SOME NEW FIXED POINT THEOREMS FOR $\mathcal{F}$-METRIC SPACES

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Abstract

The present paper introduces some new fixed point and coupled fixed point theorems for $\mathcal{F}$-metric spaces. We introduce a new fixed point theorem, a Kannan type theorem for asymptotically regular operators and basic coupled fixed point theorems along with some consequences.

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1 Introduction

Over the past years, generalizations of the metric space concept have been intensively studied. Starting with the emergence of $b$-metric spaces in Czerwik paper [3], many more new concepts of extensions and generalizations of metric spaces have appeared. Fagin et al. introduced the notion of $s$-relaxed $p$-metric in [5]. There have also been generalizations on an $X \times X \times X$ product set, like Gähler’s 2-metric in [6], and in Dhage’s paper [4] where the $D$-metric is defined.

Jleli and Samet [9] have recently introduced a new type of metric space, called an $\mathcal{F}$-metric space. Thereby, let $\mathcal{F}$ be the set of functions $f : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

$(\mathcal{F}_1)$ $f$ is non-decreasing, i.e. $0 < s < t \Rightarrow f(s) \leq f(t)$;

$(\mathcal{F}_2)$ for every sequence $\{t_n\} \subset (0, \infty)$, we have

$$\lim_{n \to \infty} t_n = 0 \iff \lim_{n \to \infty} f(t_n) = -\infty.$$ 

The concept of metric space is thus generalized as follows.

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Definition 1. Let $X$ be a nonempty set, and let $D : X \times X \rightarrow [0, \infty)$ be a given mapping. Suppose that there exists $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ such that

$$(D_1) \quad (x,y) \in X \times X, D(x,y) = 0 \iff x = y;$$

$$(D_2) \quad D(x,y) = D(y,x), \forall (x,y) \in X \times X;$$

$$(D_3) \quad \text{For every } (x,y) \in X \times X, \text{ for every } N \in \mathbb{N}, N \geq 2 \text{ and for every } (u_i)_{i=1}^{N} \subset X \text{ with } (u_1, u_N) = (x,y), \text{ we have }$$

$$D(x,y) > 0 \Rightarrow f(D(x,y)) \leq f \left( \sum_{i=1}^{N-1} D(x_i, x_{i+1}) \right) + \alpha.$$ 

Then $D$ is said to be an $\mathcal{F}$-metric on $X$, and the pair $(X,D)$ is said to be an $\mathcal{F}$-metric space.

2 Main Results

2.1 A new fixed point theorem for $\mathcal{F}$-metric spaces

Based on the results for b-metric spaces by Huang, Deng and Radenovic from [7], we give a fixed point theorem for $\mathcal{F}$-metric spaces which involves one similar inequality.

Theorem 1. Let $(X,D)$ be a complete bounded $\mathcal{F}$-metric space and $T : X \rightarrow X$ be an operator such that there exists $\lambda_1, \lambda_2, \lambda_3 \in (0,1)$ for which:

$$D(Tx,Ty) \leq \lambda_1 D(x,y) + \frac{\lambda_2 D(x,Ty)D(y,Tx)}{1 + D(x,y)} + \frac{\lambda_3 D(x,Tx)D(x,Ty)}{1 + D(x,y)} \quad (1)$$

$\forall x,y \in X.$ Then, $T$ has a fixed point.

Proof. Let $f \in \mathcal{F}$ and $\alpha \in [0,\infty)$ such that $(D_3)$ is satisfied. By $\mathcal{F}_2$, for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < t < \delta \Rightarrow f(t) < f(\varepsilon) < f(\varepsilon) + \alpha \Rightarrow f(t) < f(\varepsilon) - \alpha. \quad (2)$$

For $x_n = Tx_{n-1} = T^nx_0$, by (1), we have

$$D(Tx_n, Tx_{n-1}) = D(x_{n+1}, x_n) \leq \lambda_1 D(x_n, x_{n-1}) + \frac{\lambda_2 D(x_n, Tx_{n-1})D(x_{n-1}, Tx_n)}{1 + D(x_n, x_{n-1})} + \frac{\lambda_3 D(x_n, Tx_{n-1})D(x_n, Tx_n)}{1 + D(x_n, x_{n-1})}$$

$$\leq \lambda_1 D(x_n, x_{n-1}) \leq \ldots \leq \lambda_1^n D(x_1, x_0).$$

Thus we have

$$\sum_{i=n}^{m-1} D(x_i, x_{i+1}) \leq \frac{\lambda_1^n}{1 - \lambda_1} D(x_0, x_1), \quad m > n \quad (3)$$
and since
\[ \lim_{n \to \infty} \frac{\lambda^n_1}{1 - \lambda_1} D(x_0, x_1) = 0 \]
there exists an \( N \in \mathbb{N} \) such that
\[ 0 \leq \frac{\lambda^n_1}{1 - \lambda_1} D(x_0, x_1) \leq \delta, \quad n \geq N. \tag{4} \]

