ν as follows:

$$\begin{aligned} h_\nu \quad &: \quad Q_\nu \times Q_\nu \to R, \quad \nu = 1,2; \\ h_1(p,q) \quad &= \quad \frac{1}{2}\left[\varepsilon(p)\varepsilon(q)(p \times \alpha q) + \varepsilon(q)\varepsilon(p)(q \times \alpha p)\right] \quad \text{for every } p,q \in R_1^3, \\ h_2(p,q) \quad &= \quad \frac{1}{2}\left[-\varepsilon(p)\varepsilon(q)(p \times \alpha q) - \varepsilon(q)\varepsilon(p)(q \times \alpha p)\right] \quad \text{for every } p,q \in R_2^4. \end{aligned}$$

Hence it is called semi-real quaternion inner product [4]. The norm of a semi-real quaternion $q = (q_1, q_2, q_3, q_4) \in \mathbb{Q}_v$ is

$$\|q\|^2 = |h_\nu(q,q)| = |\varepsilon(q)(q \times \alpha q)| = \left|-q_1^2 - q_2^2 + q_3^2 + q_4^2\right|.$$

If $h_\nu = (p,q) = 0$ then semi-real quaternions p and q are called h-orthogonal. If $\|q\| = 1$ then q is called a semi-real unit quaternion [12]. q is called a semi-real spatial quaternion whenever $q + \alpha q = 0$ [1]. Moreover, the quaternionic product of two semi-real spatial quaternions is $p \times q = \langle p, q\rangle + p \wedge_L q$. q is a semi-real temporal quaternion whenever $q - \alpha q = 0$. Any general q can be written as $q = \frac{1}{2}(q + \alpha q) + \frac{1}{2}(q - \alpha q)$ [4].

It is known that the groups of unit real quaternions and unitary matrices $SU(2)$ are isomorphic. Thus, spherical concepts in S^3 such as meridians of longitude and parallels of latitude are explained with assistance elements of $SU(2)$. Furthermore, the element of $SO(3)$ can match with each element of S^3 [11].

The four-dimensional Euclidean space E^4 is identified with the space of unit quaternions. A semi-real quaternionic sphere with origin m and radius $R > 0$ in E_2^4 is

$$S_2^3(m,R) = \{p \in Q_\nu : h(p-m, p-m) = R^2\}$$

and the semi-real quaternionic hyperbolical space is defined by

$$H_1^3(m,R) = \{p \in Q_\nu : h(p-m, p-m) = -R^2\}.$$

The semi-real quaternionic null cone with the vertex at a point m in E_2^4 is defined by

$$C(m) = \{p \in Q_\nu : h(p-m, p-m) = 0\}.$$

Theorem 1. *The three-dimensional semi-Euclidean space E_1^3 is identified with the space of spatial quaternions $\{p \in \mathbb{Q}_p \mid p + \gamma p = 0\}$ in an obvious manner. Let $I = [0,1]$ denote the unit interval of the real line $\mathbb{R}$ and*

$$\alpha : I \to \mathbb{Q}_p, \quad s \to \alpha(s) = \sum_{i=1}^{3} \alpha_i(s) e_i, \quad 1 \le i \le 3,$$

be the parameter along the smooth curve. Let the parameter s be chosen such that the tangent $t = \alpha'(s) = \sum\limits_{i=1}^{3} \alpha_i'(s) e_i$ has unit magnitude and $\{t, n_1, n_2\}$ be the Frenet *trihedron of the differentiable semi-Euclidean space curve in the semi-Euclidean space E_1^3. Then the Frenet equations are*

$$\begin{array}{rcl} t' & = & \varepsilon_{n_1} k n_1 \\ n_1' & = & -\varepsilon_t k t + \varepsilon_{n_1} r n_2 \\ n_2' & = & -\varepsilon_{n_2} r n_1 \end{array} \tag{1}$$

where k is the principal curvature, r is torsion of α and $h(t,t) = \varepsilon_t,\ h(n_1,n_1) = \varepsilon_{n_1}$ $h(n_2,n_2) = \varepsilon_{n_2}$ [4].

Theorem 2. *The 4-dimensional semi-Euclidean space R_2^4 is identified with the space of unit semi-quaternions which is denoted by $\mathbb{Q}_v$. Let*

$$\beta : I \subset R \to \mathbb{Q}_v$$
$$s \to \beta(s) = \sum_{i=1}^{4} \beta_i(s)\overrightarrow{e_i}, \quad (1 \leq i \leq 4), \quad \overrightarrow{e_4} = 1$$

be a smooth curve defined over the interval I. Let the parameter s be chosen such that the tangent $T = \beta'(s) = \sum\limits_{i=1}^{4} \beta_i'(s)\overrightarrow{e_i}$ has unit magnitude. Let $\{T, N_1, N_2, N_3\}$ be the Serret-Frenet frame in the point $\beta(s)$ of the semi-real quaternionic curve β and s be the arc-length parameter of the semi-real quaternionic curve β. Then the Frenet equations are

