Bulletin of the *Transilvania* University of Braşov • Vol 11(60), No. 1 - 2018
Series III: Mathematics, Informatics, Physics, 11-22

A SPECIAL TYPE OF QUARTER-SYMMETRIC NON-METRIC CONNECTION ON P-SASAKIAN MANIFOLDS

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Abstract

The object of the present paper is to study a special type of quarter-symmetric non-metric connection on a P-Sasakian manifold. It is shown that the first Bianchi identity of the curvature tensors on P-Sasakian manifolds admits a special type of quarter-symmetric non-metric connection. Among others we prove that if P-Sasakian manifolds admit a special type of quarter-symmetric non-metric connection, then they are Ricci-Semi-symmetric. Finally, an illustrative example is given to verify our result.

2000 *Mathematics Subject Classification:* $53C15$, $53C25$.
Key words: P-Sasakian manifold, quarter-symmetric non-metric connection, Levi-Civita connection, recurrent manifold, Ricci-semi-symmetric.

1 Introduction

In 1977, Adati and Matsumoto [2] defined Para-Sasakian and Special Para-Sasakian manifolds which are considered special cases of an almost paracontact manifold introduced by Sato [16]. Para-Sasakian manifolds have been studied by De and Pathak [7], Matsumoto, Ianus and Mihai [13], De, $\ddot{O}$zg$\ddot{u}$r, Arslan, Murathan and Yildiz [8], Yildiz, Turan and Acet [17], Barman ([3], [4]) and many others.

In 1924, Friedmann and Schouten [9] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection $\widetilde{\nabla}$ on a differentiable manifold M is said to be a semi-symmetric connection if the torsion tensor T of the connection $\widetilde{\nabla}$ satisfies $T(X,Y) = u(Y)X - u(X)Y$, where u is a 1-form and ρ is a vector field defined by $u(X) = g(X,\rho)$, for all vector fields $X, Y \in \chi(M)$, $\chi(M)$ denotes the set of all differentiable vector fields on M.

In 1932, Hayden [11] introduced the idea of semi-symmetric metric connections on a Riemannian manifold (M,g). A semi-symmetric connection $\widetilde{\nabla}$ is said to be a semi-symmetric metric connection if $\widetilde{\nabla} g = 0$.

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After a long gap the study of a semi-symmetric connection $\hat{\nabla}$ satisfying $\hat{\nabla} g \neq 0,$ was initiated by Prvanović [15] with the name pseudo-metric semi-symmetric connection and was just followed by Andonie [1]. The semi-symmetric connection $\hat{\nabla}$ is said to be a semi-symmetric non-metric connection.

In 1975, Golab [10] defined and studied quarter-symmetric connection in differentiable manifolds with affine connections. A linear connection $\bar{\nabla}$ on a Riemannian manifold M is called a quarter-symmetric connection [10] if its torsion tensor T satisfies $T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y,$ where η is a 1-form and ϕ is a (1,1) tensor field. In particular, if $\phi X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection [9]. Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection.

A quarter-symmetric connection $\breve{\nabla}$ is said to be a quarter-symmetric metric connection if $\breve{\nabla} g = 0.$ Moreover, if a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition $(\bar{\nabla}_X g)(Y,Z) \neq 0,$ then $\bar{\nabla}$ is said to be a quarter-symmetric non-metric connection, for all $X, Y, Z \in \chi(M).$

In 2012, Barman [5] studied another type of quarter-symmetric non-metric connection $\bar{\nabla}$ for which we get $(\bar{\nabla}_X g)(Y,Z) = 2\eta(X)g(Y,Z),$ where η is a non-zero 1-form. The author called this a quarter-symmetric non-metric ϕ-connection and in that paper semisymmetric and Ricci-symmetric with respect to the quarter-symmetric non-metric ϕ-connections are also investigated.

