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A FIXED POINT THEOREM IN G - METRIC SPACES FOR MAPPINGS USING AUXILIARY FUNCTIONS

Alina-Mihaela PATRICIU[*,1] **and Valeriu POPA**[2]

Abstract

In this paper we introduce a new type of implicit relation and we prove a general fixed point theorem in G - metric spaces using two auxiliary functions, generalizing Theorem 3.3 [1].

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Key words: fixed point, G - metric space, auxiliary function, implicit relation.

1 Introduction

In [2], [3] Dhage introduced a new class of generalized metric space, named D - metric spaces. Mustafa and Sims [5], [6] proved that most of the claims concerning the fundamental topological structures on D - metric spaces are incorrect and introduced an appropriate notion of generalized metric space, named G - metric space.

In fact, Mustafa, Sims and other authors studied many fixed point results for self mappings in G - metric spaces.

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit function [7], [8] and in other papers. The study of fixed points for mappings satisfying implicit relations in G - metric spaces is initiated in [9] - [11] and in other papers.

Recently, in [1], new fixed point results for mappings in G - metric spaces using a new type of auxiliary mappings are obtained.

M. S. Khan et al. [4] introduced the notion of altering distance. Some results using altering distance in metric spaces are obtained in [13], [14] and in other papers. Recently results in G - metric spaces are obtained in [12].

The purpose of this paper is to introduce a new type of implicit relation and to prove a general fixed point theorem using two auxiliary functions, generalizing Theorem 3.3 [1].

[1*] *Corresponding author* Department of Mathematics and Computer Sciences, Faculty of Sciences and Environment, *Dunărea de Jos* University of Galaţi, Romania, e-mail: Alina.Patriciu@ugal.ro
[2]Faculty of Sciences, *Vasile Alecsandri* University of Bacău, Romania, e-mail: vpopa@ub.ro

2 Preliminaries

Definition 1 ([6]). *Let X be a nonempty set and $G : X^3 \to \mathbb{R}_+$ be a function satisfying the following conditions:*
$(G_1) : G(x,y,z) = 0$ *for* $x = y = z$,
$(G_2) : G(x,x,y) > 0$ *for all* $x,y \in X$ *with* $x \neq y$,
$(G_3) : G(x,y,y) \leq G(x,y,z)$ *for all* $x,y,z \in X$ *with* $y \neq z$,
$(G_4) : G(x,y,z) = G(y,z,x) = \ldots$ *(symmetry in all three variables),*
$(G_5) : G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ *for all* $x,y,z,a \in X$ *(triangle inequality).*
The function G is called a G - metric on X and (X,G) is called a G - metric space.

Note that if $G(x,y,z) = 0$, then $x = y = z$.

Remark 1. *Let (X,G) be a G - metric space. If $y = z$, then $G(x,y,y)$ is a quasi - metric on X. Hence, (X,Q), where $Q(x,y) = G(x,y,y)$, is a quasi - metric space and since every metric space is a quasi - metric space it follows that the notion of G - metric space is a generalization of metric space.*

Definition 2 ([6]). *Let (X,G) be a G - metric space. A sequence $\{x_n\}$ in X is said to be:*
a) G - convergent if for $\varepsilon > 0$, there exist $x \in X$ and $k \in \mathbb{N}$ such that for all $m,n \in \mathbb{N}, m,n \geq k$, $G(x_n,x_m,x) < \varepsilon$.
b) G - Cauchy if for $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $m,n,p \in \mathbb{N}$, $m,n,p \geq k$, $G(x_n,x_m,x_p) < \varepsilon$, that is $G(x_n,x_m,x_p) \to 0$ as $n,m,p \to \infty$.
c) A G - metric space is said to be G - complete if every G - Cauchy sequence in X is G - convergent.

Lemma 1 ([6]). *Let (X,G) be a G - metric space. Then, the following conditions are equivalent:*
1) $\{x_n\}$ is G - convergent to x;
2) $G(x_n,x_n,x) \to 0$ as $n \to \infty$;
3) $G(x_n,x,x) \to 0$ as $n \to \infty$;
4) $G(x_n,x_m,x) \to 0$ as $n,m \to \infty$.

