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SOME NEW FIXED POINT THEOREMS FOR $\mathcal{F}$-METRIC SPACES

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Abstract

The present paper introduces some new fixed point and coupled fixed point theorems for $\mathcal{F}$-metric spaces. We introduce a new fixed point theorem, a Kannan type theorem for asymptotically regular operators and basic coupled fixed point theorems along with some consequences.

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1 Introduction

Over the past years, generalizations of the metric space concept have been intensively studied. Starting with the emergence of *b-metric* spaces in Czerwik paper [3], many more new concepts of extensions and generalizations of metric spaces have appeared. Fagin et al. introduced the notion of s-$relaxed_p$ metric in [5]. There have also been generalizations on an $X \times X \times X$ product set, like Gähler's *2-metric* in [6], and in Dhage's paper [4] where the *D-metric* is defined.

Jleli and Samet [9] have recently introduced a new type of metric space, called an $\mathcal{F}$-metric space. Thereby, let $\mathcal{F}$ be the set of functions $f : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

($\mathcal{F}_1$) f is non-decreasing, i.e. $0 < s < t \Rightarrow f(s) \leq f(t)$;

($\mathcal{F}_2$) for every sequence $\{t_n\} \subset (0, \infty)$, we have

$$\lim_{n\to\infty} t_n = 0 \Leftrightarrow \lim_{n\to\infty} f(t_n) = -\infty.$$

The concept of metric space is thus generalized as follows.

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Definition 1. *Let X be a nonempty set, and let $D : X \times X \to [0, \infty)$ be a given mapping. Suppose that there exists $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ such that*

(D_1) $(x, y) \in X \times X$, $D(x, y) = 0 \Leftrightarrow x = y$;

(D_2) $D(x, y) = D(y, x)$, $\forall (x, y) \in X \times X$;

(D_3) *For every $(x, y) \in X \times X$, for every $N \in \mathbb{N}$, $N \geq 2$ and for every $(u_i)_{i=1}^{N} \subset X$ with $(u_1, u_N) = (x, y)$, we have*

$$D(x, y) > 0 \Rightarrow f(D(x, y)) \leq f\left(\sum_{i=1}^{N-1} D(x_i, x_{i+1})\right) + \alpha.$$

Then D is said to be an $\mathcal{F}$-metric on X, and the pair (X, D) is said to be an $\mathcal{F}$-metric space.

2 Main Results

2.1 A new fixed point theorem for $\mathcal{F}$-metric spaces

Based on the results for b-metric spaces by Huang, Deng and Radenovic from [7], we give a fixed point theorem for $\mathcal{F}$-metric spaces which involves one similar inequality.

Theorem 1. *Let (X, D) be a complete bounded $\mathcal{F}$-metric space and $T : X \to X$ be an operator such that there exists $\lambda_1, \lambda_2, \lambda_3 \in (0, 1)$ for which:*

$$D(Tx, Ty) \leq \lambda_1 D(x, y) + \lambda_2 \frac{D(x, Ty)D(y, Tx)}{1 + D(x, y)} + \lambda_3 \frac{D(x, Tx)D(x, Ty)}{1 + D(x, y)} \tag{1}$$

$\forall\, x, y \in X$. Then, T has a fixed point.

Proof. Let $f \in \mathcal{F}$ and $\alpha \in [0, \infty)$ such that (D_3) is satisfied. By $\mathcal{F}_2$, for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < t < \delta \Rightarrow f(t) < f(\varepsilon) < f(\varepsilon) + \alpha \Rightarrow f(t) < f(\varepsilon) - \alpha. \tag{2}$$

For $x_n = Tx_{n-1} = T^n x_o$, by (1), we have

$$D(Tx_n, Tx_{n-1}) = D(x_{n+1}, x_n) \leq$$

$$\lambda_1 D(x_n, x_{n-1}) + \lambda_2 \frac{D(x_n, Tx_{n-1})D(x_{n-1}, Tx_n)}{1 + D(x_n, x_{n-1})} + \lambda_3 \frac{D(x_n, Tx_n)D(x_n, Tx_{n-1})}{1 + D(x_n, x_{n-1})}$$

$$\leq \lambda_1 D(x_n, x_{n-1}) \leq ... \leq \lambda_1^n D(x_1, x_0).$$

Thus we have

$$\sum_{i=n}^{m-1} D(x_i, x_{i+1}) \leq \frac{\lambda_1^n}{1 - \lambda_1} D(x_0, x_1), \quad m > n \tag{3}$$

and since

$$\lim_{n\to\infty} \frac{\lambda_1^n}{1-\lambda_1} D(x_0, x_1) = 0$$

there exists an $N \in \mathbb{N}$ such that

$$0 \leq \frac{\lambda_1^n}{1-\lambda_1} D(x_0, x_1) \leq \delta, \quad n \geq N. \tag{4}$$