In this paper we study P-Sasakian manifolds with respect to a special type of quarter-symmetric non-metric connection. The paper is organized as follows: After introduction in section 2, we give a brief account of the P-Sasakian manifolds. In section 3, we define a special type of quarter-symmetric non-metric connection on P-Sasakian manifolds. Section 4 is devoted to establishing the relation between the curvature tensors with respect to a special type of the quarter-symmetric non-metric connection and the Levi-Civita connection. In this section the covariant derivative with Levi-Civita connection on the curvature tensor of P-Sasakian manifolds admitting a special type of quarter-symmetric non-metric connection $\bar{\nabla}$ and the recurrent curvature tensor with Levi-Civita connection are also studied in this paper. In the next section, we investigate if the P-Sasakian manifold is Ricci-Semi-symmetric with respect to a special type of quarter-symmetric non-metric connection. Finally, we construct an example of 5-dimensional P-Sasakian manifold with respect to a special type of the quarter-symmetric non-metric connection, which verifies the results of Section 4 and Section 5.

2 P-Sasakian manifolds

An n-dimensional differentiable manifold M is said to be an almost para-contact structure (ϕ, ξ, η, g), if there exist ϕ a $(1,1)$ tensor field, ξ a vector field, η a 1-form and g the Riemannian metric on M which satisfy the conditions

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \quad g(X,\xi) = \eta(X), \tag{1}$$

$$\phi^2(X) = X - \eta(X)\xi, \tag{2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3}$$

$$(\nabla_X \eta)Y = \nabla_X \eta(Y) - \eta(\nabla_X Y) = g(X, \phi Y) = (\nabla_Y \eta)X, \tag{4}$$

for any vector fields X, Y on M.

Moreover, it (ϕ, ξ, η, g) satisfy the conditions

$$d\eta = 0, \quad \nabla_X \xi = \phi X, \tag{5}$$

$$\begin{aligned}(\nabla_X \phi)Y = \nabla_X \phi(Y) - \phi(\nabla_X Y) = -g(X, Y)\xi - \eta(Y)X \\ +2\eta(X)\eta(Y)\xi,\end{aligned} \tag{6}$$

then M is called a para-Sasakian manifold or briefly a P-Sasakian manifold.

In a P-Sasakian manifold the following relations hold ([2], [16]) :

$$\begin{aligned}\eta(R(X, Y)Z) = g(R(X, Y)Z, \xi) = g(X, Z)\eta(Y) \\ -g(Y, Z)\eta(X),\end{aligned} \tag{7}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{8}$$

$$R(\xi, X)\xi = X - \eta(X)\xi, \tag{9}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{10}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{11}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{12}$$

where R and S are the curvature tensor and the Ricci tensor of the Levi-Civita connection respectively.

3 Quarter-symmetric non-metric connection on P-Sasakian manifolds

Theorem 1. *The linear connection* $\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y + g(X, Y)\xi - \eta(Y)X - \eta(X)Y + \eta(X)\eta(Y)\xi$ *is a special type of quarter-symmetric non-metric connection on P-Sasakian manifolds.*

Proof. This section deals with a special type of quarter-symmetric non-metric connection on P-Sasakian manifold. Let (M, g) be a P-Sasakian Manifold with the Levi-Civita connection ∇ and we define a linear connection $\bar{\nabla}$ on M by

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y + g(X, Y)\xi - \eta(Y)X - \eta(X)Y + \eta(X)\eta(Y)\xi. \tag{13}$$

Using (13), the torsion tensor T of M with respect to the connection $\bar{\nabla}$ is given by

$$T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y] = \eta(Y)\phi X - \eta(X)\phi Y. \tag{14}$$

The linear connection $\bar{\nabla}$ satisfying (14) is a quarter-symmetric connection.

So the equation (13) with the help of (1) turns into

$$(\bar{\nabla}_X g)(Y, Z) = \bar{\nabla}_X g(Y, Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z) = 2\eta(X)g(Y, Z) \\ +2\eta(X)g(Y, \phi Z) - 2\eta(X)\eta(Y)\eta(Z) \neq 0. \tag{15}$$

Thus, the linear connection $\bar{\nabla}$ satisfying (14) and (15) is called a quarter-symmetric non-metric connection on P-Sasakian manifolds.