$$\begin{array}{rcl} T' & = & \varepsilon_{N_1} K N_1 \\ N_1' & = & -\varepsilon_t \varepsilon_{N_1} K T + \varepsilon_{n_1} k N_2 \\ N_2' & = & -\varepsilon_t k N_1 + \varepsilon_{n_1} \left(r - \varepsilon_t \varepsilon_T \varepsilon_{N_1} K\right) N_3 \\ N_3' & = & -\varepsilon_{n_2} (r - \varepsilon_t \varepsilon_T \varepsilon_{N_1} K) N_2 \end{array} \tag{2}$$

where

$$K = \varepsilon_N \left\| T' \right\|, \ N_1 = \varepsilon_t (t \times T), \quad N_2 = \varepsilon_t (n_1 \times T), \quad N_3 = \varepsilon_t (n_2 \times T),$$
$$h(T,T) = \varepsilon_T,\ h(N_1,N_1) = \varepsilon_{N_1},\ h(N_2,N_2) = \varepsilon_T \varepsilon_{n_1},\ h(N_3,N_3) = \varepsilon_T \varepsilon_{n_2}, \quad \textit{[4]}.$$

The Serret-Frenet formulae of the semi-real quaternionic curve $\beta = \beta(s)$ is obtained by making use of the Serret-Frenet formulae of the semi-real spatial quaternionic curve $\alpha = \alpha(s)$ where α is a semi-real spatial quaternionic curve associated with the semi-real quaternionic curve β and $\{t, n_1, n_2\}$ is the Frenet frame of the semi-real spatial quaternionic curve α in R_1^3. Moreover, there are relationships between curvatures of the curves β and α. These relations can be explained because the torsion of β is the principal curvature of the curve α. Also, the bitorsion of β is $(r - \varepsilon_t \varepsilon_T \varepsilon_{N_1} K)$, where r is the torsion of α and K is the principal curvature of β. These relations are only determined for quaternions, [4].

3 Position Vectors of Semi-real Spatial Quaternionic W-curves in E_1^3

Let $\alpha = \alpha(s)$ be unit speed semi-real spatial quaternionic W-curve in E_1^3 with non-zero curvatures k and r. Then the position vector of the curve $\alpha(s)$ satisfies the equation

$$\alpha(s) = \lambda(s)t(s) + \mu(s)n_1(s) + \nu(s)n_2(s) \tag{3}$$

for some differentiable functions $\lambda(s), \mu(s), \nu(s)$. Then differentiating (3) with respect to s and using the corresponding Frenet equations (1), we obtain

$$\begin{aligned} \lambda' - \varepsilon_t k\mu - 1 &= 0 \\ \varepsilon_{n_1} k\lambda + \mu' - \varepsilon_{n_2} r\nu &= 0 \\ \varepsilon_{n_1} r\mu + \nu' &= 0 \end{aligned} \tag{4}$$

From these equations in (4) we get

$$\mu'' + \varepsilon_{n_1}(\varepsilon_{n_2} r^2 + \varepsilon_t k^2)\mu + \varepsilon_{n_1} k = 0 \tag{5}$$

Here we have distinguished the following cases:

Case 1. When $\alpha(s)$ is a semi-real spatial quaternionic W-curve with spacelike principal normal n_1. Thus $\varepsilon_{n_1} = 1$, therefore from (5) we get

$$\mu'' + (\varepsilon_{n_2} r^2 + \varepsilon_t k^2)\mu + k = 0 \tag{6}$$

In this case we have the following subcases.

Case 1.1. If $\varepsilon_{n_2} r^2 + \varepsilon_t k^2 = 0,$ (6) takes the form $\mu'' + k = 0$. Then the solution of this equation is

$$\mu(s) = -k\tfrac{s^2}{2} + c_1 s + c_2 \tag{7}$$

where $c_1, c_2 \in \mathbb{R}$. By using (7), from (4) we get

$$\begin{aligned} \lambda(s) &= -\varepsilon_t k^2 \tfrac{s^3}{6} + \varepsilon_t c_1 k \tfrac{s^2}{2} + (\varepsilon_t c_2 k + 1)s, \\ \upsilon(s) &= rk\tfrac{s^3}{6} - c_1 r\tfrac{s^2}{2} - c_2 rs. \end{aligned}$$

Thus we find the position vector as;

$$\begin{aligned} \alpha(s) &= \left[k\tfrac{s^2}{2} + (\varepsilon_t c_2 k + 1)s\right] t(s) + \left[-k\tfrac{s^2}{2} + c_1 s + c_2\right] n_1(s) \\ &\quad + \left[rk\tfrac{s^3}{6} - c_1 r\tfrac{s^2}{2} - c_2 rs\right] n_2(s) \end{aligned} \tag{8}$$

