Bulletin of the *Transilvania* University of Braşov • Vol 12(61), No. 2 - 2019
Series III: Mathematics, Informatics, Physics, 409-418
https://doi.org/10.31926/but.mif.2019.12.61.2.18

CONFORMAL CHANGE OF FINSLER SPACE WITH (α, β)-METRIC

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Abstract

In this paper, we discuss the necessary and sufficient conditions for a Finsler space with (α, β)-metric of type $L = \kappa(\alpha + \beta) + \epsilon\frac{\beta^2}{\alpha}$ (where κ and ϵ are non zero constants) which is the sum of constant multiple of Randers metric and metric $\frac{\beta^2}{\alpha}$ to be a Douglas space and also to be a Berwald space, where α is Riemannian metric and β is differential 1-form. In second part of this paper, we discuss about conformal change of Douglas space with (α, β)-metric.

2010 *Mathematics Subject Classification:* 53B40, 53C60.
Key words: Finsler space; (α, β)-metric; conformal change; Douglas space; Berwald space.

1 Introduction

M. Matsumoto [7] introduced the concept of (α, β)-metric on a differentiable manifold M^n, where $\alpha^2 = a_{ij}(x)y^iy^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. The Matsumoto metric is an interesting (α, β)-metric introduced by using the gradient of slope, speed and gravity [8]. This metric formulates the model of a Finsler space. Many authors [1, 8, 12] studied this metric by different perspectives. The theory of Finsler space with (α, β)-metric has been developed into the faithful branch of Finsler geometry. Finsler space with (α, β)-metric was studied by many authors and it is an quite old concept, but it is a very important aspect of Finsler geometry and its application to physics. S. Bacso and Matsumoto [3] introduced the notion of Douglas space as a generalization Berwald space from the view point of geodesic equation. M. Matsumoto [9] obtained the condition for some Finsler space with an (α, β)-metric to be Douglas space. H. S. Park and E. S. Choi [12]

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worked on Finsler space with the 2^{nd} approximate Matsumoto metric. H. S. Park and E. S. Choi [13] worked on Finsler space with an approximate Matsumoto metric of Douglas type.

The conformal change was introduced by M. S. Kneblman [6] and deeply investigated by many authors. The conformal theory of Finsler metric based on the theory of Finsler space was developed by M. Matsumoto and M. Hasiguchi [4, 10] and studied the conformal change of a Finsler metric. S. K. Narasimhamurthy [11] worked on conformal change of Douglas space with special (α, β)-metric. In this paper, we discuss the necessary and sufficient conditions for a Finsler space with (α, β)-metric of type $L = \kappa(\alpha + \beta) + \epsilon\frac{\beta^2}{\alpha}$ (where κ and ϵ are constants) which is the sum of constant multiple of Randers metric and metric $\frac{\beta^2}{\alpha}$ to be a Douglas space and also to be a Berwald space, where α is Riemannian metric and β is differential 1-form. In second part of this paper, we discuss about conformal change of Douglas space with (α, β)-metric.

2 Preliminaries

Definition 1. *A Finsler metric on M is a function $L : TM \to [0, \infty)$ with the following properties:*

- *L is C^∞ on TM_0,*
- *L is positively 1-homogeneous on the fiber of tangent bundle TM,*
- *the Hessian of F^2 with element $g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is regular on TM_0, i.e., $det(g_{ij}) \neq 0$.*

The pair (M^n, L) is called a Finsler space. L is called fundamental function and g_{ij} is called fundamental tensor.