Conversely, we show that a linear connection $\bar{\nabla}$ defined on M satisfying (14) and (15) is given from equation (13). Let H be a tensor field of type $(1, 2)$ and we get

$$\bar{\nabla}_X Y = \nabla_X Y + H(X, Y). \tag{16}$$

Then we conclude that

$$T(X, Y) = H(X, Y) - H(Y, X). \tag{17}$$

Further, using (16), it follows that

$$(\bar{\nabla}_X g)(Y, Z) = \bar{\nabla}_X g(Y, Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z) = -g(H(X, Y), Z) \\ -g(Y, H(X, Z)). \tag{18}$$

In view of (15) and (18) it yields,

$$g(H(X, Y), Z) + g(Y, H(X, Z)) = -2\eta(X)g(Y, Z) - 2\eta(X)g(Y, \phi Z) \\ +2\eta(X)\eta(Y)\eta(Z). \tag{19}$$

Also using (19) and (17), we derive that

$$g(T(X, Y), Z) + g(T(Z, X), Y) + g(T(Z, Y), X) = 2g(H(X, Y), Z)$$

$$+2\eta(X)g(Y,Z)+2\eta(Y)g(X,Z)-2\eta(Z)g(X,Y)-2\eta(X)\eta(Y)\eta(Z).$$

From the above equation it yields,

$$g(H(X,Y),Z)=\frac{1}{2}[g(T(X,Y),Z)+g(T(Z,X),Y)+g(T(Z,Y),X)] \\ -\eta(X)g(Y,Z)-\eta(Y)g(X,Z)+\eta(Z)g(X,Y)+\eta(X)\eta(Y)\eta(Z). \tag{20}$$

Now contracting Z in (20) and using (1) and (14), it implies that

$$H(X,Y)=-\eta(X)\phi Y+g(X,Y)\xi-\eta(Y)X-\eta(X)Y \\ +\eta(X)\eta(Y)\xi. \tag{21}$$

Combining (16) and (21), it follows that

$$\bar{\nabla}_X Y=\nabla_X Y-\eta(X)\phi Y+g(X,Y)\xi-\eta(Y)X-\eta(X)Y+\eta(X)\eta(Y)\xi.$$

Therefore Theroem 1 is proved. □

4 Curvature tensor of a P-Sasakian manifold with respect to the quarter-symmetric non-metric connection

In this section we obtain the expressions of the curvature tensor and Ricci tensor of M with respect to the quarter-symmetric non-metric connections on P-Sasakian manifolds defined by (13).

Analogous to the definitions of the curvature tensor of M with respect to the Levi-Civita connection ∇, we define the curvature tensor $\bar{R}$ of M with respect to the quarter-symmetric non-metric connections $\bar{\nabla}$ by

$$\bar{R}(X,Y)Z=\bar{\nabla}_X\bar{\nabla}_Y Z-\bar{\nabla}_Y\bar{\nabla}_X Z-\bar{\nabla}_{[X,Y]}Z, \tag{22}$$

where $X,Y,Z\in\chi(M)$.

Using (2) and (13) in (22), we obtain

$$\bar{R}(X,Y)Z=R(X,Y)Z+\eta(X)(\nabla_Y\phi)(Z)-\eta(Y)(\nabla_X\phi)(Z)+g(Y,Z)\nabla_X\xi \\ -g(X,Z)\nabla_Y\xi+(\nabla_Y\eta)(Z)X-(\nabla_X\eta)(Z)Y+(\nabla_X\eta)(Z)\eta(Y)\xi \\ -(\nabla_Y\eta)(Z)\eta(X)\xi+\eta(Y)\eta(Z)\nabla_X\xi-\eta(X)\eta(Z)\nabla_Y\xi+\eta(X)g(Y,\phi Z)\xi \\ -\eta(Y)g(X,\phi Z)\xi+\eta(X)g(Y,Z)\xi-\eta(Y)g(X,Z)\xi+\eta(X)\eta(Z)\phi Y \\ -\eta(Y)\eta(Z)\phi X+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y. \tag{23}$$

By making use of (4), (5) and (6) in (23), we have

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z + g(Y,\phi Z)X - g(X,\phi Z)Y + g(Y,Z)\phi X \\ -g(X,Z)\phi Y + \eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi + g(X,Z)Y \\ -g(Y,Z)X + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y. \end{aligned} \quad (24)$$

So equation (24) turns into

$$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z$$

and

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0. \quad (25)$$

We call (25) the first Bianchi identity with respect to a special type quarter-symmetric non-metric connection on P-Sasakian manifolds.