From (2) and (4) we get
\[ f \left( \sum_{i=n}^{m-1} D(x_i, x_{i+1}) \right) \leq f \left( \frac{\lambda^n_1}{1 - \lambda_1} D(x_0, x_1) \right) \leq f(\varepsilon) - \alpha, \quad m > n \geq N. \tag{5} \]

Using \((D_3)\) we have for \( D(x_n, x_m) > 0, m > n \geq N \)
\[ f(D(x_n, x_m)) \leq f \left( \sum_{i=n}^{m+1} D(x_i, x_{i+1}) \right) + \alpha < f(\varepsilon). \]

Thus, we get \( D(x_n, x_m) < \varepsilon \) for \( n > m \geq N \) which proves that the sequence \( \{x_n\} \) is \( \mathcal{F} \)-Cauchy, and because \( X \) is \( \mathcal{F} \)-complete, then \( \{x_n\} \) is \( \mathcal{F} \)-convergent, so there exists \( x^* \in X \) such that \( \lim_{n \to \infty} x_n = x^* \).

We prove that \( x^* \) is a fixed point of \( T \). If we suppose that \( D(Tx^*, x^*) > 0 \) then
\[ f(D(Tx^*, x^*)) \leq f(D(Tx^*, Tx_n) + D(Tx_n, x^*)) + \alpha \]
and using the inequality from (1) we have
\[ f(D(Tx^*, x^*)) \leq f \left( \lambda_1 D(x_n, x^*) + \lambda_2 \frac{D(x^*, Tx_n)D(x_n, Tx^*)}{1 + D(x_n, x^*)} + \lambda_3 \frac{D(x^*, Tx^*)D(x^*, Tx_n)}{1 + D(x_n, x^*)} + D(Tx_n, x^*) + \alpha \right). \]

Since \( \lim_{n \to \infty} D(x_n, x^*) = 0 \) and \( Tx_n = x_{n+1} \) we get
\[ \lim_{n \to \infty} f \left( \lambda_1 D(x_n, x^*) + \lambda_2 \frac{D(x^*, Tx_n)D(x_n, Tx^*)}{1 + D(x_n, x^*)} + \lambda_3 \frac{D(x^*, Tx^*)D(x^*, Tx_n)}{1 + D(x_n, x^*)} + D(Tx_n, x^*) + \alpha \right) = -\infty \]
and thus \( D(x^*, Tx^*) = 0 \) which implies \( x^* = Tx^* \), meaning that \( x^* \) is a fixed point of \( T \).

\[ \square \]

**Remark 1.** Moreover, if \( \lambda_1 + \lambda_2 < 1 \), the fixed point is unique.
Proof. We suppose that there exists $x^*, y^* \in X$ such that $Tx^* = x^*$ and $Ty^* = y^*$. We get

$$D(Tx^*, Ty^*) = D(x^*, y^*) \leq \lambda_1 D(x^*, y^*) + \lambda_2 \frac{D(x^*, Ty^*) D(y^*, Tx^*)}{1 + D(x^*, y^*)} + \lambda_3 \frac{D(x^*, Tx^*) D(y^*, Ty^*)}{1 + D(x^*, y^*)}$$

and so

$$D(x^*, y^*) \leq \lambda_1 D(x^*, y^*) + \lambda_2 \frac{D(x^*, y^*) D(y^*, x^*)}{1 + D(x^*, y^*)} + \lambda_3 \frac{D(x^*, x^*) D(y^*, y^*)}{1 + D(x^*, y^*)}$$

and grouping by $D(x^*, y^*)$

$$D(x^*, y^*) \leq D(x^*, y^*) \left( \lambda_1 + \lambda_2 \frac{D(x^*, y^*)}{1 + D(x^*, y^*)} \right). \quad (6)$$