Case 1.2. If $\varepsilon_{n_2} r^2 + \varepsilon_t k^2 > 0$, then the solution of the equation (5) is

$$\mu(s) = c_1 \cos(\sqrt{\varepsilon_{n_2} r^2 + \varepsilon_t k^2}s) + c_2 \sin(\sqrt{\varepsilon_{n_2} r^2 + \varepsilon_t k^2}s) - \tfrac{k}{\varepsilon_{n_2} r^2 + \varepsilon_t k^2} \tag{9}$$

where $c_1, c_2 \in \mathbb{R}$. By using (9), from (4) we get

$$\begin{aligned} \lambda(s) &= (1 - \tfrac{\varepsilon_t k^2}{\varepsilon_{n_2} r^2 + \varepsilon_t k^2})s + \tfrac{\varepsilon_t c_1 k}{\sqrt{\varepsilon_{n_2} r^2 + \varepsilon_t k^2}} \sin(\sqrt{\varepsilon_{n_2} r^2 + \varepsilon_t k^2}s) \\ &\quad - \tfrac{\varepsilon_t c_2 k}{\sqrt{\varepsilon_{n_2} r^2 + \varepsilon_t k^2}} \cos(\sqrt{\varepsilon_{n_2} r^2 + \varepsilon_t k^2}s) \\ \upsilon(s) &= -\tfrac{c_1 r}{\sqrt{\varepsilon_{n_2} r^2 + \varepsilon_t k^2}} \sin(\sqrt{\varepsilon_{n_2} r^2 + \varepsilon_t k^2}s) \\ &\quad + \tfrac{c_2 r}{\sqrt{\varepsilon_{n_2} r^2 + \varepsilon_t k^2}} \cos(\sqrt{\varepsilon_{n_2} r^2 + \varepsilon_t k^2}s) + \tfrac{rks}{\varepsilon_{n_2} r^2 + \varepsilon_t k^2} \end{aligned}$$

Thus we find the position vector as;

$$\begin{aligned}\alpha(s) &= \Big[(1-\tfrac{\varepsilon_t k^2}{\varepsilon_{n_2}r^2+\varepsilon_t k^2})s+\tfrac{\varepsilon_t c_1 k}{\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}}\sin(\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}s)\\ &\quad -\tfrac{\varepsilon_t c_2 k}{\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}}\cos(\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}s)\Big]t(s)\\ &\quad +\Big[c_1\cos(\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}s)+c_2\sin(\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}s)-\tfrac{k}{\varepsilon_{n_2}r^2+\varepsilon_t k^2}\Big]n_1(s)\\ &\quad +\Big[-\tfrac{c_1 r}{\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}}\sin(\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}s)\\ &\quad +\tfrac{c_2 r}{\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}}\cos(\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}s)+\tfrac{rks}{\varepsilon_{n_2}r^2+\varepsilon_t k^2}\Big]n_2(s)\end{aligned} \tag{10}$$

Case 1.3. If $\varepsilon_{n_2}r^2+\varepsilon_t k^2<0$, we get

$$\mu(s)=c_1e^{-s\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}+c_2e^{s\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}-\tfrac{k}{\varepsilon_{n_2}r^2-\varepsilon_t k^2} \tag{11}$$

where $c_1, c_2\in\mathbb{R}$. By using (11), from (4) we get

$$\begin{aligned}\lambda(s) &= (1-\tfrac{\varepsilon_t k^2}{\varepsilon_{n_2}r^2+\varepsilon_t k^2})s-\tfrac{\varepsilon_t c_1 k}{\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}e^{-s\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}\\ &\quad +\tfrac{\varepsilon_t c_2 k}{\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}e^{s\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}\\ \upsilon(s) &= \tfrac{c_1 r}{\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}e^{-s\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}\\ &\quad -\tfrac{c_2 r}{\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}e^{s\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}+\tfrac{rks}{\varepsilon_{n_2}r^2+\varepsilon_t k^2}\end{aligned}$$

Thus we find the position vector as;

$$\begin{aligned}\alpha(s) &= \Big[(1-\tfrac{\varepsilon_t k^2}{\varepsilon_{n_2}r^2+\varepsilon_t k^2})s-\tfrac{\varepsilon_t c_1 k}{\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}e^{-s\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}\\ &\quad +\tfrac{\varepsilon_t c_2 k}{\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}e^{s\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}\Big]t(s)\\ &\quad +\Big[c_1e^{-s\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}+c_2e^{s\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}-\tfrac{k}{\varepsilon_{n_2}r^2-\varepsilon_t k^2}\Big]n_1(s)\\ &\quad +\Big[\tfrac{c_1 r}{\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}e^{-s\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}\\ &\quad -\tfrac{c_2 r}{\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}e^{s\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}+\tfrac{rks}{\varepsilon_{n_2}r^2+\varepsilon_t k^2}\Big]n_2(s)\end{aligned} \tag{12}$$

Case 2. When $\alpha(s)$ is a semi-real spatial quaternionic W-curve with timelike principal normal n_1. Thus $\varepsilon_{n_1}=-1$ and so $\varepsilon_{n_2}=\varepsilon_t=1$. In that case $r^2+k^2>0$ and (5) takes the form $\mu''-(r^2+k^2)\mu-k=0$. Then the solution of this equation is

$$\mu(s)=c_1e^{-s\sqrt{r^2+k^2}}+c_2e^{s\sqrt{r^2+k^2}}-\tfrac{k}{r^2+k^2} \tag{13}$$

where $c_1, c_2 \in \mathbb{R}$. By using (13), from (4) we get

$$\begin{aligned}
\lambda(s) &= (1-\tfrac{k^2}{r^2+k^2})s - \tfrac{c_1 k}{\sqrt{r^2+k^2}}e^{-s\sqrt{r^2+k^2}} + \tfrac{c_2 k}{\sqrt{r^2+k^2}}e^{s\sqrt{r^2+k^2}}, \\
\upsilon(s) &= -\tfrac{c_1 r}{\sqrt{r^2+k^2}}e^{-s\sqrt{r^2+k^2}} + \tfrac{c_2 r}{\sqrt{r^2+k^2}}e^{s\sqrt{r^2+k^2}} - \tfrac{rks}{r^2+k^2}.
\end{aligned}$$