Definition 2. *Let (M^n, L) be a Finsler space, where M^n is an n-dimensional C^∞ manifold and $L(x, y)$ is a Finsler metric function. If $\sigma(x)$ is a function in each coordinate neighborhood of M^n, the change $L(x, y) \to e^{\sigma(x)}\overline{L}(x, y)$ is called a conformal change.*

Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ be a Riemannian metric, $\beta = b_i y^i$ is a 1-form and let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where ϕ is a positive C^∞ function defined in a neighborhood of the origin $s = 0$. It is well known that $F = \alpha\phi\left(\frac{\beta}{\alpha}\right)$ is a Finsler metric for any α and β with $b = \|\beta\|_\alpha < b_0$ if and only if

$$\phi(s) > 0,\ \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0,\ (|s| \leq b < b_0).$$

For a given Finsler metric $L = L(x, y)$, the geodesic of L is given by

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0, \tag{1}$$

where $G^i = G^i(x,y)$ are called the geodesic coefficients, which are given by

$$G^i = \frac{g^{il}}{4}\Big\{[L^2]_{x^m y^l} y^m - [L^2]_{x^l}\Big\}. \tag{2}$$

A Finsler space F^n is said to be Douglas space [3] if

$$D^{ij} = G^i(x,y)y^j - G^j(x,y)y^i, \tag{3}$$

where D^{ij} are homogeneous polynomial in (y^i) of degree 3. In [3] it is proved that the Finsler space F^n is of Douglas type if and only if the Douglas tensor

$$D^h_{ijk} = C^h_{ijk} - \frac{1}{n+1}(G_{ijk}y^h + G_{ij}\delta^h_k + G_{jk}\delta^h_i + G_{ki}\delta^h_j)$$

vanishes identically, where $G^h_{ijk} = \dot\partial_k G^h_{ij}$ is the hv-curvature tensor of the Berwald connection $B\Gamma$.

Let $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ be the Cartan tensor. Consider the Finsler space $F^n = (M^n, L)$ equipped with an (α,β)-metric $L(\alpha,\beta)$. Let γ^i_{jk} denote the Christoffel symbols in the Riemannian space (M^n,α). Denote $b_{i|j}$ the covariant derivative of the vector field b_i with respect to Riemannian connection γ^i_{jk}, i.e., $b_{i|j} = \frac{\partial b_i}{\partial x^j} - b_k\gamma^k_{ij}$. Let $\nabla\beta = b_{i|j}dx^i \otimes dx^j$ be the covariant derivative of β with respect to α. Denote

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}) \;\; s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}).$$

β is closed if and only if $s_{ij} = 0$ [14]. Let $s_j = b^i s_{ij}$, $s^i_j = a^{il}s_{lj}$, $s_0 = s_i y^i$, $s^i_0 = s^i_j y^j$ and $r_{00} = r_{ij}y^i y^j$.

The functions G^i of F^n with an (α,β)-metric are written in the form [7]

$$2G^i = \gamma^i_{00} + 2B^i, \tag{4}$$

$$B^i = \frac{\alpha L_\beta}{L_\alpha}s^i_0 + C^*\left\{\frac{\beta L_\beta}{\alpha L}y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}\left(\frac{1}{\alpha}y^i - \frac{\alpha}{\beta}b^i\right)\right\}, \tag{5}$$

provided $\beta^2 + L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0$, where $\gamma^2 = b^2\alpha^2 - \beta^2$ and $C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}$. The subscript 0 means contraction by y^i. We shall denote the homogeneous polynomials in (y^i) of degree r by $hp(r)$ for brevity.
From (4) the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ of F^n with an (α,β)-metric is given by [7]

$$G^i_j = \dot\partial_j G^i = \gamma^i_{0j} + B^i_j, \tag{6}$$

$$G^i_{jk} = \dot\partial_k G^i_j = \gamma^i_{jk} + B^i_{jk}, \tag{7}$$

where $B^i_j = \dot\partial_j B^i$ and $B^i_{jk} = \dot\partial_k B^i_j$. On account of [7], B^i_{jk} is determined by

$$L_\alpha B^k_{ji}y^j y_t + \alpha L_\beta(B^k_{ji}b_t - b_{j;i})y^j = 0, \tag{8}$$

where $y_k = a_{ik}y^i$.

