Bulletin of the *Transilvania* University of Braşov • Vol 12(61), No. 2 - 2019
Series III: Mathematics, Informatics, Physics, 435-442
https://doi.org/10.31926/but.mif.2019.12.61.2.21

NEW FIXED POINT THEOREM FOR GENERALIZED CONTRACTIONS

Alexandrina Maria PROCA[1]

Abstract

In this paper we give a Ciric type fixed point theorem in a complete metric space; this theorem extends other well-known fixed point theorems ([7], [8], [9]). Two examples are given to demonstrate the importance of our work.

2000 *Mathematics Subject Classification:* 47H10, 54H25.
Key words: fixed point theory, contraction.

1 Introduction and preliminaries

Definition 1. *Let (X,d) be a metric space. A mapping $T: X \to X$ is a contraction if there exists a number q, $0 \leq q \leq 1$, such that the condition*

$$d(Tx,Ty) \leq q \cdot d(x,y)$$

holds, for all $x, y \in X$.

The well-known Banach contraction principle (BCP) [1] is the following:

Theorem 1. *If the $T: X \to X$ is a contraction mapping of a complete metric space, then:*
(i) $(\exists!)\ x^* \in X$, *fixed point for* T*;*
(ii) $\{T^n x\} \to x^*$ *for* $n \to \infty, (\forall)\ x \in X$;
(iii) $d(T^n x, x^*) \leq \dfrac{q^n}{1-q}\ d(x,Tx)$.

Because of its importance in mathematical theory, many authors gave generalisations of it in many directions (see [1]-[18]). One of the most well-known generalisation of the BCP is Ciric fixed point theorem (**see [7],[8], [9]**).

[1]Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania, e-mail: alexproca@unitbv.ro

Definition 2. *Let (X,d) be a metric space. An operator $T : X \to X$ is a Picard operator if and only if*

(i) $(\exists!)\, x^* \in X$, *fixed point for T;*

(ii) $\{T^n x\} \to x^*$ *for* $n \to \infty, (\forall)\ x \in X$.

Ciric gives the next theorem in [7] , which is a very important result in fixed point theory:

Theorem 2. *[7] Let (X, d) be a complete metric space,and an operator $T : X \to X$. If there exists $a \in [0, 1)$ such that*

$$d\,(Tx, Ty) \leq a \cdot \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}$$

$(\forall)\, x, y \in X$, *then T is a Picard operator.*

After then, many authors give important generalizations of Ciric's theorem in a complete metric space ([11], [15], [16]) or in partial metric space [13].

2 Main results

In this paper we give a generalisation of Ciric type fixed point theorem, by replacing the value of the maximum with

$$\begin{aligned} M^*(x,y) \;=\; & \max\{d(x,y) + |d\,(x,Tx) - d\,(y,Ty)|\,, \\ & d(x,Tx) + |d\,(x,y) - d\,(y,Ty)|\,, \\ & d(y,Ty) + |d\,(x,y) - d\,(x,Tx)|\,, \\ & \left.\frac{d(x,Ty) + d(y,Tx) + |d(x,Tx) - d(y,Ty)|}{2}\right\} \end{aligned} \tag{1}$$

Theorem 3. *Let (X,d) be a complete metric space, $T : X \to X$ such that there exist $a \in [0, 1)$ and*

$$d(Tx, Ty) \leq a \cdot M^*(x, y), \;\; (\forall)\, x, y \in X, \tag{2}$$

where $M^*(x, y)$ is defined in (1). Then, T is a Picard operator.

Proof. Let $x_0 \in X$. Put $x_n = T^n x_0,\ x_0 \in X, (\forall)\, n \in \mathbb{N}$. If there exists $n \in \mathbb{N}$ such that $x_{n+1} = x_n$, then $x_{n+1} = T^n x_0 = Tx_n = x_n$, then, by induction $x_{n+p} = x_n, \forall p \in \mathbb{N}$. That is x_n is a fixed point of T. Now, we suppose that $x_{n+1} \neq x_n$, for all $n \in \mathbb{N}$. Then, $d(x_n, x_{n+1}) > 0$, for all $n \in \mathbb{N}$.