Thus we find the position vector as;

$$\begin{aligned}
\alpha(s) &= \left[(1-\tfrac{k^2}{r^2+k^2})s - \tfrac{c_1 k}{\sqrt{r^2+k^2}}e^{-s\sqrt{r^2+k^2}} + \tfrac{c_2 k}{\sqrt{r^2+k^2}}e^{s\sqrt{r^2+k^2}}\right] t(s) \\
&+ \left[c_1 e^{-s\sqrt{r^2+k^2}} + c_2 e^{s\sqrt{r^2+k^2}} - \tfrac{k}{r^2+k^2}\right] n_1(s) \qquad (14) \\
&+ \left[-\tfrac{c_1 r}{\sqrt{r^2+k^2}}e^{-s\sqrt{r^2+k^2}} + \tfrac{c_2 r}{\sqrt{r^2+k^2}}e^{s\sqrt{r^2+k^2}} - \tfrac{rks}{r^2+k^2}.\right] n_2(s)
\end{aligned}$$

Corollary 1. *Let $\alpha = \alpha(s)$ be a unit speed semi-real spatial quaternionic W-curve in E_1^3 with spacelike principal normal n_1 and the curvatures $k(s) > 0, r(s) \neq 0$ for each $s \in IR$.*

***a)** If $\varepsilon_{n_2} r^2 + \varepsilon_t k^2 = 0,$ then the position vector of the curve is given by the equation (8).*

***b)**If $\varepsilon_{n_2} r^2 + \varepsilon_t k^2 > 0,$ then the position vector of the curve is given by the equation (10).*

***c)**If $\varepsilon_{n_2} r^2 + \varepsilon_t k^2 < 0, t$hen the position vector of the curve is given by the equation (12).*

Corollary 2. *Let $\alpha = \alpha(s)$ be a unit speed semi-real spatial quaternionic W-curve in E_1^3 with timelike principal normal n_1 and the curvatures $k(s) > 0, r(s) \neq 0$ for each $s \in IR$. Then the position vector of the curve is given by the equation (14).*

4 Semi-Real Spatial Quaternionic W-curves on S_1^2 and H_0^2 in E_1^3

In this section we give some characterizations for the semi-real spatial quaternionic W-curves whose image lies on the semi-real spatial quaternionic sphere S_1^2 and semi-real quaternionic hyperbolic space H_0^2.

Theorem 3. *Let $\alpha = \alpha(s)$ be a unit speed semi-real spatial quaternionic W-curve in E_1^3 with the spacelike principal normal n_1 and non-zero curvatures $k(s)$, $r(s)$.*

***a)** If $\varepsilon_{n_2} r^2 + \varepsilon_t k^2 = 0$, then α lies on a semi-real spatial quaternionic sphere S_1^2 if and only if for each $s \in I \subset R$ the curvatures satisfy the following equalities:*

$$\begin{aligned}
0 &= -\varepsilon_t k^2 \tfrac{s^3}{6} + \varepsilon_t c_1 k \tfrac{s^2}{2} + \varepsilon_t (c_2 k s + 1)s \qquad (15) \\
-\tfrac{\varepsilon_t}{k} &= -k\tfrac{s^2}{2} + c_1 s + c_2 \\
0 &= rk\tfrac{s^3}{6} - c_1 r \tfrac{s^2}{2} - c_2 r s
\end{aligned}$$

***b)** If $\varepsilon_{n_2}r^2+\varepsilon_t k^2>0$, then α lies on a semi-real spatial quaternionic sphere S_1^2 if and only if for each $s\in I\subset R$ the curvatures satisfy the folowing equalities:*

$$\begin{aligned}
0 &= (1-\tfrac{\varepsilon_t k^2}{\varepsilon_{n_2}r^2+\varepsilon_t k^2})s+\tfrac{\varepsilon_t c_1 k}{\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}}\sin(\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}s)\\
&\quad -\tfrac{\varepsilon_t c_2 k}{\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}}\cos(\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}s)\\
-\tfrac{\varepsilon_t}{k} &= -c_1\cos(\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}s)-c_2\sin(\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}s)+\tfrac{k}{\varepsilon_{n_2}r^2+\varepsilon_t k^2}\\
0 &= -\tfrac{c_1 r}{\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}}\sin(\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}s)\\
&\quad +\tfrac{c_2 r}{\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}}\cos(\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}s)+\tfrac{rks}{\varepsilon_{n_2}r^2+\varepsilon_t k^2}
\end{aligned}$$