If we suppose that $\lambda_1 + \lambda_2 \frac{D(x^*, y^*)}{1 + D(x^*, y^*)} \geq 1$ we get

$$\lambda_1 - 1 \geq (1 - \lambda_1 - \lambda_2) D(x^*, y^*). \quad (7)$$

However, since $\lambda_1 \in (0, 1)$ and $\lambda_1 + \lambda_2 < 1$, the inequality (7) holds only if $D(x^*, y^*) < 0$ which is impossible. Thus, we have $\lambda_1 + \lambda_2 \frac{D(x^*, y^*)}{1 + D(x^*, y^*)} < 1$ and from (6) we get $D(x^*, y^*) = 0$ and so $x^*$ is the unique fixed point of $T$. \hfill $\square$

Example 1. Let $X = [0, 1]$ with the metric $D : X \times X \to [0, \infty)$ defined as $D(x, y) = |x - y|$ for all $(x, y) \in X \times X$. As stated in [2], for $f(t) = \ln t$ and $\alpha = 0$, $(X, D)$ is an $\mathcal{F}$-metric space which is also $\mathcal{F}$-complete.

Taking $Tx = \frac{x}{2}$ we can easily see that for $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \lambda_3 = \frac{1}{3}$ we have

$$D(Tx, Ty) = \left| \frac{x}{2} - \frac{y}{2} \right| \leq \frac{1}{2} |x - y| + \frac{1}{3} |x - \frac{x}{2}| \frac{|y - \frac{x}{2}|}{1 + |x - y|} + \frac{1}{3} |x - \frac{x}{2}| \frac{|x - \frac{x}{2}|}{1 + |x - y|}$$

which obviously holds since

$$\frac{1}{3} \frac{|x - \frac{x}{2}|}{1 + |x - y|} + \frac{1}{3} \frac{|x - \frac{x}{2}|}{1 + |x - y|} \geq 0.$$

Since $T$ satisfies the conditions in Theorem 1, and also $\frac{1}{2} + \frac{1}{3} \leq 1$, the operator has a unique fixed point $x^* = 0 \in [0, 1]$.

2.2 Kannan type theorem for an asymptotically regular operator

Bera et al. gave a Kannan type [11] fixed point result for $\mathcal{F}$-metric spaces in [2]. We now give a similar result, but for asymptotically regular operators.

Definition 2. For $T$ a continuous operator, we say that $T$ is orbitally continuous if for all $x \in X$ and for all $\varepsilon > 0$, there exists $\delta > 0$, such that $D(T^i x, T^j x) < \varepsilon + \delta$ implies $D(T^{i+1} x, T^{j+1} x) \leq \varepsilon$, $\forall i, j \in \mathbb{N}$. 
Definition 3. Let \((X, D)\) be an \(\mathcal{F}\)-metric space. The operator \(T : X \to X\) is asymptotically regular if for every \(x \in X\)

\[\lim_{n \to \infty} (T^n x, T^{n+1} x) = 0.\]

Theorem 2. Let \((X, D)\) be a complete \(\mathcal{F}\)-metric space and \(T : X \to X\) be an asymptotically regular operator which is orbitally continuous such that there exists an \(a \in (0, 1)\) for which

\[D(Tx, Ty) \leq a[D(x, Tx) + D(y, Ty)], \quad \forall x, y \in X.\]  \(\text{(8)}\)

Then, \(T\) has an unique fixed point.

Proof. Let \(f \in \mathcal{F}\) and \(\alpha \in [0, \infty]\) such that \((D_3)\) is satisfied. By \(\mathcal{F}_2\), for \(\varepsilon > 0\), there exists \(\delta > 0\) such that

\[0 < t < \delta \Rightarrow f(t) < f(\varepsilon) < f(\varepsilon) + \alpha \Rightarrow f(t) < f(\varepsilon) - \alpha\]  \(\text{(9)}\)

\[D(x_n, x_{n+p}) = D(Tx_{n-1}, Tx_{n+p-1}) \leq a[D(x_{n-1}, Tx_{n-1}) + D(x_{n+p-1}, Tx_{n+p-1})] = a[D(x_{n-1}, x_n) + D(x_{n+p-1}, x_{n+p})] = a[D(T^{n-1}x, T^n x) + D(T^{n+p-1}x, T^{n+p} x)].\]

As \(T\) is asymptotically regular, this implies \(D(x_n, x_{n+p}) \leq 0\) which means that the sequence \(\{x_n\}\) is \(\mathcal{F}\)-Cauchy, and because \(X\) is \(\mathcal{F}\)-complete, therefore \(\{x_n\}\) is \(\mathcal{F}\)-convergent.