Lemma 2 ([6]). *Let (X,G) be a G - metric space. Then, the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

Definition 3 ([6]). *A G - metric on a nonempty set X is said to be symmetric if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$. Then, (X,G) is said to be symmetric G - metric space.*

Lemma 3 ([1]). *Let (X,G) be a G - metric space and $\{x_n\}$ be a sequence in X such that $G(x_n,x_{n+1},x_{n+1})$ is decreasing and $\lim_{n\to\infty} G(x_n,x_{n+1},x_{n+1}) = 0$. If $\{x_{2n}\}$ is not a Cauchy sequence, then there exists $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that*

$$\lim_{n\to\infty} G(x_{2n_k+1}, x_{2m_k}, x_{2m_k}) = \varepsilon,$$

$$\lim_{n\to\infty} G\left(x_{2n_k}, x_{2m_k-1}, x_{2m_k-1}\right) = \varepsilon,$$
$$\lim_{n\to\infty} G\left(x_{2m_k-1}, x_{2n_k+1}, x_{2n_k+1}\right) = \varepsilon,$$
$$\lim_{n\to\infty} G\left(x_{2n_k}, x_{2m_k}, x_{2m_k}\right) = \varepsilon.$$

Definition 4 ([4])**.** *An altering distance is a function $\psi : [0,\infty) \to [0,\infty)$ satisfying:*
(ψ_1) : *ψ is increasing and continuous;*
(ψ_2) : *$\psi(t) = 0$ if and only if $t = 0$.*

The set of all altering distances is denoted by Ψ.

In the following we denote by Φ the set of all continuous nondecreasing functions $\varphi : [0,\infty) \to [0,\infty)$.

Lemma 4 ([1])**.** *If $\varphi \in \Phi$ and $\psi \in \Psi$ such that $\psi(t) > \varphi(t)$ for $t > 0$, then $\varphi(0) = 0$.*

The following theorem is proved in [1].

Theorem 1 ([1])**.** *Let (X, G) be a complete G - metric space and $T : X \to X$ be a mapping. If there exist $\varphi \in \Phi$ and $\psi \in \Psi$ with condition $\psi(t) > \varphi(t)$ for $t > 0$ such that*

$$\psi\left(G\left(Tx, Ty, Tz\right)\right) \leq \varphi \max\left(\left\{\begin{array}{c} G(x,y,y), G(x,Tx,Tx), \\ G(y,Ty,Ty), G(z,Tz,Tz), \\ \alpha G(y,Tx,Tx) + (1-\alpha) G(z,Ty,Ty), \\ \beta G(x,Tx,Tx) + (1-\beta) G(y,Ty,Ty), \end{array}\right\}\right) \tag{1}$$

for all $x, y, z \in X$, where $\alpha, \beta \in (0,1)$. Then T has a unique fixed point.

Remark 2. *Since*

$$\beta G(x,Tx,Tx) + (1-\beta) G(y,Ty,Ty) \leq \max\left\{G(x,Tx,Tx), G(y,Ty,Ty)\right\},$$

then $\beta G(x,Tx,Tx) + (1-\beta) G(y,Ty,Ty)$ is redundant in the inequality (1).

3 $\psi - \phi -$ implicit relations

Let $\mathfrak{F}_5$ be the set of all continuous functions $F : \mathbb{R}_+^5 \to \mathbb{R}$ such that:
(F_1) : F is decreasing in variables t_2 and t_4,
(F_2) : for all $u, v \geq 0$, $F(u, v, v, u, 0) \leq 0$ implies $u \leq v$,
(F_3) : $F(t, t, 0, 0, t') \leq 0$ implies $t \leq t'$ for $t, t' > 0$.

Remark 3. *1) In the following examples $\psi \in \Psi$, $\phi \in \Phi$ and $\psi(t) > \phi(t), \forall t > 0$.*
2) Since $\phi(t)$ is nondecreasing, then

$$\phi\left(\max\left\{t_1, t_2, t_3, t_4\right\}\right) = \max\left\{\phi(t_1), \phi(t_2), \phi(t_3), \phi(t_4)\right\}.$$

3) In the following examples, the proofs of property (F_1) is obviously.

