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THE UNIVALENCE OF AN INTEGRAL OPERATOR

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Abstract

In this paper we define an integral operator for analytic functions in the open unit disk and we determine univalence criteria of this integral operator.

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1 Introduction

Let A be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

normalized by $f(0) = f'(0) - 1 = 0$, which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

We denote by $\mathcal{S}$ the subclass of A consisting of functions $f \in A$, which are univalent in U.

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U. For $a \in \mathbb{C}$ and $n \in \mathbb{N} - \{0\}$ we note

$$H[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + \cdots\}$$

and

$$\mathcal{A}_n = \left\{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \cdots \right\},$$

with $\mathcal{A}_1 = A$.

In this paper we consider the integral operator $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta,\gamma}(z)$ defined by

$$G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta,\gamma}(z) = \left\{ \gamma \int_0^z u^{\gamma-1} \prod_{j=1}^{n} \left(\frac{u}{g_j(u)} \right)^{\alpha_j} \left(h_j'(u)\right)^{\beta} du \right\}^{\frac{1}{\gamma}} \tag{1}$$

$\alpha_j, \gamma, \beta \in \mathbb{C}$, $a = Re\gamma > 0$, $\beta \neq 0$, $g_j, h_j \in \mathcal{A}_n$, $j = \overline{1, n}$.

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2 Preliminaries

We need the following lemmas.

Lemma 1 (Pascu [3]). *Let α be a complex number, $Re\alpha > 0$ and $f \in A$. If*

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{2}$$

for all $z \in U$, then the function

$$F_\alpha(z) = \left[\alpha \int_0^z t^{\alpha-1} f'(t)dt\right]^{\frac{1}{\alpha}} \tag{3}$$

is regular and univalent in U.

Lemma 2 (Mocanu and Şerb, [2]). *Let $M_0 = 1,5936\ldots$ be the positive solution of equation*

$$(2-M)e^M = 2.$$

If $f \in A$ and

$$\left|\frac{f''(z)}{f'(z)}\right| \leq M_0,\ z \in U, \tag{4}$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, z \in U. \tag{5}$$

The edge M_0 is sharp.

Lemma 3 (General Schwarz Lemma, [1]). *Let f the function regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If the function f has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m,\ z \in U_R, \tag{6}$$

the equality (in the inequality (6) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

In this paper we determine univalence criteria of integral operator $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta,\gamma}(z)$ defined in (1).

3 Main results

Theorem 1. *Let α_j, β, γ be complex numbers, $\beta \neq 0$, $a = Re\gamma > 0$ and the functions $g_j, h_j \in \mathcal{A}_n$, M_j, L_j the positive real numbers, $j = \overline{1, n}$.*

If

$$\left| \frac{zg_j'(z)}{g_j(z)} - 1 \right| < M_j, z \in U,\ j = \overline{1,n}\ , \tag{7}$$

$$\left| \frac{zh_j''(z)}{h_j'(z)} \right| < L_j, z \in U,\ j = \overline{1,n} \tag{8}$$

and

$$\sum_{j=1}^{n} |\alpha_j| \cdot M_j + \sum_{j=1}^{n} |\beta| L_j \leq \frac{(2a+n)^{\frac{n+2a}{2a}}}{2n^{\frac{n}{2a}}} \tag{9}$$

then the function $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta,\gamma}(z)$ belongs to the class S.

Proof. From (1) we consider the function

$$f(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{u}{g_j(u)} \right)^{\alpha_j} \left(h_j'(u) \right)^{\beta} du, z \in U. \tag{10}$$

The function f defined in (10) is regular in U and $f(0) = f'(0) - 1 = 0$.
We have

$$\frac{zf''(z)}{f'(z)} = \sum_{j=1}^{n} \alpha_j \cdot \left[1 - \frac{zg_j'(z)}{g_j(z)} \right] + \beta \cdot \sum_{j=1}^{n} \frac{zh_j''(z)}{h_j'(z)}, z \in U \tag{11}$$

and hence, we obtain

$$\frac{1-|z|^{2a}}{a} \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{1-|z|^{2a}}{a} \left[\sum_{j=1}^{n} |\alpha_j| \cdot \left| 1 - \frac{zg_j'(z)}{g_j(z)} \right| + |\beta| \cdot \sum_{j=1}^{n} \left| \frac{zh_j''(z)}{h_j'(z)} \right| \right], \tag{12}$$

for all $z \in U$.