We denote by $d_n = d(x_n, x_{n+1})$. For any $n \in \mathbb{N}$, we have

$$d\,(Tx_n, Tx_{n+1}) = d\,(x_{n+1}, x_{n+2}) = d_{n+1} \tag{3}$$

and

$$\begin{aligned} M^*(x_n, x_{n+1}) &= \max\Big\{ d(x_n, x_{n+1}) + |d(x_n, Tx_n) - d(x_{n+1}, Tx_{n+1})|, \\ & d(x_n, Tx_n) + |d(x_n, x_{n+1}) - d(x_{n+1}, Tx_{n+1})|, \\ & d(x_{n+1}, Tx_{n+1}) + |d(x_n, x_{n+1}) - d(x_n, Tx_n)|, \\ & \frac{1}{2}\left(d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n) + \right. \\ & \left. + |d(x_n, Tx_n) - d(x_{n+1}, Tx_{n+1})|\right) \Big\} \quad (4) \\ &= \max\Big\{ d(x_n, x_{n+1}) + |d(x_n, x_{n+1}) - d(x_{n+1}, x_{n+2})|, \\ & d(x_n, x_{n+1}) + |d(x_n, x_{n+1}) - d(x_{n+1}, x_{n+2})|, \\ & d(x_{n+1}, x_{n+2}) + |d(x_n, x_{n+1}) - d(x_n, x_{n+1})|, \\ & \frac{1}{2}\left(d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1}) + \right. \\ & \left. + |d(x_n, x_{n+1}) - d(x_{n+1}, x_{n+2})|\right) \Big\} \end{aligned}$$

If $d_{n+1} \geq d_n$, then $|d_n - d_{n+1}| = d_{n+1} - d_n$ and from triangle inequality $d(x_n, x_{n+2}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})$ we obtain

$$\begin{aligned} M^*(x_n, x_{n+1}) &\leq \max\left\{ d_{n+1}, \frac{d_n + d_{n+1} + d_{n+1} - d_n}{2} \right\} \\ &= d_{n+1.} \end{aligned}$$

From the assumption of the theorem, we get

$$\begin{aligned} d_{n+1} &= d(x_{n+1}, x_{n+2}) \\ &= d(Tx_n, Tx_{n+1}) \leq a \cdot M^*(x_n, x_{n+1}) \\ &\leq a \cdot d_{n+1} \\ \Leftrightarrow (1-a) \cdot d_{n+1} \leq 0, \end{aligned}$$

which is false, because $a \in [0, 1)$. So, $d_{n+1} < d_n$, $(\forall)\, n \in \mathbb{N}$.
For $d_{n+1} < d_n$, we have $|d_n - d_{n+1}| = d_n - d_{n+1}$ and

$$\begin{aligned} M^*(x_n, x_{n+1}) &\leq \max\{2d_n - d_{n+1}, d_{n+1}, \frac{1}{2}(d_n + d_{n+1} + d_n - d_{n+1})\} \quad (5) \\ &= \max\{2d_n - d_{n+1}, d_{n+1}, d_n\} \end{aligned}$$

Combining (2), (3) and (5), for $d_{n+1} < d_n$ we obtain

$$\begin{aligned} d_{n+1} &= d(Tx_n, Tx_{n+1}) \leq a \cdot M^*(x_n, x_{n+1}) \\ &\leq a \cdot \max\{2d_n - d_{n+1}, d_{n+1}, d_n\} \\ &= a \cdot (2d_n - d_{n+1}) \end{aligned}$$

(because $2d_n - d_{n+1} > d_n > d_{n+1}$). Hence

$$d_{n+1} \leq \frac{2a}{a+1} \cdot d_n = k \cdot d_n, \tag{6}$$

if we denote by $k = \frac{2a}{a+1} < 1$, $(\forall)\, a \in [0,1)$. This implies that $\{x_n\}$ is Cauchy sequence. By completeness of (X,d), the sequence $\{x_n\}$ converges to some point $x^* \in X$.