Example 1. $F(t_1,...,t_5) = \psi(t_1) - \phi(\max\{t_2,t_3,t_4,t_5\})$.

(F_2) Let $u,v \geq 0$ be and $F(u,v,v,u,0) = \psi(u) - \phi(\max\{u,v\}) \leq 0$. If $u > v$, then $\psi(u) - \phi(u) \leq 0$. Hence, $\psi(u) \leq \phi(u) < \psi(u)$, a contradiction. Hence, $u \leq v$.
(F_3) Let $t,t' > 0$ and $F(t,t,0,0,t') = \psi(t) - \phi(\max\{t,t'\}) \leq 0$. If $t > t'$, then $\psi(t) - \phi(t') \leq 0$, which implies $\psi(t) \leq \phi(t) < \psi(t)$, a contradiction. Hence, $t \leq t'$.

Example 2. $F(t_1,...,t_5) = \psi(t_1) - \phi(\max\{t_2,t_3,t_4,\alpha t_5 + (1-\alpha)t_4\})$, *where* $\alpha \in (0,1)$.

$$\psi(t_1) - \phi(\max\{t_2,t_3,t_4,\alpha t_5 + (1-\alpha)t_4\}) \leq 0$$

implies

$$\psi(t_1) \leq \phi(\max\{t_2,t_3,t_4,\max\{t_4,t_5\}\}) = \phi(\max\{t_2,t_3,t_4,t_5\}).$$

So, Example 2 is reduced to Example 1.

Example 3. $F(t_1,...,t_5) = \psi(t_1) - \phi\left(\max\left\{t_2, \frac{t_3+t_4}{2}, t_5\right\}\right)$.

Since $\frac{t_3+t_4}{2} \leq \max\{t_3,t_4\}$, Example 3 is reduced to Example 1.

Example 4. $F(t_1,...,t_5) = \psi(t_1) - \phi(at_2 + bt_3 + ct_4 + dt_5)$, *where* $a,b,c,d \geq 0$ *and* $a+b+c+d < 1$.

Since $\phi(at_2 + bt_3 + ct_4 + dt_5) \leq \phi((a+b+c+d)\max\{t_2,t_3,t_4,t_5\})$, the study of Example 4 is reduced to the study of Example 1.

Example 5. $F(t_1,...,t_5) = [\psi(t_1)]^2 - a\phi(t_2)\phi(t_3) - b\phi(t_3)\phi(t_4) - c\phi^2(t_5)$, *where* $a,b,c \geq 0$ *and* $a+b+c < 1$.

(F_2) Let $u,v \geq 0$ be and $F(u,v,v,u,0) = [\psi(u)]^2 - a\phi(u)\phi(v) - b\phi(u)\phi(v) \leq 0$. If $u > v$, then $[\psi(u)]^2 - (a+b)[\phi(u)]^2 \leq 0$, which implies $[\psi(u)]^2 \leq (a+b)\phi^2(u) \leq \phi^2(u) < \psi^2(u)$, a contradiction. Hence, $u \leq v$.
(F_3) Let $t,t' > 0$ and $F(t,t,0,0,t') = \psi^2(t) - c\phi^2(t') \leq 0$. If $t > t'$, then $\psi(t) \leq \sqrt{c}\phi(t') \leq \phi(t) < \psi(t)$, a contradiction. Hence, $t \leq t'$.

Example 6. $F(t_1,...,t_5) = \psi(t_1) - a\max\{\phi(t_2),\phi(t_3),\phi(t_4)\} - b\phi(t_5)$, *where* $a,b \geq 0$ *and* $a+b < 1$.