From (2) and (4) we get

$$f\left(\sum_{i=n}^{m-1} D(x_i, x_{i+1})\right) \leq f\left(\frac{\lambda_1^n}{1-\lambda_1} D(x_0, x_1)\right) \leq f(\varepsilon) - \alpha, \quad m > n \geq N. \tag{5}$$

Using (D_3) we have for $D(x_n, x_m) > 0, m > n \geq N$

$$f(D(x_n, x_m)) \leq f\left(\sum_{i=n}^{m+1} D(x_i, x_{i+1})\right) + \alpha < f(\varepsilon).$$

Thus, we get $D(x_n, x_m) < \varepsilon$ for $n > m \geq N$ which proves that the sequence $\{x_n\}$ is $\mathcal{F}$-Cauchy, and because X is $\mathcal{F}$-complete, then $\{x_n\}$ is $\mathcal{F}$-convergent, so there exists $x^* \in X$ such that $\lim\limits_{n\to\infty} x_n = x^*$.

We prove that x^* is a fixed point of T. If we suppose that $D(Tx^*, x^*) > 0$ then

$$f(D(Tx^*, x^*)) \leq f(D(Tx^*, Tx_n) + D(Tx_n, x^*)) + \alpha$$

and using the inequality from (1) we have

$$f(D(Tx^*, x^*)) \leq f\left(\lambda_1 D(x_n, x^*) + \lambda_2 \frac{D(x^*, Tx_n) D(x_n, Tx^*)}{1 + D(x_n, x^*)} + \right.$$

$$\left. \lambda_3 \frac{D(x^*, Tx^*) D(x^*, Tx_n)}{1 + D(x_n, x^*)} + D(Tx_n, x^*) + \alpha \right).$$

Since $\lim\limits_{n\to\infty} D(x_n, x^*) = 0$ and $Tx_n = x_{n+1}$ we get

$$\lim_{n\to\infty} f\left(\lambda_1 D(x_n, x^*) + \lambda_2 \frac{D(x^*, Tx_n) D(x_n, Tx^*)}{1 + D(x_n, x^*)} + \right.$$

$$\left. \lambda_3 \frac{D(x^*, Tx^*) D(x^*, Tx_n)}{1 + D(x_n, x^*)} + D(Tx_n, x^*) \right) + \alpha = -\infty$$

and thus $D(x^*, Tx^*) = 0$ which implies $x^* = Tx^*$, meaning that x^* is a fixed point of T. □

Remark 1. *Moreover, if $\lambda_1 + \lambda_2 < 1$, the fixed point is unique.*

Proof. We suppose that there exists $x^*,\ y^* \in X$ such that $Tx^* = x^*$ and $Ty^* = y^*$.
We get

$$D(Tx^*, Ty^*) = D(x^*, y^*) \leq \lambda_1 D(x^*, y^*) + \lambda_2 \frac{D(x^*, Ty^*)D(y^*, Tx^*)}{1 + D(x^*, y^*)} +$$

$$+\lambda_3 \frac{D(x^*, Tx^*)D(x^*, Ty^*)}{1 + D(x^*, y^*)}$$

and so

$$D(x^*, y^*) \leq \lambda_1 D(x^*, y^*) + \lambda_2 \frac{D(x^*, y^*)D(y^*, x^*)}{1 + D(x^*, y^*)} + \lambda_3 \frac{D(x^*, x^*)D(x^*, y^*)}{1 + D(x^*, y^*)}$$

and grouping by $D(x^*, y^*)$

$$D(x^*, y^*) \leq D(x^*, y^*) \left(\lambda_1 + \lambda_2 \frac{D(x^*, y^*)}{1 + D(x^*, y^*)} \right). \tag{6}$$