Putting $X = \xi$ in (24) and using (1) and (8), we get

$$\bar{R}(\xi,Y)Z = -g(Y,Z)\xi + \eta(Z)Y + g(Y,\phi Z)\xi - \eta(Z)\phi Y. \quad (26)$$

Taking the inner product of (24) with U, it follows that

$$\begin{aligned}\widetilde{\bar{R}}(X,Y,Z,U) = \widetilde{R}(X,Y,Z,U) + g(Y,\phi Z)g(X,U) - g(X,\phi Z)g(Y,U) \\ +g(Y,Z)g(\phi X,U) - g(X,Z)g(\phi Y,U) + g(X,Z)g(Y,U) \\ -g(Y,Z)g(X,U) + \eta(X)\eta(U)g(Y,Z) - \eta(Y)\eta(U)g(X,Z) \\ +\eta(Y)\eta(Z)g(X,U) - \eta(X)\eta(Z)g(Y,U), \end{aligned} \quad (27)$$

where $U \in \chi(M)$, $\widetilde{\bar{R}}(X,Y,Z,U) = g(\bar{R}(X,Y)Z,U)$ and $\widetilde{R}(X,Y,Z,U) = = g(R(X,Y)Z,U)$.

From equation (27) it yields,

$$\widetilde{\bar{R}}(X,Y,Z,U) = -\widetilde{\bar{R}}(X,Y,U,Z).$$

Let $\{e_1, ..., e_n\}$ be a local orthonormal basis of the tangent space at a point of the manifold M. Then by putting $X = U = e_i$ in (27) and taking summation over i, $1 \leq i \leq n$ and also using (1), we get

$$\begin{aligned}\bar{S}(Y,Z) = S(Y,Z) + (n-2)g(Y,\phi Z) + (\alpha + 2 - n)g(Y,Z) \\ +(n-2)\eta(Y)\eta(Z), \end{aligned} \quad (28)$$

where $\bar{S}$ and S denote the Ricci tensor of M with respect to $\bar{\nabla}$ and ∇ respectively and $\alpha = g(e_i, \phi e_i)$, $g(e_i, \phi Z)g(Y, e_i) = g(Y, \phi Z)$, $g(e_i, Z)g(Y, e_i) = g(Y, Z)$, $\eta(e_i)\eta(e_i) = 1$ and $\eta(e_i)g(e_i, Z) = \eta(Z)$.

From (28), it implies that

$$\bar{S}(Y,Z) = \bar{S}(Z,Y).$$

Again putting $Z = \xi$ in (28) and using (1) and (11), we get

$$\bar{S}(Y,\xi) = (\alpha + 1 - n)\eta(Y). \tag{29}$$

Summing up all of the above equations we can state the following proposition:

Proposition 1. *For a P-Sasakian manifold M with respect to a special type of quarter-symmetric non-metric connection $\bar{\nabla}$*

(i) The curvature tensor $\bar{R}$ is given by
$\bar{R}(X,Y)Z = R(X,Y)Z + g(Y,\phi Z)X - g(X,\phi Z)Y + g(Y,Z)\phi X - g(X,Z)\phi Y + \eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi + g(X,Z)Y - g(Y,Z)X + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y,$

(ii) The Ricci tensor $\bar{S}$ is given by
$\bar{S}(Y,Z) = S(Y,Z) + (n-2)g(Y,\phi Z) + (\alpha + 2 - n)g(Y,Z) + (n-2)\eta(Y)\eta(Z),$

(iii)$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z$*,*

(iv)$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0,$

(v) The Ricci tensor $\bar{S}$ is symmetric,

(vi) $\tilde{\bar{R}}(X,Y,Z,U) = -\tilde{\bar{R}}(X,Y,U,Z).$

Definition 1. *A P-Sasakian manifold M with respect to the Levi-Civita connection is said to be recurrent [14] if its curvature tensor R satisfies the condition*

$$(\nabla_U R)(X,Y)Z) = \eta(U)R(X,Y)Z, \tag{30}$$

where η is a non-zero 1-form and $X,Y,Z,U \in \chi(M)$.