We have proven that \(x_n \to x^*\), and from \(T\) being orbitally continuous we have that \(\{Tx_n\}\) tends to \(Tx^*\) as \(n\) approaches infinity. Since \(Tx_n = x_{n+1}\) and from the fact that the limit of the sequence is unique, we get that \(Tx^* = x^*\) which means that \(x^*\) is a fixed point of \(T\).

We prove that \(x^*\) is the only fixed point of \(T\) and supposing that there exists another \(y^* \in F_T\) and we calculate the distance

\[D(Tx^*, Ty^*) \leq a[D(Tx^*, x^*) + D(Ty^*, y^*)] = a[D(y^*, y^*) + D(x^*, x^*)] = 0\]

which implies \(D(x^*, y^*) = 0\) and so \(x^* = y^*\). We have proven that \(T\) has a unique fixed point. \(\square\)

2.3 Common fixed point theorems

The following results are an equivalent for \(\mathcal{F}\)-metric spaces of the results given and proved by Jungck in \([10]\).

Lemma 1. Let \((X, D)\) be an \(\mathcal{F}\)-complete \(\mathcal{F}\) metric space and \(\{x_n\}_{n \in \mathbb{N}}\) be a sequence in \(X\). If there exists an \(\alpha \in (0, 1)\) such that \(D(x_n, x_{n+1}) \leq \alpha D(x_{n-1}, x_n)\), for all \(n \in \mathbb{N}\), then \(\{x_n\}\) is convergent.
Theorem 3. Let $D(x_n, x_{n+1}) \leq \alpha D(x_{n-1}, x_n) \leq \alpha^2 D(x_{n-2}, x_{n-1}) \leq \ldots \leq \alpha^n D(x_1, x_0)$ and thus we get

$$\sum_{i=n}^{m+1} D(x_i, x_{i+1}) \leq \frac{\alpha^n}{1-\alpha} D(x_0, x_1), \quad m > n.$$ 

On the other hand

$$\lim_{n \to \infty} \frac{\alpha^n}{1-\alpha} D(x_0, x_1) = 0$$

which means that there exists $N \in \mathbb{N}$ such that

$$0 \leq \frac{\alpha^n}{1-\alpha} D(x_0, x_1) \leq \delta, \quad n \geq N.$$ 

From property $\mathcal{F}_2$ and (2.3) we get

$$f \left( \sum_{i=n}^{m+1} D(x_i, x_{i+1}) \right) \leq f \left( \frac{\alpha^n}{1-\alpha} D(x_0, x_1) \right) \leq f(\varepsilon) - \alpha, \quad m > n \geq N.$$ 

Using the definition from (D_3) we have that $D(x_n, x_m) > 0, m > n \geq N$ implies

$$f(D(x_n, x_m)) \leq f \left( \sum_{i=n}^{m+1} D(x_i, x_{i+1}) \right) + \alpha < f(\varepsilon),.$$ 

Thus, we get $D(x_n, x_m) < \varepsilon$ for $n > m \geq N$ which proves that the sequence $\{x_n\}$ is $\mathcal{F}$-Cauchy, and because $X$ is $\mathcal{F}$-complete, then $\{x_n\}$ is $\mathcal{F}$-convergent. \qed

Theorem 3. Let $(X, D)$ be an $\mathcal{F}$-complete $\mathcal{F}$ metric space and $T : X \to X$ be a continuous operator. $T$ has a fixed point if and only if there exists $\alpha \in (0, 1)$ and an operator $S : X \to X$ such that $T(Sx) = S(Tx), \forall x \in X$ and satisfies $S(X) \subset T(X)$ and $D(Sx, Sy) \leq \alpha D(Tx, Ty), \forall x, y \in X$. Moreover, $T$ and $S$ have a unique common fixed point.

Proof. Let us suppose that $T$ with the Picard iteration $T^n x = x_{n+1}$ has a fixed point $a \in X$ such that $Ta = a$ and let $S : X \to X$ with $Sx = a, \forall x \in X$. Then, we have $S(Tx) = T(Sx) = a, \forall x \in X$ and $Sx = Tx = a, \forall x \in X$ from which $S(X) \subset T(X)$.

If we have $\alpha \in (0, 1)$, from the hypothesis we have

$$D(Sx, Sy) = D(a, a) = 0 \leq \alpha D(Tx, Ty), \forall x, y \in X$$

so the condition holds.