A Finsler space F^n with an (α, β)-metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ is $hp(3)$ [3].
From (5) B^{ij} is written as follows

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^*(b^i y^j - b^j y^i). \tag{9}$$

Lemma 1. *[5] If $\alpha^2 \equiv 0(mod \beta)$, i.e., $a_{ij}(x)y^i y^j$ contains $b_i y^i$ as a factor, then the dimension is equal to 2 and b^2 vanishes. In this case, we have 1-form $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

3 Finsler Space with (α, β)-metric of Berwald Type

In this section, we establish a condition for a Finsler space F^n with the metric $L = \kappa(\alpha + \beta) + \epsilon\frac{\beta^2}{\alpha}$ to be Berwald space.
For the (α, β)-metric $L = \kappa(\alpha + \beta) + \epsilon\frac{\beta^2}{\alpha}$, we have

$$\begin{cases} L_\alpha = \kappa - \epsilon\frac{\beta^2}{\alpha^2}, \\ L_\beta = \kappa + 2\epsilon\frac{\beta}{\alpha}, \\ L_{\alpha\alpha} = 2\epsilon\frac{\beta^2}{\alpha^3}. \end{cases} \tag{10}$$

In view of (8) and (10), we have

$$(\kappa\alpha^2 - \epsilon\beta^2)B_{jki}y^j y^k + \alpha^2(\kappa\alpha + 2\epsilon\beta)(B_{jki}b^k - b_{j|i})y^j = 0, \tag{11}$$

where $B_{jki} = a_{kr}B^r_{ji}$.

According to [2], we suppose that F^n is a Berwald space, then B^i_{jk} and $b_{j|i}$ are functions of position alone. Then (11) is separated as rational and irrational terms in (y^i) as follows

$$\begin{aligned} &(\kappa\alpha^2 - \epsilon\beta^2)B_{jki}y^j y^k + 2\epsilon\alpha^2\beta(B_{jki}b^k - b_{j|i})y^j \\ &+ \alpha\left[\kappa\alpha^2(B_{jki}b^k - b_{j|i})y^j\right] = 0, \end{aligned} \tag{12}$$

which yields two equations

$$(\kappa\alpha^2 - \epsilon\beta^2)B_{jki}y^j y^k + 2\epsilon\alpha^2\beta(B_{jki}b^k - b_{j|i})y^j = 0, \tag{13}$$

$$\kappa\alpha^2(B_{jki}b^k - b_{j|i})y^j = 0. \tag{14}$$

Using (14) in (13), we have $B_{jki}y^j y^k = 0$ and hence $B_{jki} + B_{kji} = 0$. Since B_{jki} is symmetric in (j, i), we get $B_{jki} = 0$ easily and from (13) or (14), we have

$$b_{j|i} = 0. \tag{15}$$

Conversely, if $b_{j|i} = 0$, then $B_{jki} = 0$ are uniquely determined from (11).
Thus we have

Theorem 1. *A Finsler space with (α, β)-metric of type $L = \kappa(\alpha+\beta)+\epsilon\frac{\beta^2}{\alpha}$, where κ and ϵ are non zero constants, is a Berwald space if and only if $b_{j|i} = 0$.*

4 Finsler Space with (α, β)-metric of Douglas Type

In this section, we give a necessary and sufficient condition for a Finsler space F^n with a (α, β)-metric to be Douglas space. In view of (9) and (10), we have

$$\begin{aligned}&(\kappa\alpha^2 - \epsilon\beta^2)\Big\{\alpha^2(\kappa + 2\epsilon b^2) - 3\epsilon\beta^2\Big\}B^{ij} - \alpha^2(\kappa\alpha + 2\epsilon\beta)\\ &\times \Big\{\alpha^2(\kappa + 2\epsilon b^2) - 3\epsilon\beta^2\Big\}(s^i_0 y^j - s^j_0) - \epsilon\alpha^2\Big\{r_{00}(\kappa\alpha^2 - \epsilon\beta^2)\\ &- 2\alpha^2 s_0(\kappa\alpha + 2\epsilon\beta)\Big\}(b^i y^j - b^j y^i) = 0\end{aligned} \tag{16}$$