If we suppose that $\lambda_1 + \lambda_2 \frac{D(x^*, y^*)}{1 + D(x^*, y^*)} \geq 1$ we get

$$\lambda_1 - 1 \geq (1 - \lambda_1 - \lambda_2) D(x^*, y^*). \tag{7}$$

However, since $\lambda_1 \in (0, 1)$ and $\lambda_1 + \lambda_2 < 1$, the inequality (7) holds only if $D(x^*, y^*) < 0$ which is impossible. Thus, we have $\lambda_1 + \lambda_2 \frac{D(x^*, y^*)}{1 + D(x^*, y^*)} < 1$ and from (6) we get $D(x^*, y^*) = 0$ and so x^* is the unique fixed point of T. □

Example 1. *Let $X = [0, 1]$ with the metric $D : X \times X \to [0, \infty)$ defined as $D(x, y) = |x - y|$ for all $(x, y) \in X \times X$. As stated in [2], for $f(t) = \ln t$ and $\alpha = 0$, (X, D) is an $\mathcal{F}$-metric space which is also $\mathcal{F}$-complete.*

Taking $Tx = \frac{x}{2}$ we can easily see that for $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \lambda_3 = \frac{1}{3}$ we have

$$D(Tx, Ty) = \left| \frac{x}{2} - \frac{y}{2} \right| \leq \frac{1}{2}|x - y| + \frac{1}{3} \frac{\left|x - \frac{y}{2}\right| \left|y - \frac{x}{2}\right|}{1 + |x - y|} + \frac{1}{3} \frac{\left|x - \frac{x}{2}\right| \left|x - \frac{y}{2}\right|}{1 + |x - y|}$$

which obviously holds since $\frac{1}{3} \frac{\left|x - \frac{y}{2}\right| \left|y - \frac{x}{2}\right|}{1 + |x - y|} + \frac{1}{3} \frac{\left|x - \frac{x}{2}\right| \left|x - \frac{y}{2}\right|}{1 + |x - y|} \geq 0.$

Since T satisfies the conditions in Theorem 1, and also $\frac{1}{2} + \frac{1}{3} \leq 1$, the operator has a unique fixed point $x^* = 0 \in [0, 1]$.

2.2 Kannan type theorem for an asymptotically regular operator

Bera et al. gave a Kannan type [11] fixed point result for $\mathcal{F}$-metric spaces in [2]. We now give a similar result, but for asymptotically regular operators.

Definition 2. *For T a continuous operator, we say that T is orbitally continuous if for all $x \in X$ and for all $\varepsilon > 0$, there exists $\delta > 0$, such that $D(T^i x, T^j x) < \varepsilon + \delta$ implies $D(T^{i+1} x, T^{j+1} x) \leq \varepsilon$, $\forall\, i, j \in \mathbb{N}$.*

Definition 3. *Let (X, D) be an $\mathcal{F}$-metric space. The operator $T : X \to X$ is asymptotically regular if for every $x \in X$*

$$\lim_{n\to\infty}(T^n x, T^{n+1}x) = 0.$$

Theorem 2. *Let (X, D) be a complete $\mathcal{F}$-metric space and $T : X \to X$ be an asymptotically regular operator which is orbitally continuous such that there exists an $a \in (0, 1)$ for which*

$$D(Tx, Ty) \leq a[D(x, Tx) + D(y, Ty)], \quad \forall\, x, y \in X. \tag{8}$$

Then, T has an unique fixed point.

Proof. Let $f \in \mathcal{F}$ and $\alpha \in [0, \infty]$ such that (D_3) is satisfied. By $\mathcal{F}_2$, for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < t < \delta \Rightarrow f(t) < f(\varepsilon) < f(\varepsilon) + \alpha \Rightarrow f(t) < f(\varepsilon) - \alpha \tag{9}$$

$$D(x_n, x_{n+p}) = D(Tx_{n-1}, Tx_{n+p-1}) \leq a[D(x_{n-1}, Tx_{n-1}) + D(x_{n+p-1}, Tx_{n+p-1})]$$

$$= a[D(x_{n-1}, x_n) + D(x_{n+p-1}, x_{n+p})] = a[D(T^{n-1}x, T^n x) + D(T^{n+p-1}x, T^{n+p}x)].$$

As T is asymptotically regular, this implies $D(x_n, x_{n+p}) \leq 0$ which means that the sequence $\{x_n\}$ is $\mathcal{F}$-Cauchy, and because X is $\mathcal{F}$-complete, therefore $\{x_n\}$ is $\mathcal{F}$-convergent.