Conversely, let us suppose that there exists an operator $S : X \to X$ such that $T(Sx) = S(Tx), \forall x \in X$ and $D(Sx, Sy) \leq \alpha D(Tx, Ty), \forall x, y \in X$.

Let $x_0 \in X$ and $x_1$ such that $Tx_1 = Sx_0$. Because we have $S(X) \subset T(X)$, we choose $x_n$ such that $Tx_n = Sx_{n-1}$. As a consequence, we have

$$D(Tx_n, Tx_{n+1}) \leq \alpha D(Tx_{n-1}, Tx_n), \forall x \in X$$
and from Lemma 1 we have that $T x_n \to x^*$ and as a result from $T x_n = S x_{n-1}$ we obtain $S x_n \to x^*$. Moreover, since $T$ is continuous, $S$ is continuous, which along with the previous result implies $S(T x_n) \to S x^*$ and $T(S x_n) \to T x^*$. Since $T$ and $S$ commute, we get $T(S x_n) = S(T x_n)$ which means that $S x^* = T x^*$. Moreover, $T(T x^*) = T(S x^*) = S(S x^*)$, and we can state that

$$D(S x^*, S(S x^*)) \leq \alpha D(T x^*, T(S x^*)) = \alpha D(S x^*, S(S x^*)).$$

Since $\alpha \in (0, 1)$, and $D(S x^*, S(S x^*))(1-\alpha) \leq 0$, we get $S x^* = S(S x^*)$. Thus, we proved that $S x^* = S(S x^*) = T(S x^*)$, which means that $T$ and $S$ have a common fixed point.

To prove that $x^*$ is the unique fixed point of $S$ and $T$, we argue by contradiction. We suppose that there exists $x^*, y^* \in X$ such that $x^* = T x^* = S x^*$ and $y^* = T y^* = S y^*$. Then we have $D(x, y) = D(S x^*, S y^*) \leq \alpha D(T x^*, T y^*) = \alpha D(y^*, y^*)$, which implies $D(x, y)(1-\alpha) \leq 0$ and for $\alpha \in (0, 1)$ we have $x^* = y^*$. □

**Corollary 1.** Let $(X, D)$ be an $F$-complete $F$ metric space and $T, S : X \to X$ be commuting operators. For $T$ continuous and $S(X) \subset T(X)$, if there exists $\alpha \in (0, 1)$ and $k \in \mathbb{N}$ such that

$$D(S^k x, S^k y) \leq \alpha D(T x, T y), \forall x, y \in X$$

then $S$ and $T$ have a unique common fixed point.

**Proof.** Since $S$ and $T$ commute, it is clear that $S^k(X) \subset S(X) \subset T(X)$ and we can apply Theorem 3 to $T$ and $S^k$ such that there is a unique point $x^*$ so that $S^k x^* = T x^* = x^*$. We get $S x^* = S(T x^*) = T(S x^*) = S^k(S x^*)$ which means that $S x^*$ is the common fixed point for $S^k x^*$ and $T$. But since $x^*$ is unique, we get $S x^* = T x^* = x^*$. □

**Remark 2.** As a consequence of Corollary 1, we can get the proof for the Banach contraction principle that is presented in [9].

We can give another consequence of Corollary 1.

**Corollary 2.** Let $(X, D)$ be an $F$-complete $F$ metric space and $T : X \to X$ be a continuous onto operator. If there exists $K > 1$ and $n \in \mathbb{N}$ such that

$$D(T^n x, T^n y) \geq K D(x, y), \forall x, y \in X$$

then $T$ has a unique fixed point.

**Proof.** Let $T = T^n$ and $S$ the identity operator such that $S x = x, \forall x \in X$.

It is true that $T$ and $S$ commute and $T^n x = S(T^n x) = T^n(S x) = T^n x, \forall x \in X$ and we have $X = S(X) \subset T^n(X)$ since $T$ is onto.

We can apply Theorem 3, for $T = T^n$ and $S x = x D(x, y) = D(S x, S y) \leq \alpha D(T x, T y), \forall x, y \in X$ and $\alpha \in (0, 1)$ implies that $S$ and $T^n$ have a unique fixed point.
Thus, taking \( K = \frac{1}{\alpha} > 1 \), in Theorem 3 we have

\[
D(T^n x, T^n y) \geq KD(Sx, Sy) = KD(x, y)
\]

which implies that \( T \) has a unique fixed point.

\[\Box\]

References