***c)** If $\varepsilon_{n_2}r^2+\varepsilon_t k^2<0$, then α lies on a semi-real spatial quaternionic sphere S_1^2 if and only if for each $s\in I\subset R$ the curvatures satisfy the folowing equalities:*

$$\begin{aligned}
0 &= (1-\tfrac{\varepsilon_t k^2}{\varepsilon_{n_2}r^2+\varepsilon_t k^2})s-\tfrac{\varepsilon_t c_1 k}{\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}e^{-s\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}\\
&\quad +\tfrac{\varepsilon_t c_2 k}{\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}e^{s\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}}\\
-\tfrac{\varepsilon_t}{k} &= c_1e^{-s\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}+c_2e^{s\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}-\tfrac{k}{\varepsilon_{n_2}r^2+\varepsilon_t k^2}\\
0 &= \tfrac{c_1 r}{\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}e^{-s\sqrt{\varepsilon_{n_2}r^2+\varepsilon_t k^2}}\\
&\quad -\tfrac{c_2 r}{\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}e^{s\sqrt{-\varepsilon_{n_2}r^2-\varepsilon_t k^2}}+\tfrac{rks}{\varepsilon_{n_2}r^2+\varepsilon_t k^2}
\end{aligned}$$

Proof. a)Let us first suppose that α lies on a semi-real spatial quaternionic sphere S_1^2 with center m

$$h(\alpha-m,\alpha-m)=a^2 \tag{16}$$

for every $s\in I\subset R$. Differentiation in s gives

$$h(t,\alpha)=0 \tag{17}$$

By a new differentiation, we find that

$$h(n_1,\beta)=-\tfrac{\varepsilon_t}{k}. \tag{18}$$

Then one more differentiation in s gives

$$h(n_2,\beta)=0. \tag{19}$$

By using Eqs. 16, 17, 18 and 19 in Eq. 8 , we find equations in 15. Conversely, we assume that equations in 15 holds for each $s\in I\subset R$ then from Eq. 8 we find the position vector of the curve $\alpha=-\frac{\varepsilon_t}{k}n_1$ which satisfies the equation $h(\alpha,\alpha)=\left(\frac{1}{k}\right)^2=a^2$ which means that the curve lies in S_1^2. The proofs of (b) and (c) are analogous to the proof of (a). $\square$

Theorem 4. *Let $\alpha=\alpha(s)$ be a unit speed semi-real spatial quaternionic W-curve in E_1^3 with the timelike principal normal n_1 and non-zero curvatures $k(s)$, $r(s)$. Then α lies on a semi-real spatial quaternionic hyperbolic space H_0^2 if and only if for each $s \in I \subset R$ the curvatures satisfy the following equalities:*

$$\begin{array}{rcl} 0 & = & (1-\frac{k^2}{r^2+k^2})s-\frac{c_1k}{\sqrt{r^2+k^2}}e^{-s\sqrt{r^2+k^2}}+\frac{c_2k}{\sqrt{r^2+k^2}}e^{s\sqrt{r^2+k^2}} \\ -\frac{1}{k} & = & c_1e^{-s\sqrt{r^2+k^2}}+c_2e^{s\sqrt{r^2+k^2}}-\frac{k}{r^2+k^2} \\ 0 & = & -\frac{c_1r}{\sqrt{r^2+k^2}}e^{-s\sqrt{r^2+k^2}}+\frac{c_2r}{\sqrt{r^2+k^2}}e^{s\sqrt{r^2+k^2}}-\frac{rks}{r^2+k^2} \end{array}$$

Proof. The proof is analogous to the proof of Theorem 3. □

Corollary 3. *There is no semi-real spatial quaternionic W-curve with the spacelike principal normal n_1 and non-zero curvatures $k(s)$ and $r(s)$ whose image lies on the semi-real spatial quaternionic hyperbolic space H_0^2 in E_1^3.*

Corollary 4. *There is no semi-real spatial quaternionic W-curve with the timelike principal normal n_1 and non-zero curvatures $k(s)$ and $r(s)$ whose image lies on the semi-real spatial quaternionic sphere S_1^2 in E_1^3.*

Corollary 5. *Let $\alpha(s)$ be a unit speed semi-real spatial quaternionic W-curve in E_1^3 with spacelike or timelike principal normal n_1 and non-zero curvatures $k(s)$ and $r(s)$. If α is a semi-real spatial quaternionic spherical curve or semi-real spatial quaternionic hyperbolical curve, then the radius of S_1^2 or H_0^2 is $r=\frac{1}{k}$.*

5 Position Vector of Semi-Real Quaternionic W-curve in E_2^4

Let $\beta=\beta(s)$ be unit speed semi-real quaternionic W-curve in E_2^4 with non- zero curvatures K, k and $(r-\varepsilon_t\varepsilon_T\varepsilon_{N_1}K)$. Then the position vector of the curve $\beta(s)$ satisfies the equation

$$\beta(s)=\lambda(s)T(s)+\mu(s)N_1(s)+\gamma(s)N_2(s)+\sigma(s)N_3(s) \tag{20}$$

for some differentiable functions $\lambda(s),\mu(s),\gamma(s),\sigma(s)$. These functions are called component functions (or simply components) of the position vector.