From the assumption of Theorem 3, for $x = x_n$ and $y = x^*$, we have:

$$\begin{aligned} d(x_{n+1}, Tx^*) &= d(Tx_n, Tx^*) = d(x_{n+1}, Tx^*) \leq \\ &\leq a \cdot \max \Big\{ d(x_n, x^*) + |d(x_n, Tx_n) - d(x^*, Tx^*)|, \\ &\quad d(x_n, Tx_n) + |d(x_n, x^*) - d(x^*, Tx^*)|, \\ &\quad d(x^*, Tx^*) + |d(x_n, x^*) - d(x_n, Tx_n)|, \\ &\quad \frac{1}{2}\left(d(x_n, Tx^*) + d(x^*, Tx_n) + |d(x_n, Tx_n) - d(x^*, Tx^*)|\right) \Big\} \qquad (7) \\ &= a \cdot \max \Big\{ d(x_n, x^*) + |d(x_n, x_{n+1}) - d(x^*, Tx^*)|, \\ &\quad d(x_n, x_{n+1}) + |d(x_n, x^*) - d(x^*, Tx^*)|, \\ &\quad d(x^*, Tx^*) + |d(x_n, x^*) - d(x_n, x_{n+1})|, \\ &\quad \frac{1}{2}\left(d(x_n, Tx^*) + d(x^*, x_{n+1}) + |d(x_n, x_{n+1}) - d(x^*, Tx^*)|\right) \Big\}. \end{aligned}$$

Taking the limit as $n \to \infty$ in (7), we deduce

$$d(x^*, Tx^*) \leq a \cdot d(x^*, Tx^*).$$

Since $a \in [0,1)$ it results that

$$d(x^*, Tx^*) = 0,$$

that is x^* is a fixed point of T.

Finally, we prove that the fixed point of T is unique. For this, let x^*, y^* be two fixed points of T, and suppose that $x^* \neq y^*$. It follows, from the assumption of the theorem:

$$d(x^*, y^*) \leq a \cdot d(x^*, y^*),$$

so $d(x^*, y^*) = 0$. Hence $x^* = y^*$. Therefore, T has a unique fixed point. □

Example 1. *Let* $X = \{A, B, C, D\}$, *d the usual distance,* $d(A,B) = d(B,C) = 8$, $d(A,C) = d(B,D) = 10$, $d(A,D) = 5$, $d(C,D) = 7$ *and* $T : X \to X$ *such that* $TA = A$, $TB = C$, $TC = D$, $TD = A$. *We observe that X is a metric space.*

For $x = A$ and $y = B$, we have $d(TA, TB) = d(A, C) = 10$ and

$$\begin{aligned} M(A,B) &= \max\{d(A,B), d(A,TA), d(B,TB), \\ &\quad \frac{1}{2}(d(A,TB) + d(B,TA))\} \\ &= \max\left\{8, 0, 8, \frac{10+8}{2}\right\} = 9. \end{aligned}$$

Therefore, T does not satisfy conditions from Theorem 2 . Next, we prove that T satisfies hypothesis of Theorem 3.

1. *For $x = A$ and $y = B$, $d(Tx, Ty) \leq a \cdot M^*(x, y)$ we have $d(TA, TB) = d(A, C) \leq a \cdot M^*(A, B)$, where $M^*(A, B) = 26$, so we obtain $10 \leq a \cdot 26$*
2. *For $x = A$ and $y = C$, $d(Tx, Ty) \leq a \cdot M^*(x, y)$ we have $d(TA, TC) = d(A, D) \leq a \cdot M^*(A, B)$, where $M^*(A, C) = 17$ and $d(A, D) = 5$, so we obtain $5 \leq a \cdot 17$*
3. *For $x = A$ and $y = D$, we have $TA = TD = A$, and the relation $d(Tx, Ty) \leq a \cdot M^*(x, y)$ hold for $\forall a \in [0, 1)$*
4. *For $x = B$ and $y = C$,, the relation $d(Tx, Ty) \leq a \cdot M^*(x, y)$ is $7 \leq a \cdot 9$*
5. *For $x = B$ and $y = D$,, the relation $d(Tx, Ty) \leq a \cdot M^*(x, y)$ is $10 \leq a \cdot 13$*
6. *For $x = C$ and $y = D$,, the relation $d(Tx, Ty) \leq a \cdot M^*(x, y)$ is $5 \leq a \cdot 9$*

It is sufficient that $a = 7/9$, and we can apply Theorem 3. We deduce that T is a Picard mapping. Therefore, we have a real generalisation of Ciric's theorem.