Suppose that F^n is a Douglas space, then B^{ij} are $hp(3)$. Separating the rational and irrational terms of y^i in (16), gives

$$\begin{aligned}&\Big\{\alpha^2(\kappa + 2\epsilon b^2) - 3\epsilon\beta^2\Big\}\Big[(\kappa\alpha^2 - \epsilon\beta^2)B^{ij} - 2\epsilon\alpha^2\beta(s^i_0 y^j - s^j_0 y^i)\Big]\\ &- \alpha^2\Big\{\epsilon r_{00}(\kappa\alpha^2 - \epsilon\beta^2) - 4\epsilon^2\alpha^2\beta s_0\Big\}(b^i y^j - b^j y^i)\\ &- \alpha\Big[\kappa\alpha^2\Big\{\alpha^2(\kappa + 2\epsilon b^2) - 3\epsilon\beta^2\Big\}(s^i_0 y^j - s^j_0 y^i) - 2\kappa\epsilon\alpha^4 s_0(b^i y^j - b^j y^i)\Big] = 0,\end{aligned} \tag{17}$$

or

$$U + \alpha V = 0, \tag{18}$$

where

$$\begin{aligned}U = &\Big\{\alpha^2(\kappa + 2\epsilon b^2) - 3\epsilon\beta^2\Big\}\Big[(\kappa\alpha^2 - \epsilon\beta^2)B^{ij} - 2\epsilon\alpha^2\beta(s^i_0 y^j - s^j_0 y^i)\Big]\\ &- \alpha^2\Big\{\epsilon r_{00}(\kappa\alpha^2 - \epsilon\beta^2) - 4\epsilon^2\alpha^2\beta s_0\Big\}(b^i y^j - b^j y^i),\end{aligned} \tag{19}$$

$$V = \alpha^2\Big\{\alpha^2(\kappa + 2\epsilon b^2) - 3\epsilon\beta^2\Big\}(s^i_0 y^j - s^j_0 y^i) - 2\epsilon\alpha^4 s_0(b^i y^j - b^j y^i). \tag{20}$$

The left hand side of (18) is a polynomial in y^i, such that U and V are rational in y^i and α is irrational. Therefore we must have

$$U = 0 \quad \textit{and} \quad V = 0$$

which implies that

$$\begin{aligned}&\Big\{\alpha^2(\kappa + 2\epsilon b^2) - 3\epsilon\beta^2\Big\}\Big[(\kappa\alpha^2 - \epsilon\beta^2)B^{ij} - 2\epsilon\alpha^2\beta(s^i_0 y^j - s^j_0 y^i)\Big]\\ &- \alpha^2\Big\{\epsilon r_{00}(\kappa\alpha^2 - \epsilon\beta^2) - 4\epsilon^2\alpha^2\beta s_0\Big\}(b^i y^j - b^j y^i) = 0,\end{aligned} \tag{21}$$

$$\alpha^2\Big\{\alpha^2(\kappa + 2\epsilon b^2) - 3\epsilon\beta^2\Big\}(s^i_0 y^j - s^j_0 y^i) - 2\epsilon\alpha^4 s_0(b^i y^j - b^j y^i) = 0. \tag{22}$$

Using (22) in (21), we get

$$\Big[\alpha^2(\kappa+2\epsilon b^2)-3\epsilon\beta^2\Big]B^{ij}-\alpha^2\epsilon r_{00}(b^iy^j-b^jy^i)=0. \tag{23}$$