Theorem 2. *If the covariant derivative of the curvature tensor on P-Sasakian manifolds admits a special type of quarter-symmetric non-metric connection $\bar{\nabla}$ with Levi-Civita connection and the recurrent of the curvature tensor admits a Levi-Civita connection, then the manifold is flat.*

Proof. The equation (23) turns into

$$\begin{aligned}(\nabla_U \bar{R})(X,Y)Z = (\nabla_U R)(X,Y)Z + g(X,\phi U)g(Y,Z)\xi - g(Y,\phi U)g(X,Z)\xi \\ +\eta(Z)g(Y,\phi U)X - \eta(Z)g(X,\phi U)Y + \eta(Y)g(Z,\phi U)X \\ -\eta(X)g(Z,\phi U)Y + \eta(X)g(Y,Z)\phi U - \eta(Y)g(X,Z)\phi U.\end{aligned} \tag{31}$$

If $(\nabla_U \bar{R})(X,Y)Z = 0$ and using (30) in (31), we get

$$\eta(U)R(X,Y)Z + g(X,\phi U)g(Y,Z)\xi - g(Y,\phi U)g(X,Z)\xi + \eta(Z)g(Y,\phi U)X$$
$$-\eta(Z)g(X,\phi U)Y + \eta(Y)g(Z,\phi U)X - \eta(X)g(Z,\phi U)Y + \eta(X)g(Y,Z)\phi U$$
$$-\eta(Y)g(X,Z)\phi U = 0. \quad (32)$$

Putting $U = \xi$ in (32) and using (1), it follows that

$$R(X,Y)Z = 0.$$

Hence the proof of Theorem 2 is completed.

□

5 P-Sasakian manifolds with respect to a special type quarter-symmetric non-metric connection $\bar{\nabla}$ is Ricci-Semi-symmetric

Theorem 3. *If P-Sasakian manifolds admit a special type of quarter-symmetric non-metric connection, then they are Ricci-Semi-symmetric.*

Proof. We characterize Ricci-Semi-symmetric on a P-Sasakian manifold admitting a special type of quarter-symmetric non-metric connection $\bar{\nabla}$.

$$\bar{R}\cdot\bar{S} = (\bar{R}(X,Y)\cdot\bar{S})(Z,U).$$

Then from the above equation, we can write

$$\bar{R}\cdot\bar{S} = \bar{S}(\bar{R}(X,Y)Z,U) + \bar{S}(Z,\bar{R}(X,Y)U). \quad (33)$$

Putting $X = \xi$ in (33), it follows that

$$\bar{R}\cdot\bar{S} = \bar{S}(\bar{R}(\xi,Y)Z,U) + \bar{S}(Z,\bar{R}(\xi,Y)U). \quad (34)$$

Using (1) and (26) in (34), we obtain

$$\bar{R}\cdot\bar{S} = \eta(Z)\bar{S}(Y,U) + \eta(U)\bar{S}(Z,Y) - g(Y,Z)\bar{S}(\xi,U) - g(Y,U)\bar{S}(Z,\xi)$$
$$+g(Y,\phi Z)\bar{S}(\xi,U) + g(Y,\phi U)\bar{S}(Z,\xi) - \eta(Z)\bar{S}(\phi Y,U)$$
$$-\eta(U)\bar{S}(Z,\phi Y). \quad (35)$$

We take $Z = \xi$ in (35) and using (1) and (29), we get

$$\bar{R}\cdot\bar{S} = \bar{S}(Y,U) - \bar{S}(\phi Y,U) - (\alpha+1-n)g(Y,U)$$
$$+(\alpha+1-n)g(Y,\phi U). \quad (36)$$

Again putting $U = \xi$ in (37) and also using (1) and (29), it implies that

$$\bar{R} \cdot \bar{S} = (\alpha + 1 - n)\eta(Y) - (\alpha + 1 - n)\eta(Y) = 0. \tag{37}$$

This means that the P-Sasakian manifold is Ricci-Semi-symmetric with respect to a special type of quarter-symmetric non-metric connection. This completes the proof. □

6 Example

Now, we give an example of a 5-dimensional P-Sasakian manifold admitting a special type of quarter-symmetric non-metric connection $\bar{\nabla}$, which verifies the skew-symmetric property and the first Bianchi identity of the curvature tensors $\bar{R}$ of $\bar{\nabla}$.