(F_2) Let $u,v \geq 0$ be and $F(u,v,v,u,0) = \psi(u) - a\max\{\phi(u),\phi(v)\} \leq 0$. If $u > v$, then $\psi(u) - a\phi(v) \leq 0$, which implies $\psi(u) \leq a\phi(u) < \psi(u)$, a contradiction. Hence, $u \leq v$.
(F_3) Let $t,t' > 0$ and $F(t,t,0,0,t') = \psi(t) - a\phi(t) - b\phi(t') \leq 0$. If $t > t'$, then $\psi(t) - (a+b)\phi(t) \leq 0$, which implies $\psi(t) \leq (a+b)\phi(t) \leq \phi(t) < \psi(t)$, a contradiction. Hence, $t \leq t'$.

Example 7. $F(t_1, ..., t_5) = \psi(t_1) - a\phi(t_2) - b\max\{2\phi(t_3), \phi(t_4) + \phi(t_5)\}$, *where* $a, b \geq 0$ *and* $a + 2b < 1$.

(F_2) Let $u, v \geq 0$ be and $F(u, v, v, u, 0) = \psi(u) - a\phi(v) - b\max\{2\phi(v), \phi(u)\} \leq 0$. If $u > v$, then $\psi(u) \leq (a+2b)\phi(u) \leq \phi(u) < \psi(u)$, a contradiction. Hence, $u \leq v$.
(F_3) Let $t, t' > 0$ and $F(t, t, 0, 0, t') = \psi(t) - a\phi(t) - b\phi(t') \leq 0$. If $t > t'$, then $\psi(t) \leq (a+b)\phi(t) \leq \phi(t) < \psi(t)$, a contradiction. Hence, $t \leq t'$.

Example 8. $F(t_1, ..., t_5) = \psi(t_1) - a\phi(t_2) - b\max\{\phi(t_3) + \phi(t_4), 2\phi(t_5)\}$, *where* $a, b \geq 0$ *and* $a + 2b < 1$.

The proof is similar to the proof of Example 7.

4 Main results

Theorem 2. *Let* (X, G) *be a complete* G *- metric spaces and*

$$F\left(\begin{array}{c}\psi(G(fx, fy, fy)), \phi(G(x, y, y)), \phi(G(x, fx, fx)), \\ \phi(G(y, fy, fy)), \phi(G(y, fx, fx))\end{array}\right) \leq 0, \tag{2}$$

for all $x, y \in X$, $\psi \in \Psi$, $\phi \in \Phi$ *with* $\psi(t) > \phi(t)$ *for* $t > 0$.
Then f *has a unique fixed point.*

Proof. Let $x_0 \in X$ be and $x_n = fx_{n-1}$ for $n = 1, 2, ...$. If there exists n_0 such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of f. We suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, by (2) we obtain

$$F\left(\begin{array}{c}\psi(G(fx_{n-1}, fx_n, fx_n)), \phi(G(x_{n-1}, x_n, x_n)), \\ \phi(G(x_{n-1}, fx_{n-1}, fx_{n-1})), \\ \phi(G(x_n, fx_n, fx_n)), \phi(G(x_n, fx_{n-1}, fx_{n-1}))\end{array}\right) \leq 0,$$

$$F\left(\begin{array}{c}\psi(G(x_n, x_{n+1}, x_{n+1})), \phi(G(x_{n-1}, x_n, x_n)), \\ \phi(G(x_{n-1}, x_n, x_n)), \phi(G(x_n, x_{n+1}, x_{n+1})), 0\end{array}\right) \leq 0.$$

Since $\phi(G(x_n, x_{n+1}, x_{n+1})) < \psi(G(x_n, x_{n+1}, x_{n+1}))$, then by (F_1) we obtain

$$F\left(\begin{array}{c}\psi(G(x_n, x_{n+1}, x_{n+1})), \phi(G(x_{n-1}, x_n, x_n)), \\ \phi(G(x_{n-1}, x_n, x_n)), \psi(G(x_n, x_{n+1}, x_{n+1})), 0\end{array}\right) \leq 0.$$

By (F_2) we obtain

$$\psi(G(x_n, x_{n+1}, x_{n+1})) \leq \phi(G(x_{n-1}, x_n, x_n)) < \psi(G(x_{n-1}, x_n, x_n)). \tag{3}$$