Then differentiating (20) with respect to s and using the corresponding Frenet equations (2), we obtain

$$\begin{array}{rcl} \lambda'-\varepsilon_t\varepsilon_{N_1}K\mu-1 & = & 0, \\ \varepsilon_{N_1}K\lambda+\mu'-\varepsilon_tk\gamma & = & 0, \\ \varepsilon_{n_1}k\mu+\gamma'-\varepsilon_{n_2}[r-\varepsilon_t\varepsilon_T\varepsilon_{N_1}K]\sigma & = & 0, \\ \varepsilon_{n_1}[r-\varepsilon_t\varepsilon_T\varepsilon_{N_1}K]\gamma+\sigma' & = & 0. \end{array} \tag{21}$$

From the first three equations in (21) we get

$$\mu=\frac{\varepsilon_t\varepsilon_{N_1}(\lambda'-1)}{K} \tag{22}$$

$$\gamma = \frac{\varepsilon_t \varepsilon_{N_1}}{kK}(K^2\lambda + \varepsilon_t \lambda'') \tag{23}$$

and

$$\sigma = \frac{\varepsilon_t \varepsilon_{n_2} \varepsilon_{N_1}}{kK[r - \varepsilon_t \varepsilon_T \varepsilon_{N_1} K]} \left[\varepsilon_t \lambda''' + (\varepsilon_{n_1} k^2 + K^2)\lambda' - \varepsilon_{n_1} k^2\right]. \tag{24}$$

By using (24) in the last equation in (21) we easily obtain the differential equation

$$\lambda^{(4)} - \left[\varepsilon_{n_2} k^2 - \varepsilon_t K^2 + \varepsilon_t [r - \varepsilon_t \varepsilon_T \varepsilon_{N_1} K]^2\right] \lambda'' - K^2 [r - \varepsilon_t \varepsilon_T \varepsilon_{N_1} K]^2 \lambda = 0. \tag{25}$$

The solution of the previous equation is

$$\lambda(s) = c_1 \cos(Q_1 s) + c_2 \sin(Q_1 s) + c_3 \cosh(Q_2 s) + c_4 \sinh(Q_2 s) \tag{26}$$

where

$$Q_1^2 = \frac{1}{2}\left(C + \sqrt{C^2 + 4K^2[r - \varepsilon_t \varepsilon_T \varepsilon_{N_1} K]^2}\right),$$

$$Q_2^2 = \frac{1}{2}\left(C - \sqrt{C^2 + 4K^2[r - \varepsilon_t \varepsilon_T \varepsilon_{N_1} K]^2}\right)$$

and

$$C = \varepsilon_{n_2} k^2 - \varepsilon_t K^2 + \varepsilon_t [r - \varepsilon_t \varepsilon_T \varepsilon_{N_1} K]^2.$$

Then using (26) in (22), (23) and (24) we get

$$\begin{aligned}
\mu &= \frac{\varepsilon_t \varepsilon_{N_1}}{K}\left[Q_1(-c_1 \sin(Q_1 s) + c_2 \cos(Q_1 s))\right., \\
&\quad \left. + Q_2(c_3 \sinh(Q_2 s) + c_4 \cosh(Q_2 s)) - 1\right] \\
\gamma &= \frac{\varepsilon_t \varepsilon_{N_1}}{kK}\left[\left(K^2 - \varepsilon_t Q_1^2\right)(c_1 \cos(Q_1 s) + c_2 \sin(Q_1 s))\right. \\
&\quad \left. + \left(K^2 + \varepsilon_t Q_2^2\right)(c_3 \cosh(Q_2 s) + c_4 \sinh(Q_2 s))\right]
\end{aligned}$$

and

$$\begin{aligned}
\sigma &= A\left[-\varepsilon_{n_1} k^2 + (\varepsilon_{n_1} k^2 + K^2) Q_1 - \varepsilon_t Q_1^3\right)(-c_1 \sin(Q_1 s) + c_2 \cos(Q_1 s)) \\
&\quad + \left((\varepsilon_{n_1} k^2 + K^2) Q_2 + \varepsilon_t Q_2^3\right)(c_3 \sinh(Q_2 s) + c_4 \cosh(Q_2 s)) - \varepsilon_{n_1} k^2\right].
\end{aligned}$$

where $A = \frac{\varepsilon_t \varepsilon_{n_2} \varepsilon_{N_1}}{kK[r - \varepsilon_t \varepsilon_T \varepsilon_{N_1} K]}$. Thus we find the position vector as;

$$\begin{aligned}
\beta(s) \quad = \quad & [c_1 \cos(Q_1 s) + c_2 \sin(Q_1 s) + c_3 \cosh(Q_2 s) + c_4 \sinh(Q_2 s)]\, T(s) \\
& + \tfrac{\varepsilon_t \varepsilon_{N_1}}{K} [-c_1 Q_1 \sin(Q_1 s) + c_2 Q_1 \cos(Q_1 s) \\
& + c_3 Q_2 \sinh(Q_2 s) + c_4 Q_2 \cosh(Q_2 s) - 1]\, N_1(s) \\
& + \tfrac{\varepsilon_t \varepsilon_{N_1}}{kK} \left[\left(K^2 - \varepsilon_t Q_1^2\right) (c_1 \cos(Q_1 s) + c_2 \sin(Q_1 s)) \right. \\
& \left. + \left(K^2 + \varepsilon_t Q_2^2\right) (c_3 \cosh(Q_2 s) + c_4 \sinh(Q_2 s)) \right] N_2(s) \\
& + A \left[-\varepsilon_{n_1} k^2 + (\varepsilon_{n_1} k^2 + K^2) Q_1 - \varepsilon_t Q_1^3 \right) (-c_1 \sin(Q_1 s) + c_2 \cos(Q_1 s)) \\
& \left. + \left((\varepsilon_{n_1} k^2 + K^2) Q_2 + \varepsilon_t Q_2^3 \right) (c_3 \sinh(Q_2 s) + c_4 \cosh(Q_2 s)) \right] N_3(s).
\end{aligned} \tag{27}$$