Only the term $3\epsilon\beta^4B^{ij}$ of (23) does not contain α^2. Hence, we must have v_5^{ij} of $hp(5)$, satisfying

$$3\epsilon\beta^4B^{ij}=\alpha^2v_5^{ij}. \tag{24}$$

Case-(i)**:** $\alpha^2\not\equiv 0(mod\ \beta)$
In this case (24) reduces to $B^{ij}=\alpha^2v^{ij}$, where v^{ij} are $hp(1)$. Thus (23), gives

$$\Big[\alpha^2(\kappa+2\epsilon b^2)-3\epsilon\beta^2\Big]v^{ij}-r_{00}(b^iy^j-b^jy^i)=0. \tag{25}$$

Transvecting (25) by b_iy_j and using $y_j=a_{jk}y^k$, we get

$$\alpha^2\Big\{(\kappa+2\epsilon b^2)v^{ij}b_iy_j-b^2r_{00}\Big\}=\beta^2(3\epsilon v^{ij}b_iy_j-r_{00}). \tag{26}$$

Since $\alpha^2\not\equiv 0(mod\ \beta)$, there exists a function $h(x)$ satisfying

$$(\kappa+2\epsilon b^2)v^{ij}b_iy_j-b^2r_{00}=h(x)\beta^2, \tag{27}$$

and

$$3\epsilon v^{ij}b_iy_j-r_{00}=h(x)\alpha^2. \tag{28}$$

Eliminating $v^{ij}b_iy_j$ from (27) and (28), we have

$$(\epsilon b^2-\kappa)r_{00}=h(x)\Big\{(\kappa+2\epsilon b^2)\alpha^2-3\epsilon\beta^2\Big\}. \tag{29}$$

From (29), we get

$$b_{i|j}=k\Big\{(\kappa+2\epsilon b^2)a_{ij}-3\epsilon b_ib_j\Big\}, \tag{30}$$

where $k=\frac{h(x)}{\epsilon b^2-\kappa}$. Here $h(x)$ is a scalar function.
Conversely, if (30) holds, then $s_{ij}=0$ and we get (29). Therefore (16) is written as follows

$$B^{ij}=k\Big\{\alpha^2(b^iy^j-b^jy^i)\Big\}, \tag{31}$$

where B^{ij} are $hp(3)$, i.e., F^n is a Douglas space.
Case-(ii) $\alpha^2\equiv 0(mod\ \beta)$
In this case $\alpha^2=\delta\beta$, $b^2=0$ and dimension two by Lemma 1. Therefore (24) reduces to $B^{ij}=\delta W_2^{ij}$, where w^{ij_2} are $hp(2)$. Hence (22) leads to

$$2\epsilon\delta s_0(b^iy^j-b^jy^i)-(\kappa\delta-3\epsilon\beta)(s_0^iy^j-s_0^jy^i)=0. \tag{32}$$

Transvecting (32) by $b_j y_i$, we have $s_0 = 0$. Using $s_0 = 0$ in (32), we get

$$(s^i_0 y^j - s^j_0 y^i) = 0. \tag{33}$$

Transvecting (33) by y_j, we get $s^i_0 = 0$, implies $s_{ij} = 0$. Therefore (23) reduces to

$$(\kappa\delta - 3\epsilon\beta)w^{ij}_2 - \epsilon r_{00}(b^i y^j - b^j y^i) = 0. \tag{34}$$

Transvecting (34) by $b_i y_j$, we get

$$(\kappa\delta - 3\epsilon\beta)w^{ij}_2 b_i y_j + \epsilon r_{00}\beta^2 = 0,$$

which is written as

$$\kappa\delta w^{ij}_2 b_i y_j = \beta(3\epsilon w^{ij}_2 b_i y_j - \epsilon\beta r_{00}). \tag{35}$$

Therefore, there exists an $hp(2)$, $\lambda = \lambda_{ij} y^i y^j$ such that

$$w^{ij}_2 b_i y_j = \beta\lambda, \quad 3\epsilon w^{ij}_2 b_i y_j - \epsilon\beta r_{00} = \kappa\delta\lambda.$$