We have proven that $x_n \to x^*$, and from T being orbitally continuous we have that $\{Tx_n\}$ tends to Tx^* as n approaches infinity. Since $Tx_n = x_{n+1}$ and from the fact that the limit of the sequence is unique, we get that $Tx^* = x^*$ which means that x^* is a fixed point of T.

We prove that x^* is the only fixed point of T and supposing that there exists another $y^* \in F_T$ and we calculate the distance

$$D(Tx^*, Ty^*) \leq a[D(Tx^*, x^*) + D(Ty^*, y^*)] = a[D(y^*, y^*) + D(x^*, x^*)] = 0$$

which implies $D(x^*, y^*) = 0$ and so $x^* = y^*$. We have proven that T has a unique fixed point. □

2.3 Common fixed point theorems

The following results are an equivalent for $\mathcal{F}$-metric spaces of the results given and proved by Jungck in [10].

Lemma 1. *Let (X, D) be an $\mathcal{F}$-complete $\mathcal{F}$ metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. If there exists an $\alpha \in (0, 1)$ such that $D(x_n, x_{n+1}) \leq \alpha D(x_{n-1}, x_n)$, for all $n \in \mathbb{N}$, then $\{x_n\}$ is convergent.*

Proof. Let $D(x_n, x_{n+1}) \leq \alpha D(x_{n-1}, x_n) \leq \alpha^2 D(x_{n-2}, x_{n-1}) \leq ... \leq \alpha^n D(x_1, x_0)$ and thus we get

$$\sum_{i=n}^{m+1} D(x_i, x_{i+1}) \leq \frac{\alpha^n}{1-\alpha} D(x_0, x_1), \quad m > n.$$

On the other hand

$$\lim_{n\to\infty} \frac{\alpha^n}{1-\alpha} D(x_0, x_1) = 0$$

which means that there exists $N \in \mathbb{N}$ such that

$$0 \leq \frac{\alpha^n}{1-\alpha} D(x_0, x_1) \leq \delta, \quad n \geq N.$$

From property $\mathcal{F}_2$ and (2.3) we get

$$f\left(\sum_{i=n}^{m+1} D(x_i, x_{i+1})\right) \leq f\left(\frac{\alpha^n}{1-\alpha} D(x_0, x_1)\right) \leq f(\varepsilon) - \alpha, \quad m > n \geq N.$$

Using the definition from (D_3) we have that $D(x_n, x_m) > 0, m > n \geq N$ implies

$$f(D(x_n, x_m)) \leq f\left(\sum_{i=n}^{m+1} D(x_i, x_{i+1})\right) + \alpha < f(\varepsilon), .$$

Thus, we get $D(x_n, x_m) < \varepsilon$ for $n > m \geq N$ which proves that the sequence $\{x_n\}$ is $\mathcal{F}$-Cauchy, and because X is $\mathcal{F}$-complete, then $\{x_n\}$ is $\mathcal{F}$-convergent. □

Theorem 3. *Let (X, D) be an $\mathcal{F}$-complete $\mathcal{F}$ metric space and $T : X \to X$ be a continuous operator. T has a fixed point if and only if there exists $\alpha \in (0, 1)$ and an operator $S : X \to X$ such that $T(Sx) = S(Tx), \forall x \in X$ and satisfies $S(X) \subset T(X)$ and $D(Sx, Sy) \leq \alpha D(Tx, Ty), \forall x, y \in X$. Moreover, T and S have a unique common fixed point.*

Proof. Let us suppose that T with the Picard iteration $T^n x = x_{n+1}$ has a fixed point $a \in X$ such that $Ta = a$ and let $S : X \to X$ with $Sx = a, \forall x \in X$. Then, we have $S(Tx) = T(Sx) = a, \forall x \in X$ and $Sx = Tx = a, \forall x \in X$ from which $S(X) \subset T(X)$.

If we have $\alpha \in (0, 1)$, from the hypothesis we have

$$D(Sx, Sy) = D(a, a) = 0 \leq \alpha D(Tx, Ty), \forall x, y \in X$$

so the condition holds.