Theorem 5. *Let $\beta = \beta(s)$ be a unit speed semi-real quaternionic W-curve in E_2^4 with non-zero curvatures $K(s), k(s)$ and $(r - \varepsilon_t \varepsilon_T \varepsilon_{N_1} K)(s)$ for each $s \in IR$. Then position vector of the curve is given by the equation (27).*

Next, the following theorems characterize unit semi-real quaternionic curves with respect to second curvature $k(s)$ and third curvature $(r - \varepsilon_t \varepsilon_T \varepsilon_{N_1} K)(s)$.

Theorem 6. *Let $\beta = \beta(s)$ be a unit speed semi-real quaternionic curve in E_2^4 with non-zero curvature $K(s)$. Then β has $k(s) = 0$ if and only if β lies fully in a 2-dimensional quaternionic hyperplane of E_2^4, spanned by $\{T, N_1\}$.*

Proof. If β has $k(s) = 0$, then by using the Frenet equations we obtain $\beta' = T$, $\beta'' = \varepsilon_{N_1} K N_1$, $\beta''' = \varepsilon_t K^2 T + \varepsilon_{N_1} K' N_1$. Next, all higher order derivates of β are combinations of vectors β' and β'', so by using the MacLaurin expansion for β given by

$$\beta(s) = \beta(0) + \dot{\beta}(0) s + \ddot{\beta}(0) \frac{s^2}{2!} + \dddot{\beta}(0) \frac{s^3}{3!} + ...,$$

we conclude that β lies fully in a quaternionic hyperplane of $\mathbb{Q}_v$, spanned by $\{T, N_1\}$.

Conversely, assume that β satisfies the assumptions of the theorem and lies fully in a quaternionic hyperplane π of $\mathbb{Q}_v$. Then there exist points $p, q \in E_2^4$, such that β satisfies the equation of π given by $h(\beta(s) - p, q) = 0$, where $q \in \pi^\perp$. Differentiating the last equation yields $h(T, q) = h(N_1, q) = 0$. Next differentiation of the equation $h(N_1, q) = 0$ gives $\varepsilon_{n_1} k h(N_2, q) = 0$. Since N_2 is the unit semi-real quaternionic vector perpendicular to $\{T, N_1\}$, it follows that $h(N_2, q) \neq 0$. Therefore $k = 0$. □

Theorem 7. *Let $\beta = \beta(s)$ be a unit speed semi-real quaternionic curve in E_2^4 with non-zero curvature K, k. Then β has $r = \varepsilon_t \varepsilon_T \varepsilon_{N_1} K$ if and only if β lies fully in a 3-dimensional quaternionic hyperplane of E_2^4, spanned by $\{T, N_1, N_2\}$.*

We omit the proof, as it is similar to the proof of Theorem 7.

6 Semi-real quaternionic W-curves on S_2^3, H_1^3 and $C(m)$ in E_2^4

In this section we give some characterizations for the semi-real quaternionic W-curve whose image lies on the semi-real quaternionic sphere S_2^3, semi-real quaternionic hyperbolic space H_1^3 and semi-real quaternionic null cone $C(m)$.

Theorem 8. *Let $\beta = \beta(s)$ be a unit speed semi-real quaternionic W-curve in E_2^4 with non-zero curvatures $K(s)$, $k(s)$, $(r - \varepsilon_t\varepsilon_T\varepsilon_{N_1}K)(s)$ and $\varepsilon_{N_1}\left(\frac{1}{K}\right)^2 > \varepsilon_T\varepsilon_{n_2}\left(\frac{k}{K(r-\varepsilon_t\varepsilon_T\varepsilon_{N_1}K)}\right)^2$. The image of the curve lies on a semi–real quaternionic sphere S_2^3 if and only for each $s \in I \subset R$ the curvatures satisfy the following equalities:*

$$
\begin{aligned}
&0 = c_1\cos(Q_1s) + c_2\sin(Q_1s) + c_3\cosh(Q_2s) + c_4\sinh(Q_2s),\\
&-\varepsilon_t\varepsilon_T\varepsilon_{N_1} = c_1Q_1\sin(Q_1s) - c_2Q_1\cos(Q_1s) - c_3Q_2\sinh(Q_2s) - c_4Q_2\cosh(Q_2s) - 1,\\
&0 = \left[\left(K^2 - \varepsilon_tQ_1^2\right)(c_1\cos(Q_1s) + c_2\sin(Q_1s))\right.\\
&\left. + \left(K^2 + \varepsilon_tQ_2^2\right)(c_3\cosh(Q_2s) + c_4\sinh(Q_2s))\right],\\
&-\varepsilon_{n_1}k^2 = \left[(\varepsilon_{n_1}k^2 + K^2)Q_1 - \varepsilon_tQ_1^3\right)(-c_1\sin(Q_1s) + c_2\cos(Q_1s))\\
&+ \left((\varepsilon_{n_1}k^2 + K^2)Q_2 + \varepsilon_tQ_2^3\right)(c_3\sinh(Q_2s) + c_4\cosh(Q_2s)) + \varepsilon_{n_1}k^2\right].
\end{aligned} \tag{28}
$$