Applying lemma 3, from (7) and (8) we have

$$\left| \frac{zg_j'(z)}{g_j(z)} - 1 \right| < M_j|z|^n, z \in U, j = \overline{1,n}, \tag{13}$$

$$\left| \frac{zh_j''(z)}{h_j'(z)} \right| < L_j|z|^n, z \in U, j = \overline{1,n}\ . \tag{14}$$

From (12) and (13), (14) we get

$$\frac{1-|z|^{2a}}{a} \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{1-|z|^{2a}}{a} |z|^n \left[\sum_{j=1}^{n} |\alpha_j| \cdot M_j + |\beta| \cdot \sum_{j=1}^{n} L_j \right] \tag{15}$$

for all $z \in U$.

We consider the function $Q : [0,1] \to \mathbb{R}, Q(x) = \frac{(1-x^{2a})x^n}{a}$, where $x = |z|$, $x \in [0,1]$.

We have

$$\max_{x\in[0,1]} Q(x) = \frac{2n^{\frac{n}{2a}}}{(2a+n)^{\frac{n+2a}{2a}}}. \tag{16}$$

By (9), (16) and (15) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{17}$$

for all $z \in U$.

From (17) and Lemma 1, it results that $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta,\gamma}(z)$ belongs to the class S. □

Theorem 2. *Let α_j, β, γ be a complex numbers, $\beta \neq 0$, $a = Re\gamma > 0$, the functions $g_j, h_j \in \mathcal{A}_n$, $j = \overline{1,n}$ and M_0 the positive solution of equation $(2-M)e^M = 2$.*
If

$$\left|\frac{g_j''(z)}{g_j'(z)}\right| \leq M_0,\ z \in U,\ j = \overline{1,n}\ , \tag{18}$$

$$\left|\frac{h_j''(z)}{h_j'(z)}\right| \leq M_0,\ z \in U,\ j = \overline{1,n} \tag{19}$$

and

$$\frac{1}{a}\sum_{j=1}^{n}|\alpha_j| + \frac{2}{(2a+1)^{\frac{2a+1}{2a}}}\cdot|\beta|\cdot M_0\cdot n \leq 1 \tag{20}$$

then the function $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta,\gamma}(z)$ belongs to the class S.

Proof. Using relations (10) and (11) we obtain

$$\begin{aligned}\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \quad &\leq \quad \frac{1-|z|^{2a}}{a}\left[\sum_{j=1}^{n}|\alpha_j|\cdot\left|\frac{zg_j'(z)}{g_j(z)}-1\right|+\right.\\ &+ \quad \left.|\beta|\cdot\sum_{j=1}^{n}|z|\left|\frac{h_j''(z)}{h_j'(z)}\right|\right],\end{aligned} \tag{21}$$

for all $z \in U$.

From relations (18), (19) and lemma 2 we obtain

$$\frac{1-|z|^{2a}}{a}\sum_{j=1}^{n}|\alpha_j|\cdot\left|\frac{zg_j'(z)}{g_j(z)}-1\right|+\frac{1-|z|^{2a}}{a}\cdot|\beta|\cdot\sum_{j=1}^{n}|z|\left|\frac{h_j''(z)}{h_j'(z)}\right|\leq$$
$$\leq\frac{1}{a}\sum_{j=1}^{n}|\alpha_j|+\frac{1-|z|^{2a}}{a}|z|\cdot|\beta|\cdot M_0\cdot n\ , \tag{22}$$

for all $z\in U$.

We consider the function $I:[0,1]\to\mathbb{R}$, $I(x)=\frac{(1-x^{2a})x}{a}$, where $x=|z|$, $x\in[0,1]$.

We have

$$\max_{x\in[0,1]} I(x)=\frac{2}{(2a+1)^{\frac{2a+1}{2a}}}. \tag{23}$$

By relations (23), (22), (21) and (20) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right|\leq 1,$$

for all $z\in U$.

From Lemma 1, it results that $G_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta,\gamma}(z)$ belongs to the class S. □

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