Example 2. *Let $X = \{(0, a), a \in [30, 40]\} \cup \{(0, 10)\} \cup \{(7, 0)\} \cup \{(10, 0)\} \cup \{(11, 0)\}$. We denote by*

$$\begin{aligned} A &= \{(0,a),\ a \in [30,40]\}, \\ B &= (0,10),\ C = (7,0),\ D = (10,0), \\ F &= (11,0),\ O(0,0) \end{aligned}$$

and let $T : X \to X$ with

$$Tx = \begin{cases} (0,10) & , x \in A \cup \{(10,0)\} \\ (0,0) & , x = (0,10) \\ (7,0) & , x = (0,0) \\ (11,0) & , x \in \{(7,0),(11,0)\} \end{cases}$$

F is the fixed point for mapping T, X is a complete metric space with euclidian metric and for the following cases we prove that T does not satisfy the hypothesis of Theorem 2, but satisfies Theorem 3. Therefore, from Theorem 3, T is Picard mapping.

Case 1. $x, y \in A \Rightarrow d(Tx, Ty) = 0$.

Case 2. $x \in A, y = D \Rightarrow d(Tx, Ty) = 0.$

Case 3. $x \in A, y = B \Rightarrow Tx = B = y, TB = O \Rightarrow d(Tx, Ty) = 10, d(x, y) \geq 20.$ *For* $a > 1/2$, *the hypothesis of Theorem 3 is true.*

Case 4. *For* $x \in A, y = O$, *we deduce that* $d(Tx, Ty) = \sqrt{149}, d(x, y) \geq 30$. *For* $a > 2/5$ *the hypothesis of Theorem 3 is true.*

Case 5. *For* $x \in A, y \in \{C, F\}, d(Tx, Ty) = \sqrt{10^2 + 11^2} = \sqrt{221}, d(x, Tx) \geq 20,$ *so, for* $a > \dfrac{\sqrt{221}}{20}$ *we can apply Theorem 3.*

Case 6. *For* $x = B, y = 0, d(Tx, Ty) = 7, d(x, y) = 10$. *For* $a = 0.9$, *relation (2) is true.*

Case 7. *For* $x = B, y = C, d(Tx, Ty) = 11, d(x, y) = \sqrt{10^2 + 7^2}, d(x, y) + |d(x, Tx) - d(y, Ty)| = \sqrt{149} + |10 - 4|$. *For* $a > \dfrac{11}{\sqrt{149} + 6}$, *relation (2) is true.*

Case 8. *For* $x = B, y = D, d(Tx, Ty) = 10, d(x, y) = \sqrt{200}$ *and (2) is true for* $a = 0.9$.

Case 9. *For* $x = B$ *and* $y = F$ *give us* $d(Tx, Ty) = 11, d(x, y) = \sqrt{221}$ *and (2) is true for* $a = 0.9$.

Case 10. *For* $x = O, y = C$, *we deduce* $d(Tx, Ty) = 4, d(x, y) = 7$ *and (2) is true for* $a = 0.9$.

Case 11. *For* $x = O, y = D$ *give us* $d(Tx, Ty) = \sqrt{149}, d(x, y) = 10.d(x, y) + |d(x, Tx) - d(y, Ty)| = 10 + \left|7 - \sqrt{200}\right| = \sqrt{200} + 3$. *The value* $a = 0.9$ *is right and (2) stays true.*

Case 12. *For* $x = O, y = F, d(Tx, Ty) = 4, d(x, y) = 11$ *and (2) is true for* $a = 0.9$.

Case 13. *For* $x = C, y = D, d(Tx, Ty) = \sqrt{221}, d(x, y) + |d(x, Tx) - d(y, Ty)| = 3 + \left|4 - \sqrt{200}\right| = \sqrt{200} - 1, d(y, Ty) + |d(x, y) - d(x, Tx)| = \sqrt{200} + |3 - 4| = \sqrt{200} + 1$ *so, in this case, for* $a = \dfrac{\sqrt{221}}{\sqrt{200} + 1}$, *(2) is true.*

Case 14. *For* $x = C, y = F, d(Tx, Ty) = 0$ *and (2) is true for all* $a \in [0, 1)$.