Proof. Let us first suppose that β lies on a semi-real quaternionic sphere S_2^3 with center m

$$h(\beta - m, \beta - m) = a^2 \tag{29}$$

for every $s \in I \subset R$. Differentiation in s gives

$$h(T, \beta) = 0. \tag{30}$$

By a new differentiation, we find that

$$h(N_1, \beta) = -\tfrac{\varepsilon_T\varepsilon_{N_1}}{K} \tag{31}$$

Then one more differentiation in s gives

$$h(N_2, \beta) = 0 \tag{32}$$

and

$$h(N_3, \beta) = -\varepsilon_t\varepsilon_{n_1}\varepsilon_T\varepsilon_{N_1}\frac{k}{K(r - \varepsilon_t\varepsilon_T\varepsilon_{N_1}K)}. \tag{33}$$

By using Eqs. (30), (31), (32) and (33) in Eq. (27), we find equations in (28). Conversely, we assume that equations in (28) hold for each $s \in I \subset R$ then from Eq. (27) we find the position vector of the curve

$\beta = -\frac{\varepsilon_t\varepsilon_{N_1}}{K}N_1 - \varepsilon_t\varepsilon_{n_1}\varepsilon_T\varepsilon_{N_1}\frac{k}{K(r-\varepsilon_t\varepsilon_T\varepsilon_{N_1}K)}N_3$ which satisfies the equation $h(\beta,\beta) = \varepsilon_{N_1}\left(\frac{1}{K}\right)^2 + \varepsilon_T\varepsilon_{n_2}\left(\frac{k}{K(r-\varepsilon_t\varepsilon_T\varepsilon_{N_1}K)}\right)^2 = a^2$ which means that the curve lies in S_2^3. $\square$

Theorem 9. *Let $\beta = \beta(s)$ be a unit speed speed semi-real quaternionic W- curve in E_2^4 with non-zero curvatures $K(s)$, $k(s)$, $(r - \varepsilon_t\varepsilon_T\varepsilon_{N_1}K)(s)$ and $\varepsilon_{N_1}\left(\frac{1}{K}\right)^2 < \varepsilon_T\varepsilon_{n_2}\left(\frac{k}{K(r-\varepsilon_t\varepsilon_T\varepsilon_{N_1}K)}\right)^2$. The image of the curve lies on a semi-real quaternionic hyperbolic space H_1^3 if and only if for each $s \in I \subset R$ the curvatures satisfy the equalities (28).*

Theorem 10. *Let $\beta = \beta(s)$ be a unit speed semi-real quaternionic W-curve in E_2^4 with non-zero curvatures $k(s)$, $K(s)$, $(r - \varepsilon_t\varepsilon_T\varepsilon_{N_1}K)(s)$ and $\varepsilon_{N_1}\left(\frac{1}{K}\right)^2 = \varepsilon_T\varepsilon_{n_2}\left(\frac{k}{K(r-\varepsilon_t\varepsilon_T\varepsilon_{N_1}K)}\right)^2$. The image of the curve lies on a semi-real quaternionic null cone $C(m)$ if and only if for each $s \in I \subset R$ the curvatures satisfy equalities (28).*

The proof of the Theorem 9 and Theorem 10 is analogous with the proof of Theorem 8.

Corollary 6. *Let $\beta(s)$ be a unit speed semi-real quaternionic W-curve in E_2^4 with non-zero curvatures $K(s)$, $k(s)$ and $(r - \varepsilon_t\varepsilon_T\varepsilon_{N_1}K)(s)$ for each $s \in I \subset R$. If β is a semi-real quaternionic spherical curve, then the radius of S_2^3 is $a = \sqrt{\varepsilon_{N_1}\left(\frac{1}{K}\right)^2 + \varepsilon_T\varepsilon_{n_2}\left(\frac{k}{K(r-\varepsilon_t\varepsilon_T\varepsilon_{N_1}K)}\right)^2}$.*

Corollary 7. *Let $\beta(s)$ be a unit speed semi-real quaternionic W-curve in E_2^4 with non-zero curvatures $K(s)$, $k(s)$ and $(r - \varepsilon_t\varepsilon_T\varepsilon_{N_1}K)(s)$ for each $s \in I \subset R$. If β is a semi-real quaternionic hyperbolical curve, then the radius of H_1^3 is $a = \sqrt{-\left(\varepsilon_{N_1}\left(\frac{1}{K}\right)^2 + \varepsilon_T\varepsilon_{n_2}\left(\frac{k}{K(r-\varepsilon_t\varepsilon_T\varepsilon_{N_1}K)}\right)^2\right)}$.*

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