Since ψ is nondecreasing we obtain

$$G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n).$$

Hence $\{G(x_n, x_{n+1}, x_{n+1})\}$ is a decreasing positive sequence and then $\{G(x_n, x_{n+1}, x_{n+1})\}$ is a convergent sequence. Hence, there exists $r \geq 0$ such that

$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = r$. We prove that $r = 0$. If $r > 0$, then letting n tend to infinity in the first part of (3) we obtain $\psi(r) \le \phi(r) < \psi(r)$, a contradiction. Hence, $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$. We prove that $\{x_n\}$ is a Cauchy sequence in X. Suppose that $\{x_{2n}\}$ is not a Cauchy sequence. By Lemma 3, there exists $\varepsilon > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that

$$\lim_{n\to\infty} G(x_{2n_k+1}, x_{2m_k}, x_{2m_k}) = \varepsilon,$$

$$\lim_{n\to\infty} G(x_{2n_k}, x_{2m_k-1}, x_{2m_k-1}) = \varepsilon,$$

$$\lim_{n\to\infty} G(x_{2m_k-1}, x_{2n_k+1}, x_{2n_k+1}) = \varepsilon.$$

By (2) we obtain

$$F\begin{pmatrix} \psi(G(fx_{2n_k}, fx_{2m_k-1}, fx_{2m_k-1})), \phi(G(x_{2n_k}, x_{2m_k-1}, x_{2m_k-1})), \\ \phi(G(x_{2n_k}, fx_{2n_k}, fx_{2n_k})), \\ \phi(G(x_{2m_k-1}, fx_{2m_k}, fx_{2m_k})), \phi(G(x_{2m_k-1}, fx_{2n_k}, fx_{2n_k})) \end{pmatrix} \le 0,$$

$$F\begin{pmatrix} \psi(G(x_{2n_k+1}, fx_{2m_k-1}, fx_{2m_k-1})), \phi(G(x_{2n_k}, x_{2m_k-1}, x_{2m_k-1})), \\ \phi(G(x_{2n_k}, x_{2n_k+1}, x_{2n_k+1})), \\ \phi(G(x_{2m_k-1}, x_{2m_k+1}, x_{2m_k+1})), \phi(G(x_{2m_k-1}, x_{2n_k+1}, x_{2n_k+1})) \end{pmatrix} \le 0.$$

Letting n tend to infinity we obtain

$$F(\psi(\varepsilon), \phi(\varepsilon), 0, 0, \phi(\varepsilon)) \le 0.$$

Since $\psi(\varepsilon) > \phi(\varepsilon)$, by (F_1) we obtain

$$F(\psi(\varepsilon), \psi(\varepsilon), 0, 0, \phi(\varepsilon)) \le 0.$$

By (F_3) we obtain

$$\psi(\varepsilon) \le \phi(\varepsilon) < \psi(\varepsilon),$$

a contradiction.

Hence $\{x_{2n}\}$ is a Cauchy sequence of (X, G), which implies that $\{x_n\}$ is a Cauchy sequence in (X, G). Since (X, G) is complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. We prove that u is a fixed point of f.

By (2) for $x = x_n$ and $y = u$ we obtain

$$F\begin{pmatrix} \psi(G(fx_n, fu, fu)), \phi(G(x_n, u, u)), \phi(G(x_n, fx_n, fx_n)), \\ \phi(G(u, fu, fu)), \phi(G(u, fx_n, fx_n)) \end{pmatrix} \le 0,$$

$$F\begin{pmatrix} \psi(G(x_{n+1}, fu, fu)), \phi(G(x_n, u, u)), \phi(G(x_n, x_{n+1}, x_{n+1})), \\ \phi(G(u, fu, fu)), \phi(G(u, x_{n+1}, x_{n+1})) \end{pmatrix} \le 0.$$

Letting n tend to infinity we obtain

$$F(\psi(G(u, fu, fu)), 0, 0, \phi(G(u, fu, fu)), 0) \le 0.$$

By (F_1) we obtain

$$F\left(\psi\left(G(u,fu,fu)\right),0,0,\psi\left(G(u,fu,fu)\right),0\right)\leq 0,$$

which implies $u = fu$ and u is a fixed point of f.