We consider the 5-dimensional manifold $\{(x, y, z, u, v) \in R^5\}$, where (x, y, z, u, v) are the standard coordinates in R^5.
We choose the vector fields

$$e_1 = \frac{\partial}{\partial x},\ e_2 = e^{-x}\frac{\partial}{\partial y},\ e_3 = e^{-x}\frac{\partial}{\partial z},\ e_4 = e^{-x}\frac{\partial}{\partial u},\ e_5 = e^{-x}\frac{\partial}{\partial v},$$

which are linearly independent at each point of M.
Let g be the Riemannian metric defined by

$$g(e_i, e_j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j;\ i, j = 1, 2, 3, 4, 5. \end{cases}$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z, e_1),$$

for any $Z \in \chi(M)$.
Let ϕ be the $(1, 1)$-tensor field defined by

$$\phi(e_1) = 0,\ \phi(e_2) = e_2,\ \phi(e_3) = e_3,\ \phi(e_4) = e_4,\ \phi(e_5) = e_5.$$

Using the linearity of ϕ and g, we have

$$\eta(e_1) = 1,\ \phi^2 Z = Z - \eta(Z)e_1$$

and

$$g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U),$$

for any vector fields $Z, U \in \chi(M)$. Thus for $e_1 = \xi$, the structure (ϕ, ξ, η, g) defines an almost paracontact metric structure on M.
Then we have

$$[e_1, e_2] = -e_2, [e_1, e_3] = -e_3, [e_1, e_4] = -e_4, [e_1, e_5] = -e_5,$$
$$[e_2, e_3] = [e_2, e_4] = 0, [e_2, e_5] = [e_3, e_4] = [e_3, e_5] = [e_4, e_5] = 0.$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]), \end{aligned}$$

therefore we get the following:

$$\begin{aligned} &\nabla_{e_1} e_1 = 0,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = 0,\ \nabla_{e_1} e_4 = 0,\ \nabla_{e_1} e_5 = 0, \\ &\nabla_{e_2} e_1 = e_2,\ \nabla_{e_2} e_2 = -e_1,\ \nabla_{e_2} e_3 = 0,\ \nabla_{e_2} e_4 = 0,\ \nabla_{e_2} e_5 = 0, \\ &\nabla_{e_3} e_1 = e_3,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = -e_1,\ \nabla_{e_3} e_4 = 0,\ \nabla_{e_3} e_5 = 0, \\ &\nabla_{e_4} e_1 = e_4,\ \nabla_{e_4} e_2 = 0,\ \nabla_{e_4} e_3 = 0,\ \nabla_{e_4} e_4 = -e_1,\ \nabla_{e_4} e_5 = 0, \\ &\nabla_{e_5} e_1 = e_5,\ \nabla_{e_5} e_2 = 0,\ \nabla_{e_5} e_3 = 0,\ \nabla_{e_5} e_4 = 0,\ \nabla_{e_5} e_5 = -e_1. \end{aligned}$$

In view of the above relations, we see that

$\nabla_X \xi = \phi X,\ (\nabla_X \phi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi$, for all $e_1 = \xi$.

Therefore, the manifold is a P-Sasakian manifold with the structure (ϕ, ξ, η, g).

Using (13) in the above equations, we obtain

$$\begin{aligned} &\bar{\nabla}_{e_1} e_1 = 0,\ \bar{\nabla}_{e_1} e_2 = -2e_2,\ \bar{\nabla}_{e_1} e_3 = -2e_3,\ \bar{\nabla}_{e_1} e_4 = -2e_4,\ \bar{\nabla}_{e_1} e_5 = -2e_5, \\ &\bar{\nabla}_{e_2} e_1 = 0,\ \bar{\nabla}_{e_2} e_2 = -e_1,\ \bar{\nabla}_{e_2} e_3 = 0,\ \bar{\nabla}_{e_2} e_4 = 0,\ \bar{\nabla}_{e_2} e_5 = 0, \\ &\bar{\nabla}_{e_3} e_1 = 0,\ \bar{\nabla}_{e_3} e_2 = 0,\ \bar{\nabla}_{e_3} e_3 = -e_1,\ \bar{\nabla}_{e_3} e_4 = 0,\ \bar{\nabla}_{e_3} e_5 = 0, \\ &\bar{\nabla}_{e_4} e_1 = 0,\ \bar{\nabla}_{e_4} e_2 = 0,\ \bar{\nabla}_{e_4} e_3 = 0,\ \bar{\nabla}_{e_4} e_4 = -e_1,\ \bar{\nabla}_{e_4} e_5 = 0, \\ &\bar{\nabla}_{e_5} e_1 = 0,\ \bar{\nabla}_{e_5} e_2 = 0,\ \bar{\nabla}_{e_5} e_3 = 0,\ \bar{\nabla}_{e_5} e_4 = 0,\ \bar{\nabla}_{e_5} e_5 = -e_1. \end{aligned}$$