Case 15. *For* $x = D, y = F, d(Tx, Ty) = \sqrt{221}, d(x, y) + |d(x, Tx) - d(y, Ty)| = 1 + \left|\sqrt{200} - 0\right| = \sqrt{200} + 1$, *so, for* $a = \dfrac{\sqrt{221}}{\sqrt{200} + 1}$, *relation (2) is true.*

In conclusion, for the value $a = \dfrac{\sqrt{221}}{\sqrt{200} + 1} \in [0, 1)$, *relation (2) is true,* $(\forall)\, x, y \in X$, *and, from Theorem 3, we deduce that* T *is a Picard operator. Also, we can observe that the hypothesis of Ciric's Theorem 2 is not satisfied in case 15. Because*

$$\begin{aligned} d(Tx, Ty) &= d(TD, TF) = \sqrt{221}, \\ d(x, y) &= 1, d(x, Tx) = d(D, TD) = \sqrt{200}, \\ d(y, Ty) &= d(F, TF) = 0, \\ \frac{d(x, Ty) + d(y, Tx)}{2} &= \frac{d(D, F) + d(F, B)}{2} = \frac{1 + \sqrt{221}}{2}. \end{aligned}$$

we have

$$\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}=\sqrt{200}.$$

Relation

$$d(Tx,Ty)\le a\cdot\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}$$

is false for all $a\in[0,1)$ *.*

Therefore, Theorem 3 is a real generalisation of Theorem 2.

References

[1] Banach, S., *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fundamenta Mathematical **3** (1922), 133-181.

[2] Berinde, V., *Stability of Picard iteration for contractive mappings satisfying an implicit relation*, Carpathian J. Math., **27** (2011), no. 1, 13-23.

[3] Berinde, V., *Approximating fixed points of implicit almost contractions*, Hacet. J. Math. Stat. **40** (2012) no. 1, 93-102.

[4] Berinde, V., Pacurar, M., *Fixed point theorems for nonself single-valued almost contractions*, Fixed Point Theory **14** (2013), 301-311.

[5] Berinde, V., Vetro, F., *Common fixed points of mappings satisfying implicit contractive conditions*, Fixed Point Theory and Applications **2012**, **105**(2012) doi:10.1186/1687-1812-2012-105.

[6] Berinde, V., Vetro, F., *Fixed point for cyclic* $(\psi,C)-$*contractions in* $O-$*complete metric spaces*, Filomat **27** (2013), 1405-1413.

[7] Ciric, Lj., *On contraction type mappings,* Math. Balcanica **1** (1971), 52-57.

[8] Ciric, Lj., *Generalised contractions and fixed point theorems,* Publications de l'Institut Mathematique, Nouvelle serie, **26** (1971), 19-26.

[9] Ciric, Lj., *A generalisation of Banach's contraction principle*, Proceedings of the American Mathematical Society, **45** (1974), no. 2, 267-273.

[10] Ciric, Lj., *Multi-valued nonlinear contraction mappings*, Nonlinear Anal. **71** (2009), 2716-2723.

[11] Fulga, A., Proca, A.M., *Fixed point for Geraghty* φ_E-*contractions*, J. Nonlinear Sci. Appl., **10** (2017), 51255131.

[12] Edelstein, M., An extension of Banach's contraction principle, Proc. Amer. Math. Soc.**12** (1961), No. 1, 7-10.

[13] Karapinar, E., *Weak $\varphi-$contraction on partial metric spaces*, J. Comput. Anal. Appl. **14** (2012), 206-210.

[14] Mongkolkeha, C., Cho, Y.E., Kumam, P., *Best proximity points for Geraghty's proximal contraction mappings*. Fixed Point Theory Appl. **2013**, Article ID 180 (2013).

[15] Popescu, O., *Some new fixed point theorems for $\alpha-$Geraghty contraction type maps in metric spaces*, Fixed Point Theory Appl. **2013**, Article ID 329(2013).

[16] Popescu, O., *A new type of contractive mappings in complete metric spaces*, submitted.

[17] Samet, B., Vetro, C., and Vetro,P., *Fixed point theorems for $\alpha-\psi$-contractive type mappings*, Nonlinear Anal. **75** (2012), 2154-2165;

[18] Suzuki, T., *A generalized Banach contraction principle that characterizes metric completness*, Proc.Amer.Math.Soc. **136** (2008), no. 5, 1861-1869.