Suppose that there exists another fixed point $v \neq u$. By (2) for $x = u$ and $y = v$ we obtain

$$F\left(\psi\left(G(fu,fv,fv)\right),\psi\left(G(u,v,v)\right),0,0,\phi\left(G(v,fu,fu)\right)\right)\leq 0\ ,$$

$$F\left(\psi\left(G(u,v,v)\right),\phi\left(G(u,v,v)\right),0,0,\phi\left(G(v,u,u)\right)\right)\leq 0\ .$$

By (F_1) we obtain

$$F\left(\psi\left(G(u,v,v)\right),\psi\left(G(u,v,v)\right),0,0,\phi\left(G(v,u,u)\right)\right)\leq 0.$$

By (F_3) we have

$$\psi\left(G(u,v,v)\right)\leq\phi\left(G(v,u,u)\right).$$

Similarly we obtain

$$\psi\left(G(v,u,u)\right)\leq\phi\left(G(u,v,v)\right).$$

Then

$$\psi\left(G(u,v,v)\right)\leq\phi\left(G(v,u,u)\right)\leq\psi\left(G(v,u,u)\right)\leq\phi\left(G(u,v,v)\right)<\psi\left(G(u,v,v)\right),$$

a contradiction. Hence, $u = v$ and u is the unique fixed point of f. □

Corollary 1. *Let (X,G) be a complete G - metric spaces and $f : X \to X$ be a mapping. If there exists $\psi \in \Psi$ and $\phi \in \Phi$ with $\psi(t) > \phi(t)$ for $t > 0$, such that*

$$\begin{aligned}&\psi\left(G(fx,fy,fz)\right)\\ \leq\ &\phi\left(\max\left\{G\left(x,y,y\right),G\left(x,fx,fx\right),G\left(y,fy,fy\right),G\left(y,fx,fx\right)\right\}\right)\\ =\ &\max\left\{\phi\left(G(x,y,y)\right),\phi\left(G(x,fx,fx)\right),\phi\left(G(y,fy,fy)\right),\phi\left(G(y,fx,fx)\right)\right\}\end{aligned}$$

for all $x, y \in X$, then f has a unique fixed point.

Proof. The proof it follows by Theorem 2, Example 2 and by the fact that ϕ is nondecreasing. □

Example 9. *Let $X = [0,\infty)$ and $G : X^3 \to \mathbb{R}_+$ be a G - metric on X defined by $G(x,y,z) = \max\{|x-y|,|x-z|,|y-z|\}$, for all $x,y,z \in X$. Then (X,G) is a complete metric space. Let $\psi(t) = t$, $\phi(t) = \frac{3}{4}t$, then $\psi(t) \in \psi$, $\phi(t) \in \phi$ and $\phi(t) < \psi(t)$, for all $t > 0$. Let $T : (X,G) \to (X,G)$ with $Tx = \frac{1}{2}x$. Then*

$$G\left(Tx,Ty,Ty\right)=\left|Tx-Ty\right|=\frac{1}{2}\left|x-y\right|$$

and

$$G(x,y,y) = |x-y|.$$

Hence

$$\begin{aligned} G(Tx,Ty,Ty) &= \frac{1}{2}|x-y| \le \frac{3}{4}|x-y| = \frac{3}{4}G(x,y,y) \\ &\le \frac{3}{4}\max\left\{ \begin{array}{c} G(x,y,y), G(x,Tx,Tx), \\ G(y,Ty,Ty), G(y,Tx,Tx) \end{array} \right\}. \end{aligned}$$

Hence,

$$\Psi(G(Tx,Ty,Ty)) \le \phi\left(\max\left\{ \begin{array}{c} G(x,y,y), G(x,Tx,Tx), \\ G(y,Ty,Ty), G(y,Tx,Tx) \end{array} \right\}\right).$$

By Corollary 1, f has a unique fixed point $x = 0$.

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