Eliminating $w^{ij}_2 b_i y_j$ from the above equations, we get

$$\epsilon\beta r_{00} = \lambda(3\epsilon\beta - \kappa\delta), \tag{36}$$

which implies that there exists an $hp(1)$, $v_0 = v_i(x)y^i$, such that

$$r_{00} = v_0(3\epsilon\beta - \kappa\delta), \quad \lambda = \epsilon v_0 \beta. \tag{37}$$

From (37) and $s_{ij} = 0$, we get

$$b_{i|j} = \frac{1}{2}\Big\{ v_i(3\epsilon b_j - \kappa d_j) + v_j(3\epsilon b_i - \kappa d_i) \Big\}, \tag{38}$$

where b_i is the gradient vector.
Conversely, if (38) holds, then $s_{ij} = 0$ and $r_{00} = v_0(3\epsilon\beta - \kappa\delta)$. Therefore, (16) written as follows

$$B^{ij} = -v_0\delta(b^i y^j - b^j y^i), \tag{39}$$

which are $hp(3)$. Therefore F^n is a Douglas space.
Thus, we have

Theorem 2. *A Finsler space with a (α, β)-metric of type $L = \kappa(\alpha + \beta) + \epsilon\frac{\beta^2}{\alpha}$, where κ and ϵ are constants, is a Douglas space if and only if*

- *$\alpha^2 \not\equiv 0 (mod\ \beta)$, $b^2 \neq \frac{\kappa}{\epsilon}$; $b_{i|j}$ is written in the form* (30)*,*
- *$\alpha^2 \equiv 0 (mod\ \beta)$, $n = 2$; $b_{i|j}$ is written in the form* (38)*,*

where $\alpha^2 = \delta\beta$, $\delta = d_i(x)y^i$, $v_0 = v_i(x)y^i$

5 Conformal Change of Douglas Space with (α, β)-metric

Let $F^n = (M^n, L)$ and $\overline{F}^n = (M^n, \overline{L})$ be two Finsler spaces on the same underlying manifold M^n If the angle in F^n is equal to that in $\overline{F}^n$ for any tangent vectors, then F^n is called conformal to $\overline{F}^n$ and the change $L \to \overline{L}$ of the metric is called a conformal change. In other words, if there exists a scalar function $\sigma = \sigma(x)$ such that $\overline{L} = e^\sigma L$, then the change is called conformal change.

For an (α, β)-metric $\overline{L} = e^\sigma L(\alpha, \beta)$ is equivalent to $\overline{L} = (e^\sigma \alpha, e^\sigma \beta)$ by homogeneity. Therefore, according to [4]:

$$\overline{a_{ij}} = e^{2\sigma} a_{ij}, \;\; \overline{b}_i = e^\sigma b_i, \;\; \overline{a}^{ij} = e^{-2\sigma} a^{ij}, \;\; \overline{b}^i = e^{-\sigma} b^i, \;\; b^2 = a^{ij} b_i b_j = \overline{a}^{ij} \overline{b}_i \overline{b}_j. \quad (40)$$

From (40), it follows that, the conformal change of Chritoffel symbols is given by

$$\overline{\gamma}^i_{jk} = \gamma^i_{jk} + \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma^i a_{jk}, \quad (41)$$

where $\sigma_j = \partial_j \sigma$ and $\sigma^i = a^{ij} \sigma_j$. From (41) and (5), we get the following conformal change:

$$\begin{cases} \overline{b}_{i|j} = e^\sigma (b_{i|j} + \rho a_{ij} - \sigma_i b_j), \\ \overline{r}_{ij} = e^\sigma [r_{ij} + \rho a_{ij} - \frac{1}{2}(b_i \sigma_j + b_j \sigma_i)], \\ \overline{s}_{ij} = e^\sigma [s_{ij} + \frac{1}{2}(b_i \sigma_j - b_j \sigma_i)], \\ \overline{s}^i_j = e^{-\sigma} [s^i_j + \frac{1}{2}(b^i \sigma_j - b_j \sigma^i)], \\ \overline{s}_j = s_j + \frac{1}{2}(b^2 \sigma_j - \rho b_j), \end{cases} \quad (42)$$

where $\rho = \sigma_r b^r$.