Conversely, let us suppose that there exists an operator $S : X \to X$ such that $T(Sx) = S(Tx), \forall x \in X$ and $D(Sx, Sy) \leq \alpha D(Tx, Ty), \forall x, y \in X$.

Let $x_0 \in X$ and x_1 such that $Tx_1 = Sx_0$. Because we have $S(X) \subset T(X)$, we choose x_n such that $Tx_n = Sx_{n-1}$. As a consequence, we have

$$D(Tx_n, Tx_{n+1}) \leq \alpha D(Tx_{n-1}, Tx_n), \forall x \in X$$

and from Lemma 1 we have that $Tx_n \to x^*$ and as a result from $Tx_n = Sx_{n-1}$ we obtain $Sx_n \to x^*$. Moreover, since T is continuous, S is continuous, which along with the previous result implies $S(Tx_n) \to Sx^*$ and $T(Sx_n) \to Tx^*$. Since T and S commute, we get $T(Sx_n) = S(Tx_n)$ which means that $Sx^* = Tx^*$. Moreover, $T(Tx^*) = T(Sx^*) = S(Sx^*)$, and we can state that

$$D(Sx^*, S(Sx^*)) \leq \alpha D(Tx^*, T(Sx^*)) = \alpha D(Sx^*, S(Sx^*)).$$

Since $\alpha \in (0,1)$, and $D(Sx^*, S(Sx^*))(1-\alpha) \leq 0$, we get $Sx^* = S(Sx^*)$. Thus, we proved that $Sx^* = S(Sx^*) = T(Sx^*)$, which means that T and S have a common fixed point.

To prove that x^* is the unique fixed point of S and T, we argue by contradiction. We suppose that there exists $x^*, y^* \in X$ such that $x^* = Tx^* = Sx^*$ and $y^* = Ty^* = Sy^*$. Then we have $D(x,y) = D(Sx^*, Sy^*) \leq \alpha D(Tx^*, Ty^*) = \alpha D(y^*, y^*)$, which implies $D(x,y)(1-\alpha) \leq 0$ and for $\alpha \in (0,1)$ we have $x^* = y^*$. □

Corollary 1. *Let (X, D) be an $\mathcal{F}$-complete $\mathcal{F}$ metric space and $T, S : X \to X$ be commuting operators. For T continuous and $S(X) \subset T(X)$, if there exists $\alpha \in (0,1)$ and $k \in \mathbb{N}$ such that*

$$D(S^k x, S^k y) \leq \alpha D(Tx, Ty), \forall\, x, y \in X$$

then S and T have a unique common fixed point.

Proof. Since S and T commute, it is clear that $S^k(X) \subset S(X) \subset T(X)$ and we can apply Theorem 3 to T and S^k such that there is a unique point x^* so that $S^k x^* = Tx^* = x^*$.

We get $Sx^* = S(Tx^*) = T(Sx^*) = S^k(Sx^*)$ which means that Sx^* is the common fixed point for $S^k x^*$ and T. But since x^* is unique, we get $Sx^* = Tx^* = x^*$. □

Remark 2. *As a consequence of Corollary 1, we can get the proof for the Banach contraction principle that is presented in [9].*

We can give another consequence of Corollary 1.

Corollary 2. *Let (X, D) be an $\mathcal{F}$-complete $\mathcal{F}$ metric space and $T : X \to X$ be a continuous onto operator. If there exists $K > 1$ and $n \in \mathbb{N}$ such that*

$$D(T^n x, T^n y) \geq K D(x, y), \forall\, x, y \in X$$

then T has a unique fixed point.

Proof. Let $T = T^n$ and S the identity operator such that $Sx = x$, $\forall\, x \in X$.

It is true that T and S commute and $T^n x = S(T^n x) = T^n(Sx) = T^n x$, $\forall\, x \in X$ and we have $X = S(X) \subset T^n(X)$ since T is onto.

We can apply Theorem 3, for $T = T^n$ and $Sx = x$ $D(x,y) = D(Sx, Sy) \leq \alpha D(Tx, Ty)$, $\forall\, x, y \in X$ and $\alpha \in (0,1)$ implies that S and T^n have a unique fixed point.

Thus, taking $K = \dfrac{1}{\alpha} > 1$, in Theorem 3 we have

$$D(T^n x, T^n y) \geq KD(Sx, Sy) = KD(x, y)$$

which implies that T has a unique fixed point. □

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