Now, we can easily obtain the non-zero components of the curvature tensors as follows:

$$\begin{aligned} &R(e_1,e_2)e_1 = e_2,\ R(e_1,e_2)e_2 = -e_1,\ R(e_1,e_3)e_1 = e_3,\ R(e_1,e_3)e_3 = -e_1, \\ &R(e_1,e_4)e_1 = e_4,\ R(e_1,e_4)e_4 = -e_1,\ R(e_1,e_5)e_1 = e_5,\ R(e_1,e_5)e_5 = -e_1, \\ &R(e_2,e_3)e_2 = e_3,\ R(e_2,e_3)e_3 = -e_2,\ R(e_2,e_4)e_2 = e_4,\ R(e_2,e_4)e_4 = -e_2, \\ &R(e_2,e_5)e_2 = e_5,\ R(e_2,e_5)e_5 = -e_2,\ R(e_3,e_4)e_3 = e_4,\ R(e_3,e_4)e_4 = -e_3, \\ &R(e_3,e_5)e_3 = e_5,\ R(e_3,e_5)e_5 = -e_3,\ R(e_4,e_5)e_4 = e_5,\ R(e_4,e_5)e_5 = -e_4 \end{aligned}$$

and

$$\begin{aligned} &\bar{R}(e_1,e_2)e_2 = \bar{R}(e_1,e_3)e_3 = \bar{R}(e_1,e_4)e_4 = \bar{R}(e_1,e_5)e_5 = -3e_1, \\ &\bar{R}(e_2,e_1)e_2 = \bar{R}(e_3,e_1)e_3 = \bar{R}(e_4,e_1)e_4 = \bar{R}(e_5,e_1)e_5 = 3e_1. \end{aligned}$$

With the help of the above curvature tensors with respect to a special type of quarter-symmetric non-metric connection, we find the Ricci tensors as follows:

$$\bar{S}(e_1,e_1) = 0, \bar{S}(e_2,e_2) = \bar{S}(e_3,e_3) = \bar{S}(e_4,e_4) = \bar{S}(e_5,e_5) = -3.$$

Let X, Y, Z and U be any four vector fields given by
$X = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5$, $Y = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5$, $Z = c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5$ and $W = d_1e_1 + d_2e_2 + d_3e_3 + d_4e_4 + d_5e_5$, where a_i, b_i, c_i, d_i, for all $i = 1, 2, 3, 4, 5$ are all non-zero real numbers.

Using the above curvature tensors admitting the quarter-symmetric non-metric connection, we obtain

$$\bar{R}(X,Y)Z = -3(a_1b_2c_2 + a_1b_3c_3 + a_1b_2c_2 + a_1b_4c_4 + a_1b_5c_5)e_1 = -\bar{R}(Y,X)Z.$$

Hence we also conclude that from equation(25), we get

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0.$$

Therefore, the curvature tensor of a P-Sasakian manifold admitting a special type of quarter-symmetric non-metric connection $\bar{\nabla}$ satisfies the skew-symmetric property and the first Bianchi identity of the curvature tensors $\bar{R}$ of $\bar{\nabla}$. Now, we see that the Ricci-Semi-symmetric with respect to the quarter-symmetric non-metric connections from the above relations as follows:

$$\bar{R} \cdot \bar{S} = 0.$$

Hence P-Sasakian manifolds will be Ricci-Semi-symmetric with respect to the quarter-symmetric metric connections.
The above arguments tell us that the 5-dimensional P-Sasakian manifolds with respect to the quarter-symmetric non-metric connections under consideration are in agreement with Section 5.

Acknowledgements. The author wishes to express his sincere thanks and gratitude to the referee for his valuable suggestions towards the improvement of the paper.

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