Since a Finsler space with (α, β)-metric of type $L = \kappa(\alpha + \beta) + \epsilon \frac{\beta^2}{\alpha}$ is a Douglas space if and only if

$$b_{i|j} = k \left\{ (\kappa + 2\epsilon b^2) a_{ij} - 3\epsilon b_i b_j \right\}.$$

By [15], for a conformal change, Finsler space with the (α, β)-metric of type $L = \kappa(\alpha + \beta) + \epsilon \frac{\beta^2}{\alpha}$ is a Douglas space if and only if there exists a function $k(x)$ such that $H_{ij} = 0$, where

$$H_{ij} = b_{i|j} - k \left\{ (\kappa + 2\epsilon b^2) a_{ij} - 3\epsilon b_i b_j \right\}. \quad (43)$$

From (40), (42) and (43), we get

$$\begin{aligned} \overline{H}_{ij} &= \overline{b}_{i|j} - \overline{k} \left\{ (\kappa + 2\epsilon \overline{b}^2) a_{ij} - 3\epsilon \overline{b}_i \overline{b}_j \right\} \\ &= e^\sigma \left[b_{i|j} - k \left\{ (\kappa + 2\epsilon b^2) a_{ij} - 3\epsilon b_i b_j \right\} + \rho a_{ij} - \sigma_i b_j \right], \end{aligned} \quad (44)$$

where $\overline{k} = e^{-\sigma}$. From (44), we get

$$\overline{H}_{ij} = e^\sigma [H_{ij} + \rho a_{ij} - \sigma_i b_j]. \quad (45)$$

Hence, the Douglas space with the metric $L = \kappa(\alpha + \beta) + \epsilon\frac{\beta^2}{\alpha}$ is conformally transformed to a Douglas space if and only if $a_{ij} = \sigma_i b_j$ i.e.,

$$\rho a_{ij} = \frac{1}{2}(\sigma_i b_j + \sigma_j b_i). \tag{46}$$

Transvecting (46) by b^j, we have

$$\rho b_i = \sigma_i b^2. \tag{47}$$

In view of (46) and (47), we have

$$a_{ij} = \frac{1}{b^2} b_i b_j. \tag{48}$$

Transvecting (48) with $y^i y^j$, we get $b^2\alpha^2 = \beta^2$. If $\alpha^2 \not\equiv 0 (mod\ \beta)$, then (46) is possible only when $\rho = 0$ and $\sigma_i = 0$. Thus, the transformation is homothetic. Then we state:

Theorem 3. *If $\alpha^2 \not\equiv 0(mod\ \beta)$, then a Douglas space with a (α, β)-metric of type $L = \kappa(\alpha + \beta) + \epsilon\frac{\beta^2}{\alpha}$ is accordingly transformed to a Douglas space if and only if the transformation is homothetic.*

6 Conclusion

An n-dimensional Finsler space is a Douglas space or Douglas type if and only if the Douglas tensor vanishes identically. Also, it is well known that a Douglas space is a generalization of a Berwald space from the view point of a geodesic equation. In Finsler Geometry, we generalized the various types of changes; conformal change, c-conformal change, Randers change, β-conformal change etc. The important examples of Finsler space are different types of (α, β)-metric, Randers metric, Kropina metric and other special (α, β)-metric. Many authors have shown the condition for the above spaces to be a Douglas space or Douglas type.

In this paper, we consider one of the (α, β)-metrics of type $L = \kappa(\alpha+\beta)+\epsilon\frac{\beta^2}{\alpha}$, in the first step we prove that L is Douglas type. Further, we apply the conformal change and obtain $\overline{L}$ is a Douglas metric if and only if the conformal change is homothetic.

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